A Note on a Risk-Return Measure of Hedging Effectiveness

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In a recent paper, Kuo and Chen (1995) propose a simplification of the Howard and D'Antonio (1984, 1987) model of hedging effectiveness. This note extends Kuo-Chen's suggested simplification to derive the optimal hedge ratio and second order conditions (SOCs) of the Howard-D'Antonio model. These SOCs were previously reported incorrectly by both Howard and D'Antonio (1984, 1987) and Chang and Shanker (1987). The corrected SOC is less restrictive than supposed previously and shows that there is no need to make *a priori* assumptions about risk-return relatives between spot and futures assets to ensure that SOCs conditions are satisfied. © 1998 John Wiley & Sons, Inc. Jrl Fut Mark 18:867–870, 1998

INTRODUCTION

In a recent issue of this journal, Kuo and Chen [K-C] (1995) argue that Howard and D'Antonio [H-D] (1984, 1987), in deriving a risk-return measure of hedging effectiveness, define the rate of return on futures investments inappropriately; this definition makes both the theoretical derivation and empirical estimation of the optimal hedge ratio awkward. This article shows that K-C's derivation of the optimal hedge ratio can be

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further simplified. The article also shows that the second order conditions derived by both H-D (1984) and Chang and Shanker (1987) are incorrect, and extends K-C's suggested simplification to derive the correct second order conditions.

THE OPTIMAL HEDGE RATIO

The expected rate of return on a hedged portfolio $E(r_h)$ in the K-C model (following K-C's notation) is given by:

$$E(r_h) = \mu_s + b\mu_a \tag{1}$$

where $\mu_s = E(r_s) = E(P_{s1} - P_{so})/P_{so}$; $\mu_a = E(r_a) = E(P_{f1} - P_{fo})/P_{so}$; $b = Q_f/Q_s$; P_{st}, P_{ft} represent the spot and futures price at time t and Q_s, Q_f represent the quantity of spot and futures. Now r_a is *not* a rate of return on futures but is related to it thus: $r_a = [(P_{fo}/P_{so})]\,[(P_{f1} - P_{fo})/P_{fo}] = r_0 r_f$ and $\mu_a = r_0\mu_f$ where $\mu_f = E(r_f)$. The hedger's optimization problem is to maximize the excess return to risk or Sharpe ratio (θ) which is given by:

$$\theta = \frac{E(r_h) - i}{\sigma_p} = \frac{(\mu_s - i) + b\mu_a}{(\sigma_s^2 + b^2\sigma_a^2 + 2b\sigma_{sa})^{1/2}} \tag{2}$$

where i is the risk-free rate of interest; σ_p is the standard deviation of the portfolio given in eq. (1); σ_s^2, σ_a^2 are the variances of r_s and r_a; σ_{sa} is the covariance between r_s and r_a; $\sigma_{sa} = r_{sa}\sigma_s\sigma_a$ where r_{sa} is the coefficient of correlation between r_s and r_a. Differentiating θ with respect to the hedge ratio (b) results in

$$\frac{\partial\theta}{\partial b} = \frac{\{\mu_a\sigma_p\} - \left\{\left(\frac{1}{2}\right)(\sigma_p)^{-1}2(b\sigma_a^2 + \sigma_{sa})\,((\mu_s - i) + b\mu_a)\right\}}{\sigma_p^2} \tag{3}$$

and the first order conditions imply that

$$(\mu_a)(\sigma_s^2 + b^2\sigma_a^2 + 2b\sigma_{sa}) = (b\sigma_a^2 + \sigma_{sa})((\mu_s - i) + b\mu_a)$$

Solving for the optimal hedge ratio, b^*, we get

$$b^* = \frac{\mu_a\sigma_s^2 - \sigma_{sa}(\mu_s - i)}{\sigma_a^2(\mu_s - i) - \mu_a\sigma_{sa}} \tag{4}$$

It should be noted that the derivation of b^* is clearly more straightforward than in the K-C paper.[1] It can also be easily shown that (4) can be rearranged into the form obtained by both K-C and H-D.[2]

SECOND ORDER CONDITIONS

H-D (1984) mention that the second order conditions (SOCs) for eq. (3) to attain a maximum is given by $(\mu_s - i) > 0$ and $(1 - \Gamma_0 r_{sf}) > 0$.[3] In a subsequent comment on the H-D paper, Chang and Shanker [C-S] (1987) point out that the true SOC does not restrict $(\mu_s - i) > 0$. However, both H-D (1984, 1987) and C-S (1987) are incorrect. In fact, the SOCs in the above papers are too restrictive. Using K-C's simplification, the correct SOC is derived below.

Differentiate eq. (3) to obtain

$$\frac{\partial^2\theta}{(\partial b)^2} = \frac{\left[\dfrac{\partial\ \text{Num}}{\partial\ b}\right][\sigma_p^2] - \left[\dfrac{\partial(\sigma_p^2)}{\partial\ b}\right][\text{Num}]}{[\sigma_p^2]^2} = \frac{\left[\dfrac{\partial\ \text{Num}}{\partial\ b}\right]}{[\sigma_p^2]} \tag{5}$$

where Num is the numerator of eq. (3) and the result above is implied by the first order conditions. Now

$$\frac{\partial^2\theta}{(\partial b)^2} = \frac{1}{\sigma_p^2}\left[\mu_a\left(\frac{\partial\sigma_p}{\partial b}\right) + \left(\frac{\partial\sigma_p}{\partial b}\right)^2[\theta] - [\sigma_a^2]\,[\theta] - \left(\frac{\partial\sigma_p}{\partial b}\right)\mu_a\right] \tag{6}$$

where use is made of the fact that $\partial\sigma_p/\partial b = \sigma_p^{-1}[b\sigma_a^2 + \sigma_{sa}]$. Thus,

$$\frac{\partial^2\theta}{(\partial b)^2} = \frac{\theta}{\sigma_p^2}\left[\left(\frac{\partial\sigma_p}{\partial b}\right)^2 - \sigma_a^2\right] = \frac{\theta}{(\sigma_p^2)^2}\left[(b\sigma_a^2 + \sigma_{sa})^2 - \sigma_a^2\sigma_p^2\right] \tag{7}$$

Simplifying eq. (7) leads to

[1]There are some inconsistencies in K-C's derivation of the optimal hedge ratio. The equation for the variance of the hedged position $\text{Var}(r_h)$ is incorrect (line 3, p. 43). K-C refer to a variable X_n (line 7, p. 43) which is not referenced in the paper. Presumably this variable is the same as b^*, and there is some confusion as they switch from one notation to the other.

[2]Eq. (4) can be rearranged as:

$$b^* = \frac{\Gamma - r_{sa}}{\dfrac{\sigma_a}{\sigma_s}(1 - \Gamma\, r_{sa})} = \frac{\Gamma_0 - r_{sf}}{\dfrac{r_0\sigma_f}{\sigma_s}(1 - \Gamma_0\, r_{sf})}$$

where $\Gamma = [(\mu_a/\sigma_a)/(\mu_s - i/\sigma_s)]$, $\Gamma_0 = [(\mu_f)/(\sigma_f)/(\mu_s - i/\sigma_s)]$. The first expression corresponds to the K-C model. Noting that $\sigma_a = r_0\sigma_f$, $\sigma_{sa} = r_0\sigma_{sf}$, $r_{sa} = r_{sf}$ (σ_{sf} is the covariance and r_{sf} is the correlation between r_s and r_f) leads to the H-D model.

[3]$\mu_s - i$ can be interpreted as the risk premium and Γ_0 as the risk-return relative. The value of Γ_0 represents the relative attractiveness of investing in futures versus spot assets.

$$\frac{\partial^2\theta}{(\partial\ b)^2} = \frac{\theta\sigma_s^2\sigma_a^2}{(\sigma_p^2)^2}\,[(r_{sa})^2 - 1] = \frac{\theta\sigma_s^2 r_0^2\sigma_f^2}{(\sigma_p^2)^2}\,[(r_{sf})^2 - 1] \qquad (8)$$

Clearly the sufficient condition required by eq. (2) to attain a maximum is given by $[(r_{sf})^2 - 1] < 0$.[4] In effect, this condition restricts $-1 < r_{sf} < +1$ (i.e. excludes perfect correlation), a condition that will be invariably satisfied in empirical applications. This condition should be contrasted with the SOCs in the H-D and C-S papers.[5] The SOC in the H-D paper is given by $(1 - \Gamma_0 r_{sf}) > 0$, a condition that will be automatically satisfied if $\Gamma_0 < 1$ and $r_{sf} \geq 0$. H-D in fact assume that $\Gamma_0 < 1$, an assumption that restricts the futures asset to offer less excess return per unit of risk as compared to the spot asset. The SOC derived here however indicates that no such restriction is required. Given the empirical applicability of the H-D model, it is encouraging that the empirical researcher need make no *a priori* assumptions regarding the risk-return relationship between spot and futures assets to ensure that SOCs are fulfilled.

CONCLUSION

In a recent paper, K-C propose a simplification of the H-D model of hedging effectiveness. This note extends their suggested simplification to derive the optimal hedge ratio and SOCs of the H-D model. These SOCs were previously reported incorrectly by both H-D and C-S. The corrected SOC is less restrictive than supposed previously and shows that there is no need to make *a priori* assumptions about risk-return relatives between spot and futures assets to ensure that SOCs are satisfied.

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[4]Note that θ must be positive *ex ante*. Since θ is the value of the objective function, its maximization requires that it be positive.

[5]The SOCs are not derived in either the H-D paper or the C-S paper. The objective function in the H-D paper is written in an unnecessarily complicated manner and the SOCs are similarly cumbersome to derive.