Stochastic Dominance Arguments and the Bounding of the Generalized Concave Option Price

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INTRODUCTION

Nonlinear options (options whose payoff, in case of exercise, is not linear) exist under a more or less hidden form in the world of real and financial options. Classic puts on debt instruments, such as Treasury bonds and Treasury bills, can be viewed, e.g., as calls with a concave payoff—if exercised—of the underlying interest rate (which is the independent state variable following a diffusion process), due to the fact that the bond price is a decreasing convex function of the interest rate. Studying calls where the payoff is a concave function of the underlying asset is thus essential to the understanding of bond call pricing. Similarly, a call (put) spread is nothing else but a call whose payoff is a concave function of the underlying asset. Lastly, certain structures of personal taxes can transform a linear option into a concave one for the investor with an increasing marginal tax rate.[1]

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[1]This remark is due to a referee.

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Moreover, generalized options have recently appeared in "over-the-counter" financial markets, as the so-called "power options" where the reward at expiry is a power of the difference between stock price and striking price. Hart and Ross (1994) justify the existence of nonlinear options by showing that the use of continuous strike options can result in improved hedging characteristics.

In two recent articles, Henin and Pistre (1996a,b) study the case of convex calls and puts. A power option where the power is smaller than one constitutes an example of a concave option, even if it is piecewise linear. Concave calls also exist in the domain of real options. Some types of real investments display payoffs which are concave functions of the underlying price, because of the complex interaction of fixed and variable charges.

The purpose of this article is to provide theoretical results about lower and upper bounds found on reserve buying and selling prices of concave European calls and puts, using second order stochastic dominance arguments without referring to the duplication "argument."

Black and Scholes (1973) and Merton (1973a) have published the fundamental articles on option valuation, followed by other seminal articles on option pricing such as the ones by Harrisson and Kreps (1979), Harrisson and Pliska (1981, 1983), Karatzas (1988), and El Karoui and Rochet (1989). All of these articles assume perfect and complete markets, where options can be replicated by self-financing strategies involving continual revisions in a portfolio containing the underlying security. A precise value for the premium of the option is obtained by setting some properties for the distribution of the underlying security. This allows the computation of a (unique) option price through "risk-neutral" expected cash flows discounted at the risk-free rate. However, the impossibility of continuous trading due to transaction costs, or more generally the incompleteness of the market, is the relevant economic explanation for the opening of derivatives markets. Actually, when continuous trading is excluded, or with transaction costs, Black-Scholes option prices can be generated only with very restrictive preferences (Rubinstein, 1976; Brennan, 1979). At equilibrium derivative prices should aggregate individual demands for all the assets, especially with respect to this hedging role toward a stochastic opportunity environment, as Merton (1973b), and Cox, Ingersoll, and Ross (1985) have pointed out.

A part of the literature tries to answer these questions by considering that the Black-Scholes arbitrage argument is invalidated by transaction costs (Leland, 1985; Boyle and Vorst, 1992; Jouini and Kallal, 1995). Taking a new turn, Perrakis and Ryan (1984) and Perrakis (1986) use the equilibrium formula derived by Rubinstein (1976) to evaluate any asset; comparing three strategies involving calls, riskless assets, and stocks, they determine minimum necessary bounds as equilibrium values given a consensus on stock price distributions. Ritchken (1985) and Ritchken and Kuo (1988) develop bounds for option prices based on primitive prices in incomplete markets, where state prices are not unique, by a linear programming approach placing constraints on time preferences, risk aversion, and state probabilities.

From another literature, Hadar and Russel (1969, 1971, 1974) develop the notions of first- and second-degree stochastic dominance. Second-degree stochastic dominance allows the universal ranking of distributions among risk averse individuals. Levy and Kroll (1978) and Kroll and Levy (1979) extend the use of stochastic dominance by including riskless asset in the strategies. The notion of stochastic dominance is particularly useful if neither the contingent claim nor the underlying asset is traded, which can be the case for real assets, for instance.

The academic community has lost interest in the concept of stochastic dominance, maybe because of the search for a unique price aggregating individual preferences (at equilibrium) or independent of the preferences (arbitrage). However, the limits of the arbitrage arguments in incomplete markets are mentioned in the seminal article of Harrison and Kreps (1979). The question of the diversity of transaction prices is a well-known issue in the microstructure literature: the existence of the spread (or of the transaction price) precludes the existence of a unique price. In an order driven market or a price driven market, the existence of different selling and buying prices at the same time is obvious. A traditional maximization of the individual utility has been developed first with a hedging purpose (Föllmer and Sondermann, 1986; Schäl, 1994). Some recent works deal with the pricing problem by determining the reserve (bid or ask) price of an individual for an option he or she can trade or not. This approach is employed widely in the microstructure literature, e.g., by Ho and Stoll (1981, 1983) to calculate reservation prices for a dealer. Constantinides and Zariphopoulou (1995) derive bounds for option prices under proportional transaction costs, for a (relatively) large class of utility functions, by stochastic control methods using viscosity solutions. The disadvantage of this method is that calculations are rather tough and they are done for special classes of utility functions.

On the contrary, the use of stochastic dominance arguments is very simple, even with transaction costs, because there is no optimization problem, and the criterion is universal among risk averse individuals when second order stochastic dominance is used.

Following Levy (1985), Henin and Pistre (1997) use this method to evaluate buying and selling reserve prices of classic European calls and found universal bounds for risk averse buyers and sellers. For a logarithmic Brownian motion of the underlying stock, the calculations are done for different expectations of the mean return, and they find the spread between the highest buying price and the lowest selling price, which is always tight. The market maker is thus able to define the limits of his or her potential counterparts, which do not necessarily include the Black and Scholes price in case of incomplete markets.

The purpose of this article is to use this methodology to derive bounds on option prices whose exercise payoff is a concave function of the difference between the underlying asset and striking price.

The present article develops general conditions based on second order stochastic arguments that such concave option prices must satisfy under very broad assumptions, without making the hypothesis of complete markets, but using the hypothesis that investors are risk averse. Theoretical lower and upper bounds are derived for buying and selling prices of concave calls and puts, and put/call "parity" relations are also presented.

HYPOTHESES, DEFINITIONS, AND NOTATIONS

A generalized option on a given financial instrument (security, index, future, etc) or a real asset is an instrument that gives a reward at a given date (expiry date), which is a function of the price of the underlying security vis-à-vis a certain price called the strike price. An option is said to be American if this reward can be obtained before the expiry date, European if otherwise. A generalized call (put) is an option whose reward is zero below (above) the strike price. This article focuses on the study of a particular type of options: European, continuous concave calls and puts, i.e., options with a concave reward function.

Hypotheses

H1. The options are European options, i.e., they can only be exercised at expiry.

H2. The price of the underlying security is S_0 at time 0. There exists for the underlying security a probabilistic distribution of values S_T at time T, the expiration date of the options. There is no particular restriction on the underlying distribution of the security price at time T, which is only defined between 0 and infinity by a cumulative distribution function F and its density function f. The expected final value of the stock and the same expectation stopped at stock value y are defined respectively as

$$E\ (\tilde{S}_T) = \int_0^\infty xf(x)dx$$

$$E_y\ (\tilde{S}_T) = \int_0^y xf(x)dx$$

Two other notations must be defined for values that appear in our results. The first one is the "stopped"-at-point-y expected value of the stock divided by the distribution function at this point. This can be viewed as the mean value of the distribution until point y. The second one is the complementary data; it can be viewed as the mean value of the distribution after point y:

$$E_y^*\ (\tilde{S}_T) = \frac{\int_0^y xf(x)dx}{F(y)};\quad E^y(\tilde{S}_T) = E(\tilde{S}_T) - E_y\ (\tilde{S}_T);$$

$$E^{y*}\ (\tilde{S}_T) = \frac{[E(\tilde{S}_T) - E_y\ (\tilde{S}_T)]}{[1 - F(y)]}$$

H3. The call is defined by a payoff function at expiry of the type: $C_T = \text{Max}(0, u(S_T - K))$, where $u(0) = 0$, $u' > 0$, $u'' < 0$; u can be a piecewise linear option. The put is defined by a payoff function of S_T at expiry of the type: $\text{Max}(0, u(K - S_T))$, where $u(0) = 0$, $u' < 0$, $u'' < 0$. These reward functions also have final expected values. The same notations as for the stock are defined for the call and for the put:

$$E(\tilde{C}_T) = \int_k^\infty u(x - K)f(x)dx;\ E_y(\tilde{C}_T) = \int_K^y u(x - K)f(x)dx;$$

$$E_y^*(\tilde{C}_T) = E_y(\tilde{C}_T)/F(y)$$

and

$$E^y\ (\tilde{C}_T) = E(\tilde{C}_T) - E_y\ (\tilde{C}_T);\ E^{y*}\ (\tilde{C}_T) = E^y\ (\tilde{C}_T)/[1 - F(y)]$$

Similarly,

$$E(\tilde{P}_T) = \int_0^K u(K - x)f(x)dx; \; E_y(\tilde{P}_T) = \int_y^K u(K - x)f(x)dx;$$

$$E_y^*\ (\tilde{P}_T) = E_y\ (\tilde{P}_T)/F(y)$$

and

$$E^y\ (\tilde{P}_T) = E_y\ (\tilde{P}_T) - E_y^*\ (\tilde{P}_T); \; E^{y*}\ (\tilde{P}_T) = E^y\ (\tilde{P}_T)/[1 - F(y)]$$

H4. There exists a risk-free rate r of interest during the life of the options, and borrowing and lending can be made at this rate. These conditions are very general and do not impose a given behavior on stock prices. They just ensure the existence of finite parameters for bounding the options. They do not preclude the existence of dividends—discrete or continuous—on the underlying security. Moreover, transaction costs can be included by simply calculating the expected values of gains by taking fees into account. The hypothesis that leads all the proofs now follows.

H5. Nondominance hypothesis: If two nonlosing strategies[2] have the same cost at time 0, there cannot exist a relation of stochastic dominance of the first or second order[3] between these two strategies or a fortiori, a relation of strict dominance. This last hypothesis allows us to define minimum conditions so that none of the two—same cost—strategies dominates the other, without using the hypothesis that the market is complete nor the duplication of the call.

If expectations are homogeneous, bounds on the price are universal [because the SSD criterion is universal among risk averse individuals (Fishburn, 1964; Hadar and Russel, 1969; Rothschild and Stiglitz, 1970, 1971)]. If expectations are heterogeneous, these bounds are subjective.

Some simple lemmas are used in the proofs of the article, the proofs of which can be found in Henin and Pistre (1997). The lemmas are recalled in Appendix A.

[2]It is always possible to define "losing" strategies; a losing strategy is defined as a bearish strategy if the expected return on the underlying security is larger than the risk-free rate, and is defined as a bullish strategy otherwise.

[3]If X and Y are two variables representing the gain of two assets, and F and G represent the cumulative distribution functions of X and Y, respectively, F is said to stochastically dominate G at the second order (F SSD G) if for all t:

$$\int_{-\infty}^{t} F(x)dx \leq \int_{-\infty}^{t} G(x)dx$$

with a strict inequality for at least one t:

$$\Leftrightarrow \tilde{Y} \stackrel{d}{=} \tilde{X} + \tilde{\eta} + \tilde{\varepsilon}$$

with $\tilde{\eta}$ nonpositive and $E(\tilde{\varepsilon}/\tilde{X} + \tilde{\eta}) = 0$.

UNIVERSAL BOUNDS ON CONCAVE CALL PRICE

With no particular assumptions on the shape of the distribution and the shape of the reward function of the options, it is impossible to derive exact formulas from hedging strategies, even if these strategies remain feasible conceptually.

This section aims at presenting upper and lower bounds for the premium at time 0 of a European concave call from comparisons between nondominating strategies. Strategies of the same cost are now compared to find explicit bounds which allow H5 to be verified. In order to be coherent, only strategies which are not dominated by a risk-free strategy are compared, which leads to differentiation between bullish and bearish strategies.

In case of a bullish market, selling the asset is dominated by buying the asset and borrowing at the risk-free rate. Moreover, Arrow (1965) showed that all investors have a positive proportion of their wealth invested in the risky asset.[4] To solve this problem, Levy (1985) simply supposes that in the Capital Asset Pricing Model framework, the asset has a nonnegative beta. We prefer not to adopt this framework and to make the assumption that stock buyers make their bullish hypothesis reflected in the present stock price and the expected return (greater than the risk-free rate), whereas sellers have the inverse expectation. The explicit hypothesis that buyers and sellers of the stock have different expectations and that these expectations lead the buying prices on the one hand and the selling prices on the other, almost "independently," is thus made.

H5 is somewhat similar to the nonarbitrage hypothesis. Nevertheless, to be really exploited, the latter needs the duplication of the call with a portfolio of existing instrument, which means that it needs complete markets. Here, the markets need not to be complete. If the underlying distribution is not lognormal or if there are transaction costs, they are not dynamically complete. In this case, minimum conditions are defined so that none of two strategies with the same cost dominates the other. There is no direct link with the notion of arbitrage, because the notion of risk is integrated in the notion of stochastic dominance. The risk is not eliminated, as in the Black-Scholes derivation, because it is impossible. The calculations are never made in the so-called risk-neutral probability, but in the "real" probability. Notice that the order defined by second order stochastic dominance is universal, if the expectations are homogeneous. If this is the case, the bounds are universal; if investors have different

[4]Arrow (1953) shows that if the expected return of a financial investment is larger than the risk-free rate, then every investor invests a positive proportion of his or her wealth in this asset.

expectations on the distributions of the stock price, these bounds are subjective.

Bullish Strategies

Studying bullish strategies allows us to determine bounds on buying prices for risk averse individuals willing to buy the underlying asset and/or the call option. If the investor has the following expectation:

$$E(S_T) > (1 + r)S_0 \tag{1}$$

then he or she adopts bullish strategies. Only bullish strategies are compared. Effectively, condition (1) should be satisfied for potential stock or call buyers; if it is not satisfied, investing in the risk-free rate would stochastically dominate any bullish strategies on a single stock and its options.

Three equal-cost strategies are defined to make some comparisons and derive bounds on the call premium.

At time 0, strategy 1 (s1) consists in buying one unit of the underlying security at price S_0; strategy 2 (s2) consists in buying calls at price C_0 for the same amount, i.e., a quantity S_0/C_0; and strategy 3 (s3) consists in buying α calls with $\alpha < S_0/C_0$ and investing the remaining amount ($S_0 - \alpha C_0$) in the risk-free asset with a yield of r until expiry. The three strategies have the same initial cost, S_0.

At time T, the final value of s1 is S_T, the final value of s2 is $S_0/C_0 u(S_T - K)$, and the final value of s3 is $\alpha u(S_T - K) + (S_0 - \alpha C_0)(1 + r)$.

The above strategies are now compared two by two to derive conditions of no stochastic dominance by using H5.

Comparison 1: between s1 and s2

In order to prevent strict dominance from s1 over s2, the distribution functions must have at least one intersection point (see Fig. 2). As u is concave, this implies that there exists a point at which s2 yields a larger value than s1, which can be expressed as

$$\text{Max}\left[\left(\frac{S_0}{C_0}\right)\frac{u(S_T - K)}{S_T}\right] > 1$$

or

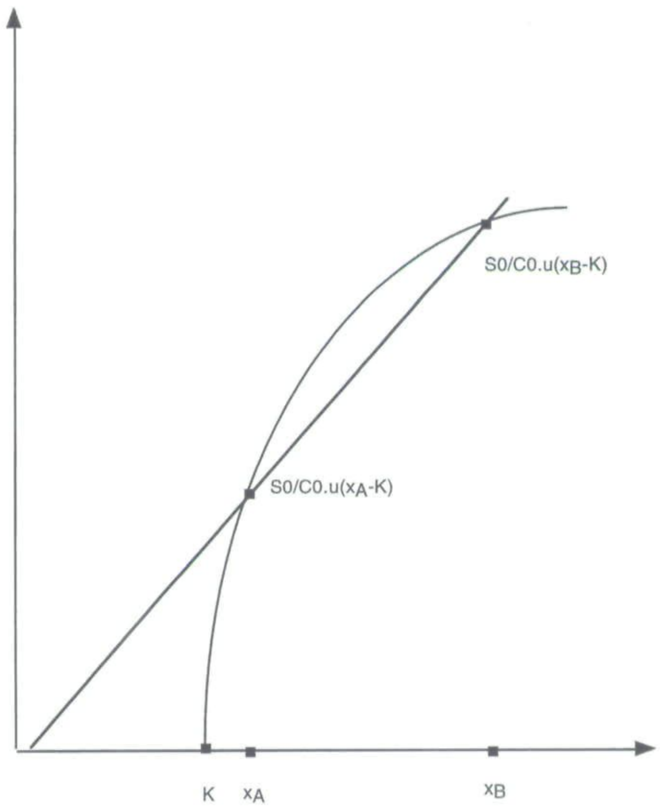

FIGURE 1
Final values of s1 and s2.

$$C_0 < S_0 \underset{S_T > K}{\mathrm{Max}} \left[\frac{u(S_T - K)}{S_T} \right] \tag{2}$$

which gives a trivial upper bound. As u is concave and increasing, there are one or two values of the underlying security where both strategies give the same final value, i.e., $S_0/C_0 u(S_T - K) = S_T$. These points correspond to A and B, the intersection points of both distribution functions, of coordinates $(x_A, F(x_A))$ and $(x_B, F(x_B))$. x_A and x_B are the fixed points of the function $S_0/C_0 u(S_T - K)$ as shown by Figures 1 and 2.

There is only one intersection point if

$$\lim_{x \to \infty} \frac{S_0}{C_0} u' \, (x - K) > 1 \tag{3}$$

In this case, x_B is infinite. It depends, in fact, on the form of the function $u(\cdot)$.

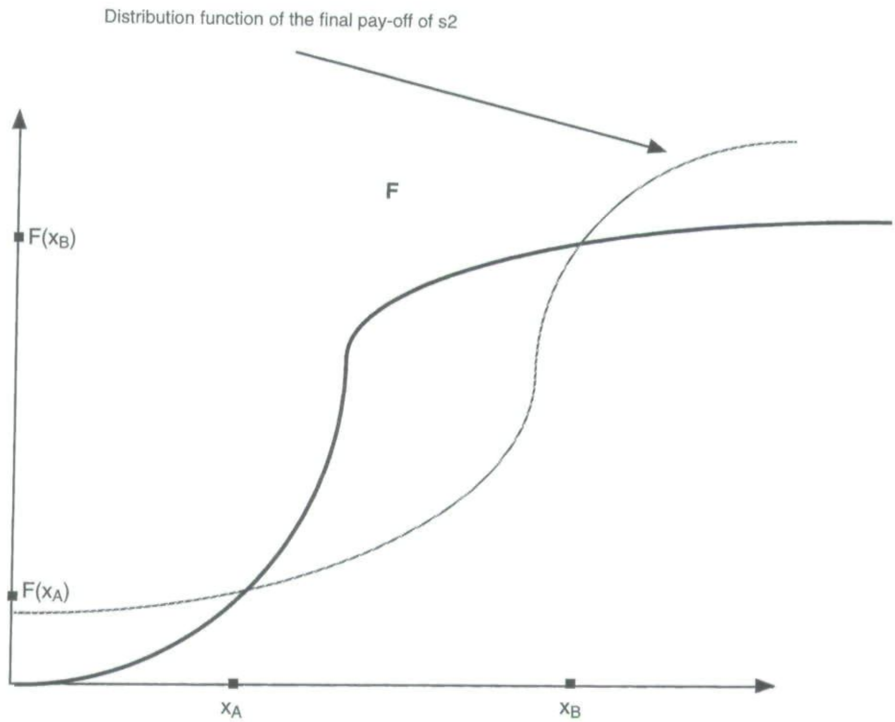

FIGURE 2
Distribution functions of s1 and s2.

By using the hypothesis of no dominance (H5) and lemma 1 of Appendix A, it is possible to define conditions which prevent s1 from dominating s2. This condition, obtained at point B (B can be infinity if there is only one intersection point), creates bounds on the call price. Notice that despite the comparison between two distributions, there is no need to know the distribution of the underlying security, to derive theoretical results. Effectively, it is possible to compare state of the world by state of the world the respective gains of the strategies, whatever the probabilities of the states, because the payoffs of the option strategies depend directly on the underlying asset value. Now s1 and s3 can be compared in the same manner.

Comparison 2: between s1 and s3

If one compares s1 and s3, one gets the following: There can be one, two, or three intersection points, but to avoid strict dominance, the existence of at least one intersection point is necessary. Notice that in order for s3 to lie above s1, for a final value of the underlying security equal to K, it is both necessary and sufficient to have:

$$(S_0 - \alpha C_0)(1 + r) > K$$

This implies that

$$\alpha < (S_0 (1 + r) - K)/(C_0 (1 + r)) \tag{4}$$

This condition is impossible for options very much out of the money, i.e., when S_0 is small compared with K. For options very much in the money, it becomes feasible as the numerator increases with an increase in K and the denominator decreases. When this condition (4) is not satisfied, there is always one (or more) intersection points as s3 cuts s1 before value K. When this condition (4) is satisfied, there cannot be more than one intersection point because of the concavity of the options. In this case, as both strategies must intersect, one must have:

$$\alpha u'(\infty) < 1$$

This condition can be made as strict as possible by replacing α by its upper bound in eq. (4) to yield:

$$C_0 > u'(\infty).\left(S_0 - \frac{K}{1 + r}\right)$$

However, because $\alpha < S_0/C_0$, the condition reduces to

$$C_0 > u'(\infty).\ S_0 \tag{5}$$

For

$$(S_0 (1 + r) - K)/(C_0 (1 + r)) < \alpha < S_0/C_0 \tag{6}$$

there is at least one intersection point. There is a unique intersection point if for any $x > K$:

$$(S_0 - \alpha C_0)(1 + r) + \alpha u(x - K) < x \Leftrightarrow$$

$$\alpha < \underset{x>K}{\mathrm{Min}} \left[\frac{x - S_0 (1 + r)}{u(x - K) - C_0 (1 + r)}\right] \tag{7}$$

If this last condition is not respected, there are two intersection points if

$$\alpha u'(\infty) > 1 \tag{8}$$

and three points otherwise. Setting α', the possible values of α [verifying eqs. (4) and (5) or eqs. (6) and (7)] such that there is a unique intersection point, $C(x_C, F(x_C))$, α'' [verifying eq. (6); eq. (8) without eq. (7)] for two intersection points, $C(x_C, F(x_C))$ and $D(x_D, F(x_D))$, and finally α''' [verifying eq. (6) without eqs. (7) or (8)] for three intersection points, $C(x_C, F(x_C))$, $D(x_D, F(x_D))$, and $E(x_E, F(x_E))$. The intersection points between the final values of s1 and s3 are such that the first coordinate is a fixed point of the function: $\alpha u(x - K) + (S_0 - \alpha C_0)(1 + r)$. If there is only one intersection point, it implies that x_D is infinite and it does not in any way affect the validity of the results obtained in the two intersection points situation. Figure 3 shows the two intersection points situation in the final payoffs and Figure 4 shows the three intersection points situation. As can be seen in Figures 3 and 4, the option strategy dominates in this case the stock strategy for small final values. Thus, the expected value of s3 must not be too high, unless s3 dominates stochastically s1. The constraint is thus the inverse of the one obtained in the previous comparison, which implies that lower price bounds are also obtained.

Theorem 1 (Proof in Appendix B)

The buying price of a concave call verifies:

$$C_0 < S_0 \frac{E_{x_B}(\tilde{C}_T)}{E_{x_B}(\tilde{S}_T)} \tag{9}$$

with x_B eventually infinite (if there is only one intersection point), and

$$C_0 > \operatorname{Max}\left[\operatorname*{Max}_{\alpha''} \frac{S_0(1 + r) + \alpha'' E^*_{x_D}(\tilde{C}_T) - E^*_{x_D}(\tilde{S}_T)}{\alpha''(1 + r)F(x_D)};\right.$$
$$\left.\operatorname*{Max}_{\alpha'} \frac{S_0(1 + r) + \alpha' E(\tilde{C}_T) - E(\tilde{S}_T)}{\alpha'(1 + r)}\right] \tag{10}$$

$$C_0 > \operatorname*{Max}_{\alpha'''} \operatorname{Min}\left[\frac{S_0(1 + r) + \alpha''' E_{x_D}(\tilde{C}_T) - E_{x_D}(\tilde{S}_T)}{\alpha'''(1 + r)F(x_D)};\right.$$
$$\left.\frac{S_0(1 + r) + \alpha''' E(\tilde{C}_T) - E(\tilde{S}_T)}{\alpha'''(1 + r)}\right] \tag{11}$$

Note that the lower bound of the call premium makes the stochastic dominance of the option upon the stock impossible, while the upper bound of the call premium makes the stochastic dominance of the stock

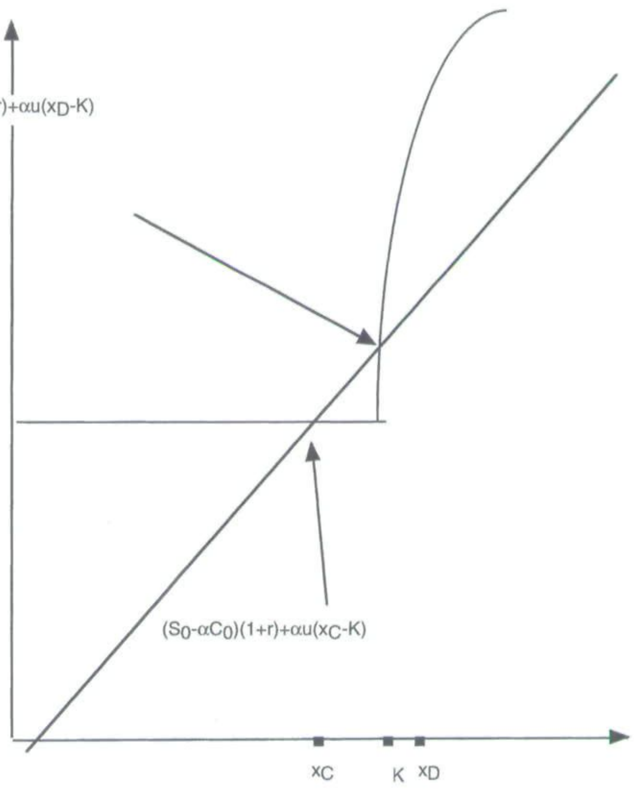

FIGURE 3
Final values of s1 and s3. Two intersection points.

upon the option impossible. To derive numerical results, it is necessary to make some hypotheses on the underlying asset distribution. Equation (9) is then a simple computation, because the intersection point is found easily. Equations (10) and (11) can be calculated by iteration: Each potential price C_0 must verify these relations and is rejected if this is not the case. For numerical results in the "linear" case, see Henin and Pistre (1997).

Bearish Strategies

Bearish strategies involve the selling of calls or stocks. In order not to be dominated by a lend at the risk-free rate, the following inequality must hold for the seller:

$$E(S_T) < S_0(1 + r) \tag{12}$$

The three former strategies are dominated by an investment in the risk-

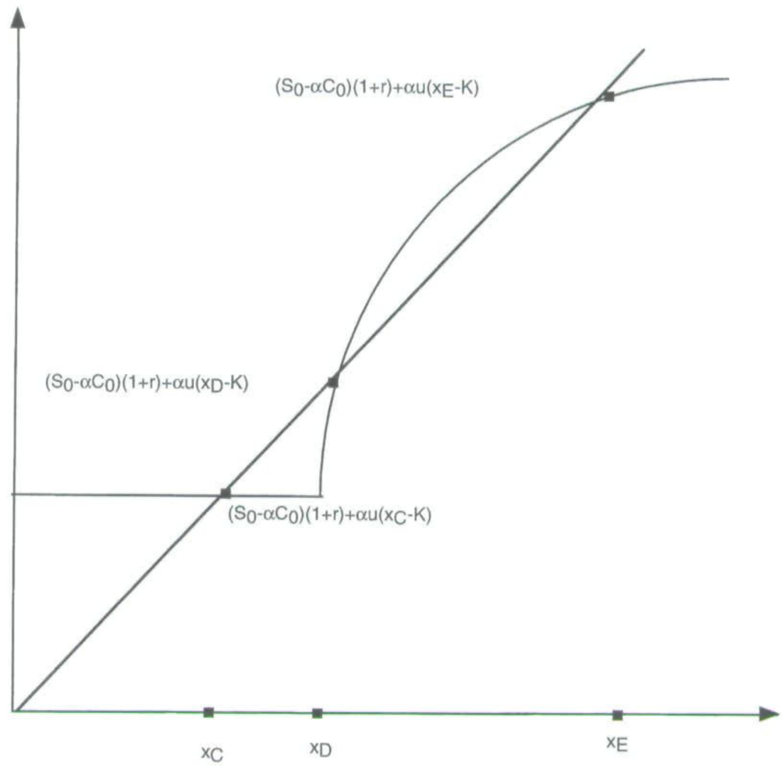

FIGURE 4
Final values of s1 and s3. Three intersection points.

free rate and must be replaced by the reverse (selling) strategies to obtain bounds on the selling price of the calls.

At time 0, s1′ consists in selling one unit of the underlying security at price S_0, s2′ in selling S_0/C_0 options to get initially the same amount S_0, and s3′ in selling α options with $\alpha < S_0/C_0$ (for an initial amount αC_0) and borrowing $(S_0 - \alpha C_0)$ at the risk-free rate to start with the same initial amount S_0. The final values are, respectively, for s1′, $-S_T$ with a distribution function F', for s2′, $-S_0/C_0 \cdot u(S_T - K)$, and for s3′, $-\alpha u(S_T - K) - (S_0 - \alpha C_0)(1 + r)$.

Comparison 3: between s1′ and s2′

If one notices that $F'(x) = P(-S_T < x) = P(S_T > -x) = 1 - F(x)$, there are one or two intersection points of the same coordinates as in the bullish situation, $(-x_A, F(x_A))$ and $(-x_B, F(x_B))$ (see Figs. 5 and 6).

To understand the intuition of the results for selling prices a general remark applies. The bearish strategies are the exact symmetry of the bull-

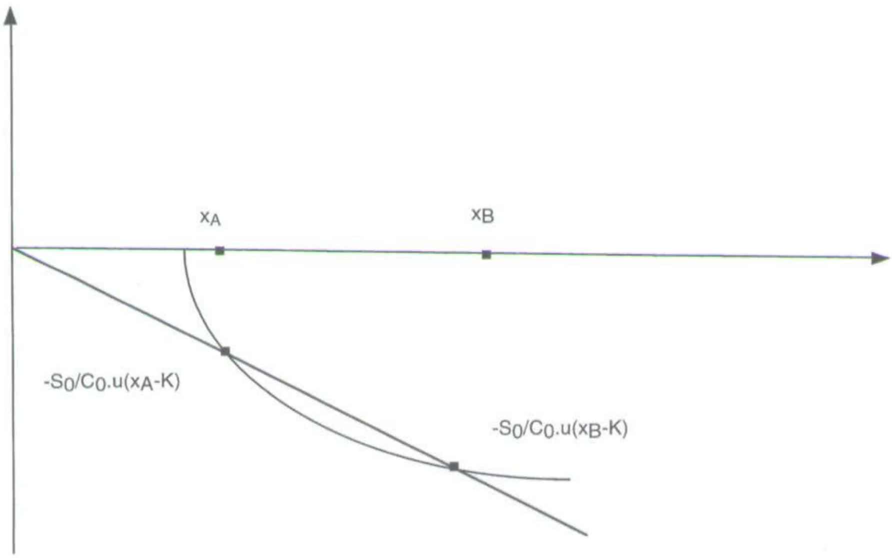

FIGURE 5
Final values of s1′ and s2′. Two intersection points.

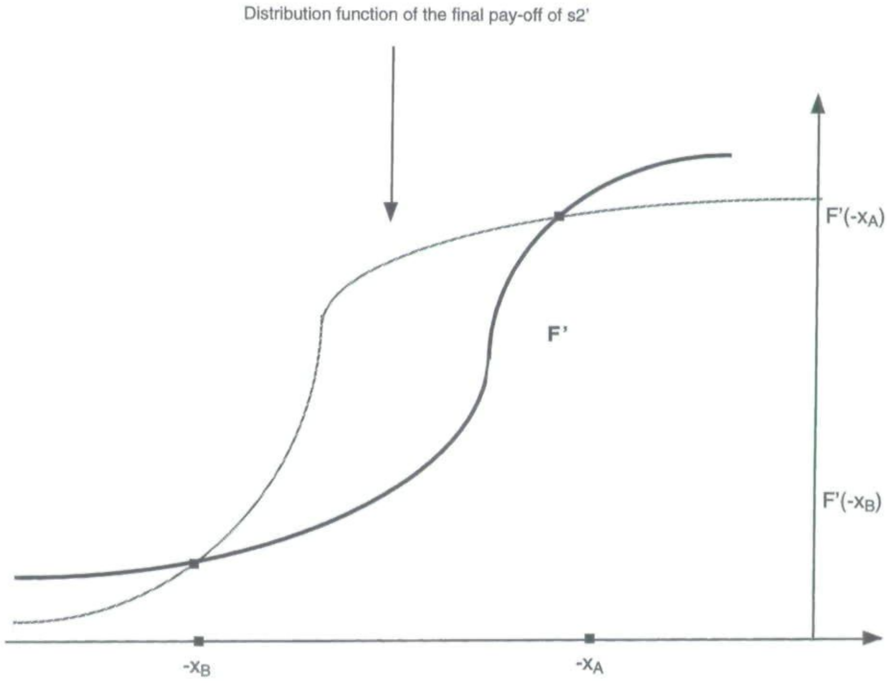

FIGURE 6
Distribution functions of s1′ and s2′.

ish strategies. However, the symmetry in the results is not trivially obtained. In the case of two intersection points, the strategy which tends to dominate for the smallest final payoffs in the bullish case (for instance, s1 in comparison 1) is dominated (in its bearish form) for the smallest final payoffs in the bearish case (s1′ in comparison 3). In addition, because of the short positions, those smallest final payoffs are obtained for the highest value of the underlying asset. In this case, the inequality between the expected values of the two strategies is inverted, but because of the negative sign of the final values, the bound on the price is finally obtained on the same side as in the bullish case. In the case of an odd number of intersection points, the strategy which tends to dominate for the smallest final payoffs in the bullish case still dominates (in its bearish form) for the smallest final payoffs in the bearish case (s1′ in comparison 3 with only one intersection point). In this case, the inequality between the expected values of the two strategies is not inverted, and because of the negative sign of the final values, the bound on the price is finally obtained on the opposite side of the bullish case. In this bearish situation, an upper bound is thus obtained for the call price if there are two intersection points in order to prevent the dominance of s2′, and a lower bound if there is one intersection price in order to prevent the dominance of s1′. Notice that the second intersection point is $(-x_A, F'(-x_A))$.

Comparison 4: between s1′ and s3′

Here again, to obtain at least one intersection between the strategies, to prevent strict dominance, one has the same upper bound (2) as for comparison 2. The same three cases happen, depending on the chosen value of α: α', α'', α'''. The previous remark applies: The bounds are obtained on the opposite side of the bullish case except in the case of two intersection points, where a lower bound is obtained again.

Theorem 2 (Proof in Appendix B)

The selling price of a concave call verifies:

$$C_0 < S_0 \frac{E^{x_A}\,(\tilde{C}_T)}{E^{x_A}\,(\tilde{S}_T)} \tag{13}$$

with x_A infinite if there is only one intersection point. For every α', α'', and α''' defined previously:

$$\underset{\alpha''}{\text{Max}}\left[\frac{S_0(1 + r) + \alpha'' E^{x_C*}(\tilde{C}_T) - E^{x_C*}(\tilde{S}_T)}{\alpha''(1 + r)}\right]$$

$$< C_0 < \underset{\alpha'}{\text{Min}}\left[\frac{S_0(1 + r) + \alpha' E(\tilde{C}_T) - E(\tilde{S}_T)}{\alpha'(1 + r)}\right] \tag{14}$$

$$C_0 < \underset{\alpha'''}{\text{Min}}\ \text{Max}\left[\frac{S_0(1 + r) + \alpha''' E^{x_D*}(\tilde{C}_T) - E^{x_D*}(\tilde{S}_T)}{\alpha'''(1 + r)};\right.$$

$$\left.\frac{S_0(1 + r) + \alpha''' E(\tilde{C}_T) - E(\tilde{S}_T)}{\alpha'''(1 + r)}\right] \tag{15}$$

In this bearish situation, the upper bound of the call premium makes the stochastic dominance of the latter upon the stock impossible (the call must not be sold too high), while the lower bound of the call premium makes the stochastic dominance of the stock impossible (the stock must not be sold too high).

BOUNDS ON CONCAVE PUT PRICE

Bounds can be obtained for concave puts as they were obtained for calls. The comparisons still use either bullish or bearish strategies depending upon the expectation on stock returns. This implies that, in the strategies, stock buying is always associated or compared with put writing and vice versa.

In case of a bullish market, two strategies (s4 and s5) are now defined, which are compared successively with s1, which stays identical. The following hypothesis is made:

$$P_0 < u(K)/(1 + r) \tag{16}$$

That is, the price P_0 for insurance is, of course, smaller than the certain cash flow $u(K)$ (the best case with certitude).

Bullish Strategies

Strategy 4 (s4) consists in buying one share and buying γ puts and borrowing γP_0 with $\gamma < 1/u'(0)$, in order for the gains of the strategy to be strictly monotonic. In strategy 5 (s5), one writes δ puts and invests $(\delta P_0 + S_0)$ in the risk-free asset. For

$$\delta_0 = S_0(1 + r)/u(K) - P_0(1 + r) \tag{17}$$

s1 and s5 have the same final value for $S_T = 0$, i.e., $-\delta_0 u(K) + (\delta_0 P_0 + S_0)(1 + r) = 0$. Strategy 5 is considered with $\delta' < \delta_0$, then with $\delta'' > \delta_0$.

Lemma 4 (Proof in Appendix C)

The gain at point $S_T = 0$ decreases when δ increases.

The final value of s4 is $\gamma u(K - S_T) + \gamma S_T - \gamma P_0(1 + r)$ and the final value of s5 is $-\delta u(K - S_T) + (\delta P_0 + S_0)(1 + r)$.

Comparison 5: between s1 and s4

Because of eq. (16), the value of the gain of s4 for $S_T = 0$ is higher than the gain of s1 and there is a unique fixed point x_M of the function: $\gamma u(K - x) + \gamma x - \gamma P_0(1 + r)$.

Thus, the distribution functions of s1 and s4 have a unique intersection point, $M(x_M, F(x_M))$, and a condition in order s4 not to dominate s1 can be found (see Figs. 7 and 8).

Comparison 6: between s1 and s5

For $\delta'' > \delta_0$, there is at least one intersection point if there exists $x < K$ such as $-\delta'' u(K - x) + (\delta'' P_0 + S_0)(1 + r) > x$, which implies the following necessary condition:

$$P_0 > \underset{\delta''>\delta_0}{\mathrm{Max}}\ \underset{x}{\mathrm{Min}}\ \frac{x + \delta'' u(K - x) - S_0(1 + r)}{\delta''(1 + r)} \tag{18}$$

In this case, there are two and only two intersection points between the distribution functions s1 and s5, $N(x_N, F(x_N))$ and $P(x_P, F(x_P))$, and x_N and x_P are the fixed points of the function $-\delta'' u(K - x) + (\delta'' P_0 + S_0)(1 + r)$ as shown in Figure 9.

For $\delta' < \delta_0$, there is at least one intersection point, but there are either one or three intersection points, depending on the form of $u(\cdot)$, as shown by Figures 10 and 11.

Theorem 3 (Proof in Appendix C)

The selling price of a concave put verifies:

$$P_0 < \mathrm{Max}\left[\underset{\delta}{\mathrm{Min}}\ \frac{E(\tilde{S}_T) + \delta' E(\tilde{P}_r) - S_0(1 + r)}{\delta' (1 + r)};\right.$$

$$\left.\underset{\delta}{\mathrm{Min}}\ \frac{E^*_{xp}(\tilde{S}_T) + \delta' E^*_{xp}(\tilde{P}_T) - S_0(1 + r)}{\delta' (1 + r)}\right] \tag{19}$$

with x_P eventually infinite if there is only one intersection point[5]:

[5]In this case, the two expressions in eq. (17) are identical.

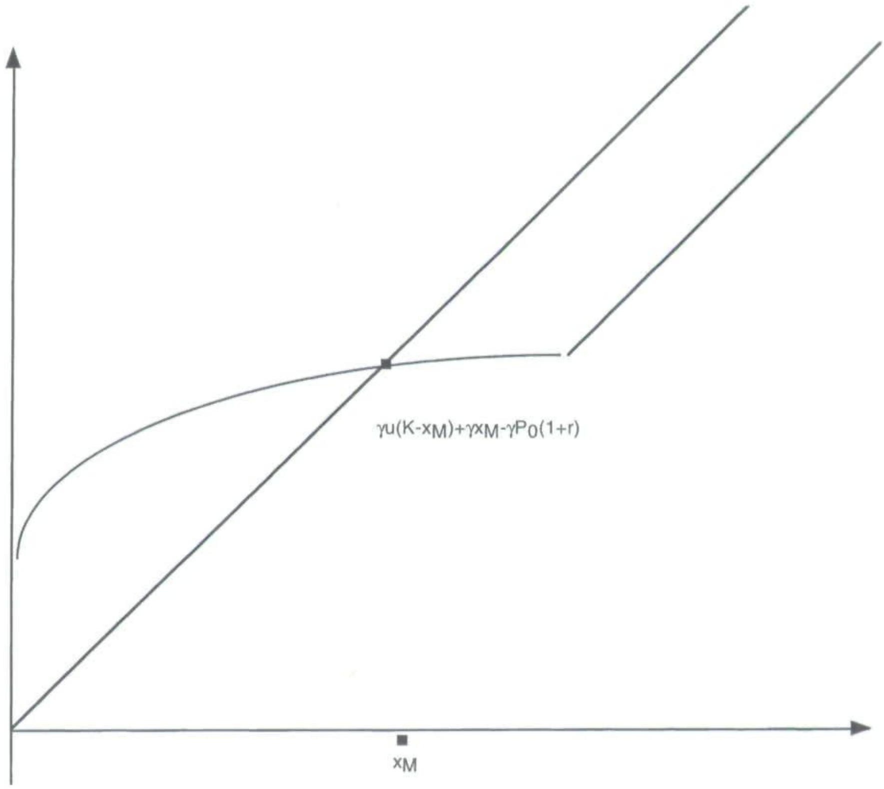

FIGURE 7
Final values of s1 and s4.

$$P_0 > \mathrm{Max}\left[\underset{\delta''}{\mathrm{Max}}\ \frac{E^*_{xp}(\tilde{S}_T) + \delta''\ E^*_{xp}(\tilde{P}_T) - S_0(1 + r)}{\delta''(1 + r)};\ \frac{E(\tilde{P}_T)}{(1 + r)}\right] \quad (20)$$

Puts are sold in both s4 and s5; eq. (19) shows that they cannot be sold at too high a price to prevent the dominance of the stock buying strategy by s4 but also at a price sufficiently high (20) to prevent them from being dominated by the stock buying strategy. In case of bearish expectations, one can make the following comparisons for selling prices.

Bearish Strategies

In this case, the following (inverse) strategies are defined. Strategy 4′ (s4′) can be defined but gives a trivial result. Strategy 5′ (s5′) consists in buying δ puts and borrowing $(\delta P_0 + S_0)$. This strategy with $\delta' < \delta_0$ and then with $\delta'' > \delta_0$ is again considered. Both strategies, as well as s1′, which is used for comparison, have an initial positive flow of S_0. The final value of s5′ is $\delta u(K - S_T) - (\delta P_0 + S_0)(1 + r)$.

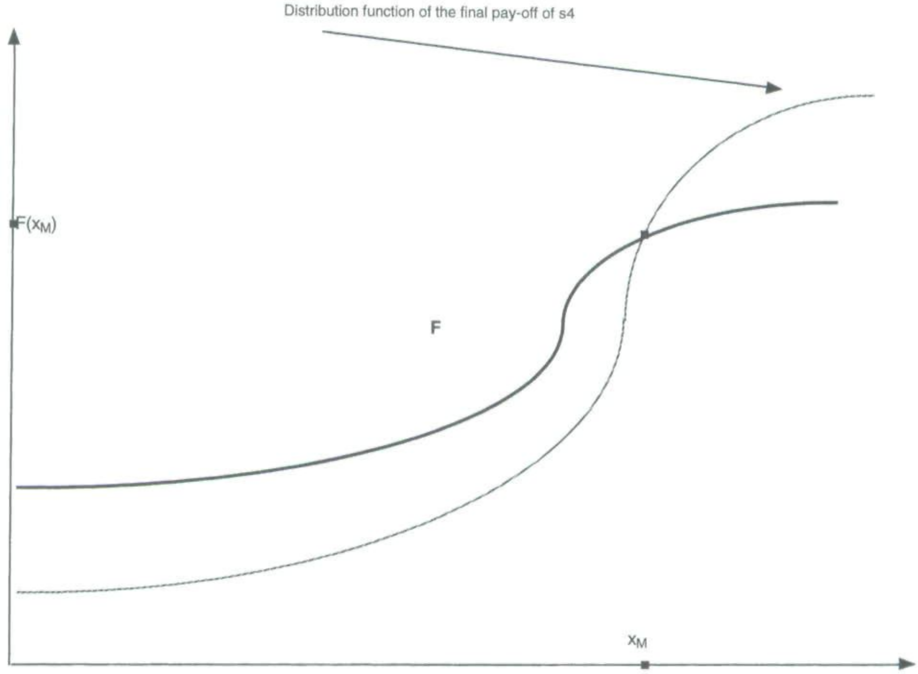

FIGURE 8
Distribution functions of s1 and s4.

Comparison 7: between s1′ and s4′

This comparison is the exact symmetry of comparison 5. Thus, the distribution functions of s1′ and s4′ have a unique intersection point, $M(-x_M, F'(-x_M))$, and a condition in order s4′ not to dominate s1′ can be found. A lower bound is obtained for the buying put price.

Comparison 8: between s1′ and s5′

The conditions for the existence (and the number) of intersection points are those of comparison 6. For $\delta'' > \delta$, the condition (18) must be verified and the two intersection points of the distribution functions s1′ and s5′ are $N(-x_N, F'(-x_N))$ and $P(-x_P, F'(-x_P))$. For $\delta' < \delta_0$, there is always at least one intersection point and there may be one or three intersection points, depending on the form of $u(\cdot)$ as in the bullish strategies. The same general remark applies with respect to the symmetry of results (toward bullish results) depending on the number of intersection points. These strategies give only lower bounds on the buying price of the put. Another strategy must be defined to find an upper bound.

Strategy 6′(s6′) is to buy one put and sell β shares with

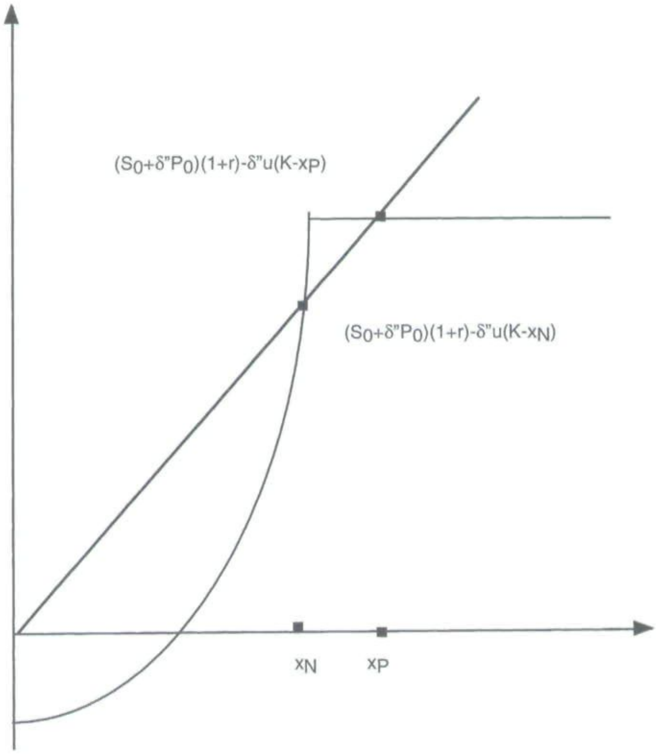

FIGURE 9
Final values of s1 and s5 for $\delta'' > \delta_0$.

$$\beta = (P_0 + S_0)/S_0 \tag{21}$$

in order to again have an initial positive flow of S_0. The final value of s6′ is $-u(K - S_T) - \beta S_T$.

Comparison 8: between s1′ and s6′

In order for s6′ not to be strictly dominated by s1′, there must exist $x < K$ such as

$$u(K - x) - \beta x > -x$$

Because of the value of β defined by eq. (21), the put price must verify:

$$P_0 < \min_x \frac{u(K - x)}{x} \tag{22}$$

which gives a first (trivial) bound. In this case, there must be one or two

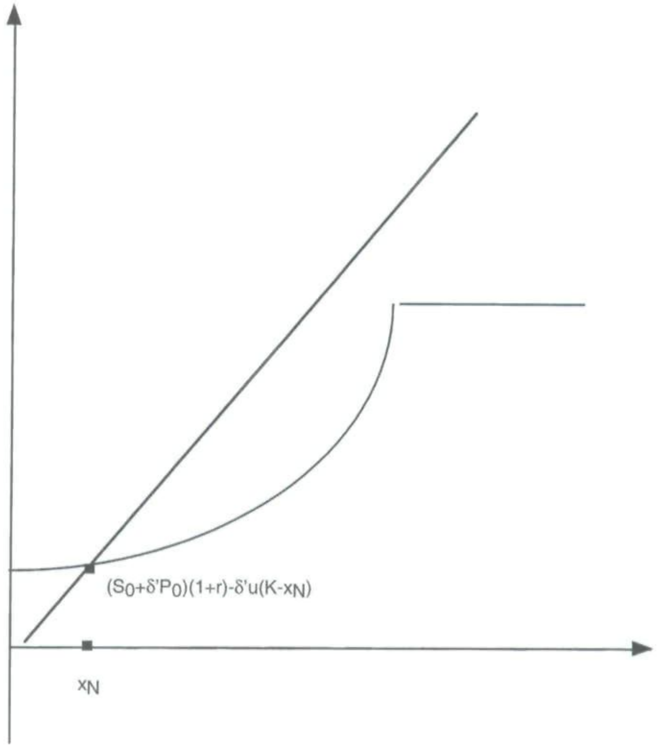

FIGURE 10
Final values of s1 and s5 for $\delta' < \delta_0$. One intersection point.

intersection points, $U(-x_U, F'(-x_U))$ and $V(-x_V, F'(-x_V))$ (see Figs. 12 and 13).

Theorem 4

The buying price of a concave put verifies:

$$P_0 < S_0 \frac{E^{x_U}(\tilde{P}_T)}{E^{x_U}(\tilde{S}_T)} \tag{23}$$

with x_u eventually infinite if there is only one intersection point, and

$$P_0 > \mathrm{Max}\left[\min_{\delta'} \frac{E(\tilde{S}_T) + \delta' E(\tilde{P}_T) - S_0(1+r)}{\delta'(1+r)};\right.$$

$$\left.\min_{\delta'} \frac{E^{x_P*}(\tilde{S}_T) + \delta' E^{x_P*}(\tilde{P}_T) - S_0(1+r)}{\delta'(1+r)}\right] \tag{24}$$

with x_P eventually infinite, and

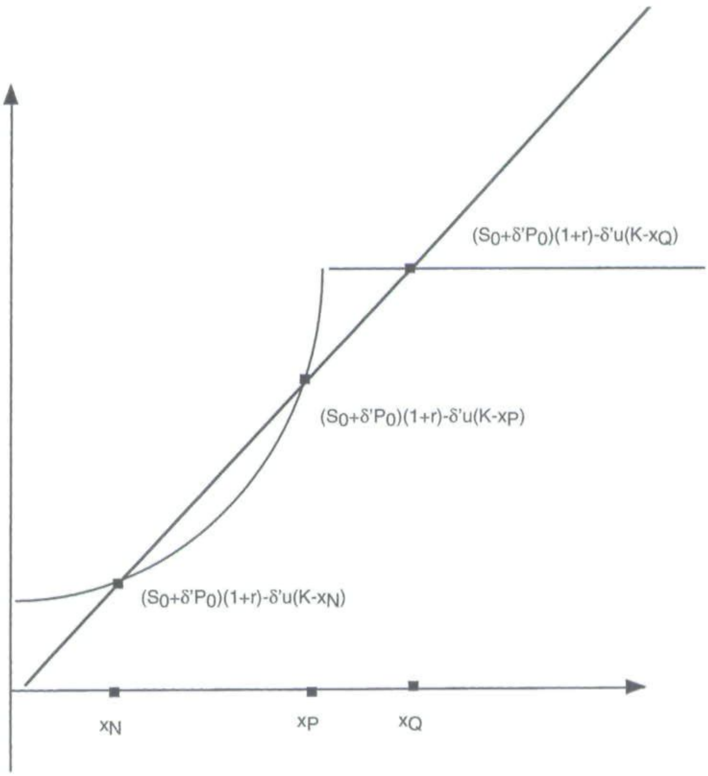

FIGURE 11
Final values of s1 and s5 for $\delta' < \delta_0$. Three intersection points.

$$P_0 > \operatorname*{Max}_{\delta''} \frac{E^{x_N*}\ (\tilde{S}_T)\ +\ \delta''\ E^{x_N*}\ (\tilde{P}_T)\ -\ S_0\ (1\ +\ r)}{\delta''\ (1\ +\ r)} \tag{25}$$

All of these bearish strategies dealing with puts involve buying puts and selling stocks. The upper bound guarantees that the put does not dominate and the lower bound guarantees that the stock does not dominate.

A PUT/CALL RELATION

As minimum relations holding at each time between call and stock prices (or put and stock prices), minimum conditions existing necessarily between call and put prices at each instant can be defined. Setting $u_c(x - K)$ the final gain of the call, which is null if $x < K$ and increasing and concave otherwise, and $u_p(K - x)$ as the final gain of the put, which is null if $x > K$ and decreasing and concave otherwise. In order for the put and call to be comparable, $u_c(x - K) = u_p(K - x)$ for $x = K + h$ and

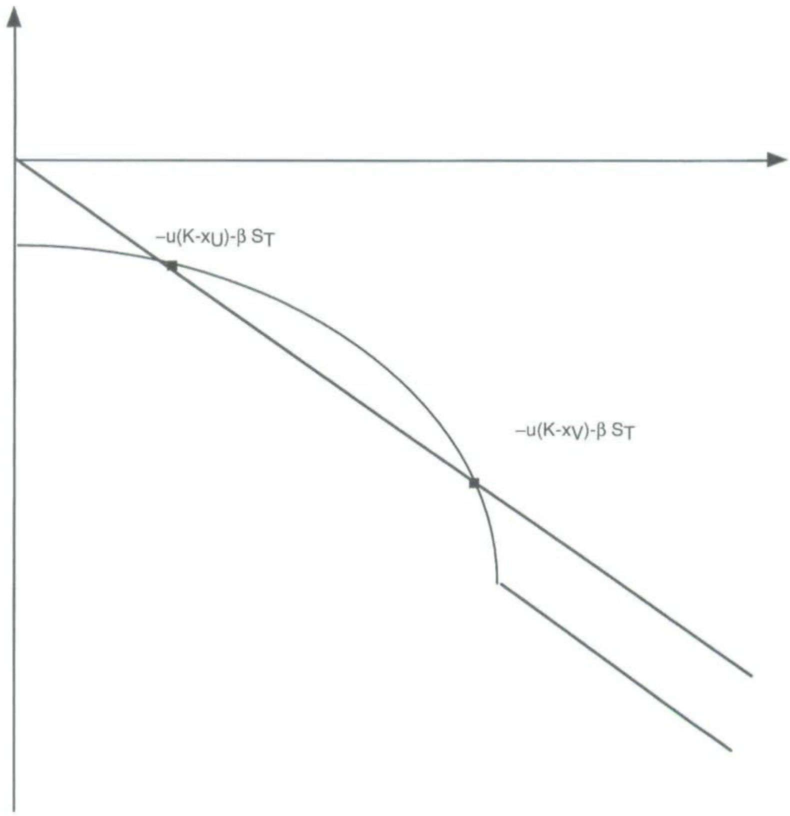

FIGURE 12
Final values of s1′ and s6′.

$x' = K - h$, with $0 < h < K$. Two new strategies are now defined for bullish expectation.

Bullish Put/Call Bounds

Strategy 7 (s7) is to write k puts, buy k calls, and invest $(kP_0 - kC_0 + S_0)$ in the risk-free asset. This strategy has the same cost S_0 as s1. The final value of s7 is

$$\text{for } x > K;\ ku_c(x - K) + (kP_0 - kC_0 + S_0)\,(1 + r)$$
$$\text{for } x < K:\ -ku_p(K - x) + (kP_0 - kC_0 + S_0)\,(1 + r)$$

Setting k_0 such that the final gain of s7 if $S_T = 0$ is 0, k_0 verifies:

$$k_0 = \frac{-S_0\,(1 + r)}{[-u_p\,(K) + (P_0 - C_0)(1 + r)]} \tag{26}$$

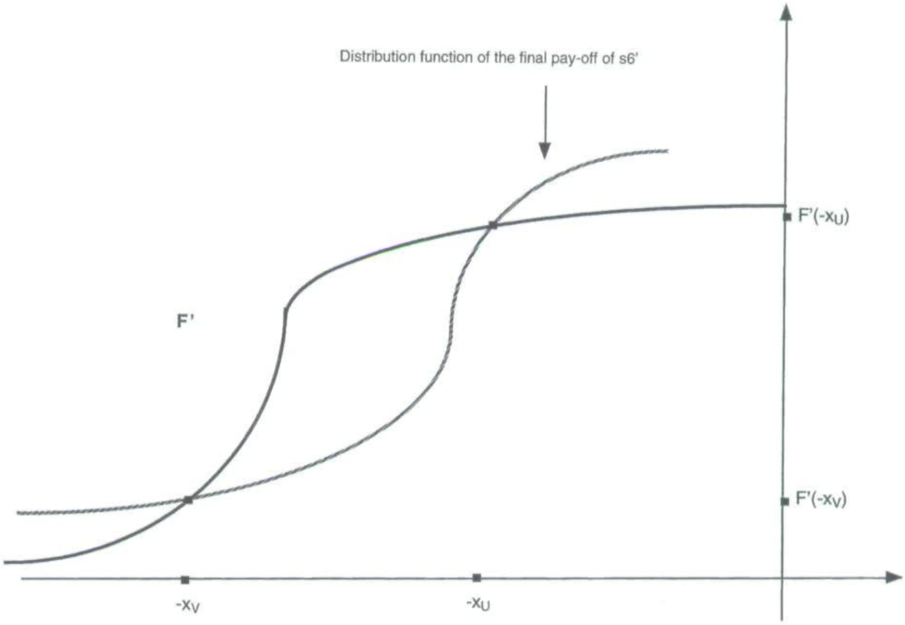

FIGURE 13
Distribution functions of s1′ and s6′.

Notice that the final gain of the strategy at point $S_T = 0$ is decreasing in k; i.e., for $k' < k_0$, it is positive and for $k'' > k_0$ it is negative.

Lemma 5

There exist some values of k', such that there is only one intersection point between s1 and s7.

Proof

If $S_0(1 + r) > K$ by taking, if $(C_0 - P_0 > 0)$:

$$k' < \frac{S_0\,(1 + r) - K}{(C_0 - P_0)(1 + r)} \tag{27}$$

or any value of k' otherwise, it follows:

$$(k'\,(P_0 - C_0) + S_0)(1 + r) > K \tag{28}$$

if $k'u_p' > -1$, the s7 gain is always higher than the s1 gain between 0 and K. And there is only one unique intersection point. If there exists a

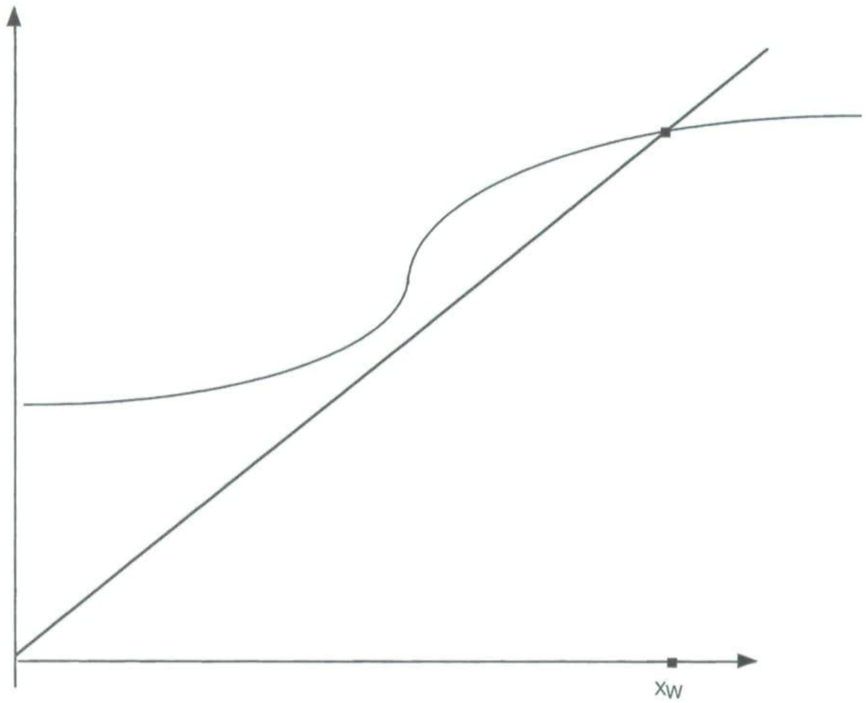

FIGURE 14
Final values of s1 and s7. One intersection point.

point such that $k'u'_p$ is not >1, setting R the point such that $-u_p{}'(x_R) = 1/k,$; x_R is also decreasing in k. In this case, to have one unique intersection point, the following inequality must hold:

$$k' < \frac{S_0\ (1 + r)}{u_P(K - x_R) + (C_0 - P_0)(1 + r)} \tag{29}$$

If the denominator is positive or any value otherwise.

If $S_0(1 + r) < K$, if $(C_0 - P_0 < 0)$ by taking k' verifying:

$$k' < \frac{S_0\ (1 + r) - K}{(C_0 - P_0)(1 + r)} \tag{30}$$

or any value of k' otherwise, it follows:

$$(k'\ (P_0 - C_0) + S_0)(1 + r) < K \tag{31}$$

If $k'u'_C < 1$, the s7 gain is always lower than the s1 gain for $S_T > K$. And there is only one unique intersection point. If there exists a point such as $k'u'_C$ is not <1, setting R' the point such that $u'_p\ (x'_R) = 1/k'$. In this

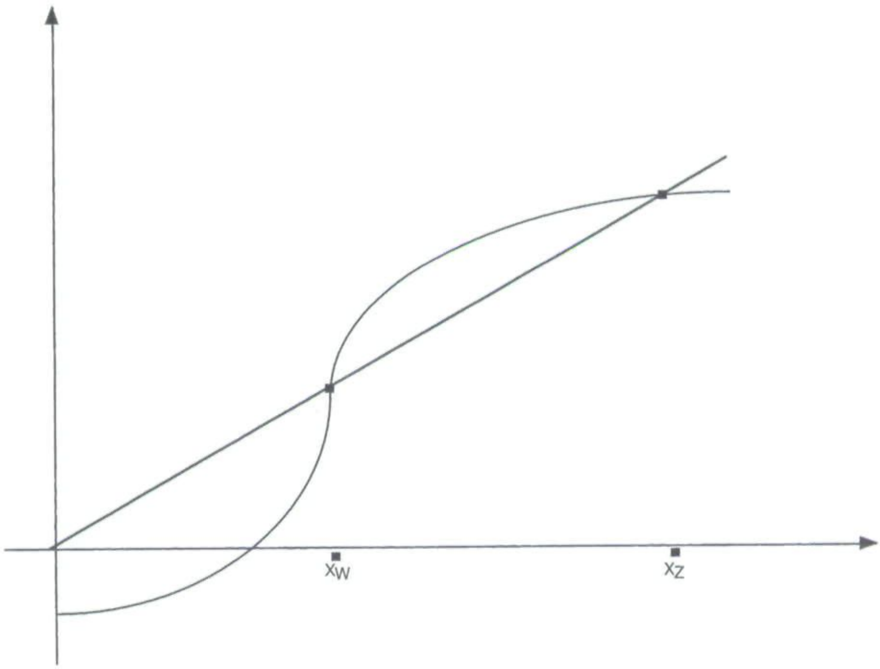

FIGURE 15
Final values of s1 and s7 for $k'' > k_0$. Two intersection points.

case, to have one unique intersection point, the following inequality must hold:

$$k' < \frac{S_0\ (1\ +\ r)}{u_C(K\ -\ x_{R'})\ +\ (C_0\ -\ P_0)(1\ +\ r)} \tag{32}$$

if the denominator is positive or any value otherwise.

As illustrated in Figures 14 and 15, a constraint must hold, so that the put/call strategy s7 does not dominate the stock strategy s1.

For $k'' > k_0$ (33), to prevent strict dominance of s1, there must exist x such that

$$k''\ u_c(x\ -\ K)\ +\ (k''\ (P_0\ -\ C_0)\ +\ S_0)(1\ +\ r) > x \tag{33}$$

$$P_0\ -\ C_0 < \underset{x}{\mathrm{Min}}\ \frac{x\ -\ S_0\ (1\ +\ r)}{u_c(x\ -\ K)\ +\ (P_0\ -\ C_0)(1\ +\ r)} \tag{34}$$

The following results are obtained.

Theorem 5

The spread between the buying price of the call and the selling price of the put verifies:

$$\operatorname*{Max}_{k''} \frac{E^*_{x_z}(\tilde{P}_T) - E^*_{x_z}(\tilde{C}_T) + E^*_{x_z}(\tilde{S}_T)}{(1 + r)} - \frac{S_0}{k''} < P_0 - C_0$$

$$< \operatorname*{Min}_{k'} \frac{E(\tilde{P}_T) - E(\tilde{C}_T) + E(\tilde{S}_T)}{(1 + r)} - \frac{S_0}{k'} \tag{35}$$

Bearish Put/Call Bounds

Defining the inverse strategy s7′ [buy k puts, write k calls, and invest $(-kP_0 + kC_0 - S_0)$ in the risk-free asset] in case of bearish expectations implies the derivation of two lower bounds.

In order to get an upper bound, it is necessary to exhibit a strategy which is dominated by s1′ for small values of gains (very negative). This is possible if

$$k'' > \lim_{x \to \infty} \frac{1}{u'_c(x - K)} \tag{36}$$

Such a value is finite only if

$$\lim_{x \to \infty} u'_c(x - K) \neq 0 \tag{37}$$

In this case, there is a unique intersection point, $L(-x_L; F'(-x_L))$, of the two strategies (see Figs. 16 and 17) and a constraint must be defined, so that the stock strategy s1′ does not dominate the put/call strategy s7′. This comparison gives a bound on the other side.

Theorem 6

The spread between the selling price of the call and the buying price of the put verifies:

$$P_0 - C_0 > \operatorname*{Max}_{k'} \frac{E(\tilde{P}_T) - E(\tilde{C}_T) + E(\tilde{S}_T)}{(1 + r)} - \frac{S_0}{k'} \tag{38}$$

If eq. (37) is verified, by taking k'' verifying eq. (36), it follows:

$$P_0 - C_0 < \operatorname*{Min}_{k''} \frac{E(\tilde{P}_T) - E(\tilde{C}_T) + E(\tilde{S}_T)}{(1 + r)} - \frac{S_0}{k''} \tag{39}$$

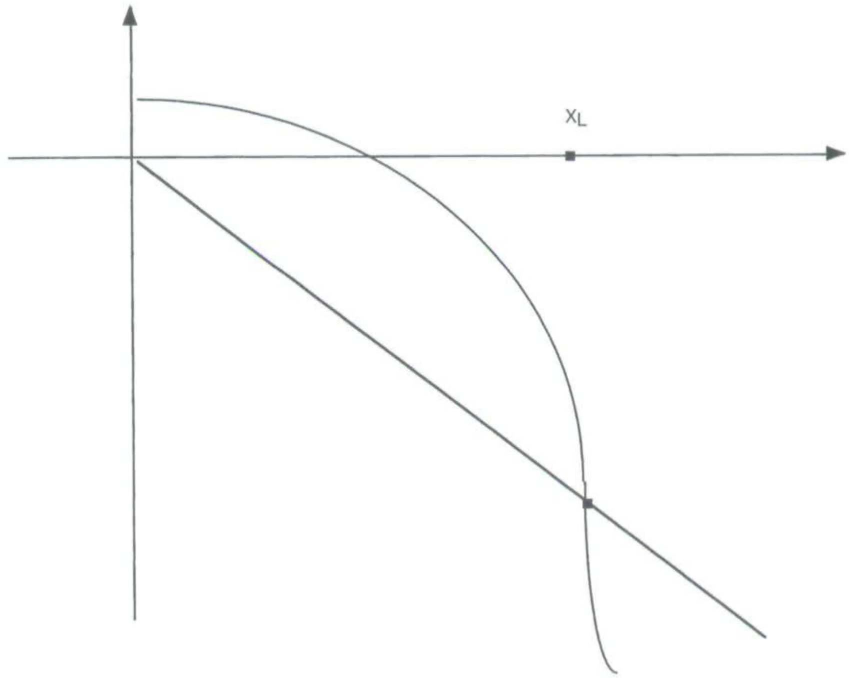

FIGURE 16
Final values of s1′ and s7′ for k''.

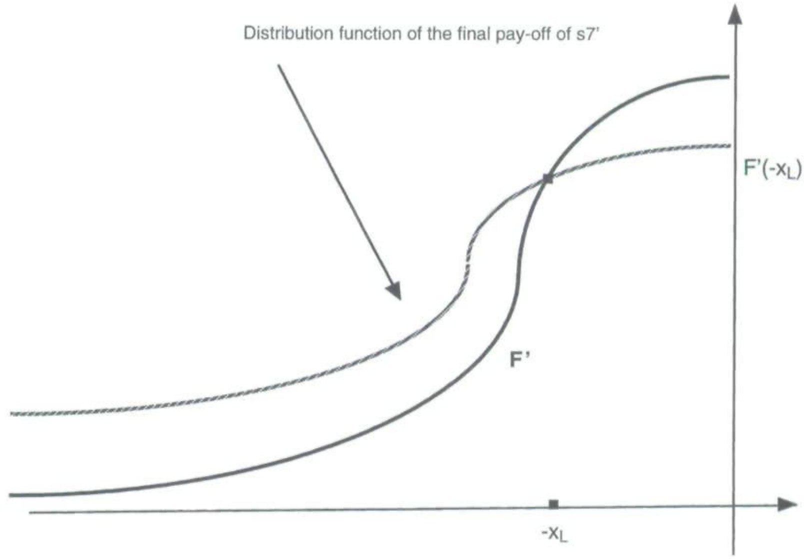

FIGURE 17
Distribution functions of s1′ and s7′.

CONCLUSIONS

The present article offers some general properties for generalized concave options. All the conditions obtained depend solely on the first moment of the distributions of the options and the underlying security, without even including the variance. This is purely formal, as the whole distribution is included from the asymmetry in the reward function of the option and from the fact that its expected value is an intrinsic part of the constraints. When the yields are normally distributed, the Black-Scholes formula makes the variance appear explicitly. In a more general fashion, computing the expected value of the call under real probability and comparing it with the distribution and expected value of the underlying security really captures the whole distribution, any eventual asymmetry being put forward by their comparison.

The bounds presented here are very general. The payment of dividends on the underlying security easily can be taken into account, because it can be directly included in the final expected value of the stock. The payment of dividends simply affects the distribution of the stock prices at expiry, thus resulting simply in a modification of the function $u(\cdot)$ to be used in the results. This is an important extension which comes from the fact that only final distribution of the stock price is used in the bounding conditions. Similarly, there is no difficulty in taking transaction costs into account, since there is no duplication strategy in our method. Once again, these costs simply affect the initial cost of the strategies (in the bullish case, S_0 plus the transaction cost) and the expected payoffs of the strategies (decreased by the transaction cost).

The practical use of these results is clear even if a unique price is not found for the option. Consider a market maker on options. This market maker is able to determine the highest price at which a bullish investor will accept to buy the option and the lowest price at which a bearish investor will accept to sell the option. The bullish higher bound on the option price can thus be calculated with respect to the actual price of the asset, i.e., the ask price of the market maker. Constraints on the spreads on options can thus be defined vis-à-vis the spreads offered on the underlying security by simply replacing the share price by its bid or ask price, depending on the sense of the transaction.

This represents a theoretical spread, of course, adjusted for the purpose of competition. In most cases, there are no analytical values of the bounds, because the various intersection points depend on the price of the option itself. For a given possible price of the call, one must check that the relations are verified. If not, one must proceed by iteration until a price is not rejected. The computations of the bounds for various concave payoffs constitute a further research development. Because pricing is not the only objective of investors, one other important research development will be the derivation—analytically or numerically—of the sensibility of the bounds with respect to various parameters, e.g., price and volatility of the underlying asset, and time to maturity.

APPENDIX A: LEMMAS OF SECOND ORDER STOCHASTIC DOMINANCE

Lemma 1 (Hammond, 1974)

When there exits $A(x_A,F(x_A))$ such as

$$F(x) < G(x)\ \forall x < x_A,\quad F(x) > G(x)\quad \forall x > x_A,\quad F(x_A) = G(x_A) \quad \text{(A1)}$$

(i.e., F intersects G once and only once from below at point A), F stochastically dominates G if and only if

$$E(X) \geq E(Y)$$

Lemma 2

If

$$\exists B(x_B,\ F(x_B)),\ \text{and}\ C(x_c,\ F(x_c))$$

with $0 < x_B < x_C$ such as

$$F(x) < G(x) \quad \forall x < x_B \text{ and } x > x_c;\ F(x) > G(x) \quad x_B < x < x_c$$

$$F(x_B) = G(x_B) \text{ and } F(x_C) = G(x_C)$$

(i.e., F intersects G twice, first from below at point B and then from above at point C), F dominates G in the SSD sense if and only if

$$\int_{-\infty}^{x_C} F(x)dx \leq \int_{-\infty}^{x_C} G(x)dx \Rightarrow \int_{-\infty}^{x_C} xf(x)dx \geq \int_{-\infty}^{x_C} xg(x)dx \qquad \text{(A2)}$$

i.e., if condition (A1) is satisfied at the second intersection point C.

Lemma 3

When F intersects G more than twice, the first time being from below, F dominates G in the SSD sense if eq. (A2) is satisfied for all even-numbered intersection points, infinity being considered such a point [i.e., condition (A2) applies to the mean if there is an odd number of intersection points].

APPENDIX B: PROOFS OF THEOREMS OBTAINED FOR CONCAVE CALLS

Proof of Theorem 1

Comparison 1

For $x < x_A$ and $x > x_B$ $S_0/C_0 u(x - K) < x$; this implies that $F(x) < G(x)$ and, for $x_A < x < x_B$, $S_0/C_0 u(x - K) > k$; hence, $F(x) > G(x)$. This implies that the conditions of lemma 2 are satisfied. Thus, if $E_{x_B}(\mathrm{s1}) \geq E_{x_B}(\mathrm{s2})$, then s1 DS_2 s2: hence a contradiction with H5. Therefore,

$$E_{x_B}\ (\mathrm{s1}) < E_{x_B}\ (\mathrm{s2})$$

which implies

$$E_{x_B}\ (\tilde{S}_t) < E_{x_B}\left(\frac{S_0}{C_0}\ u(\tilde{S}_r\ -\ K)\right)$$

and thus:

$$C_0 < S_0\ \frac{E_{x_B}\ (\tilde{C}_T)}{E_{x_B}\ (\tilde{S}_T)}$$

If there is only one intersection point, x_B is infinite and this relation becomes:

$$C_0 < S_0\ \frac{E\ (\tilde{C}_T)}{E\ (\tilde{S}_T)}$$

From this eq. (9) is obtained.

Comparison 2

Three different cases can be obtained according to the number of intersection points. The first two cases are similar because one intersection point is equivalent to two intersection points, with the second one infinite. The case of two intersection points C and D with D eventually infinite gives the following result. When $x < x_C$, then Final Value(s3) > Final Value(s1) and if H is the cumulative distribution function of s3, $H(x) < F(x)$ and for $x > x_D$, $H(x) < F(x)$. Choosing α'' verifying eq. (7) without verifying eq. (6) to obtain two intersection points and using lemma 2 to prevent stochastic dominance of s3 gives, for every α'':

$$\int_{-\infty}^{x_D} F(x)dx < \int_{-\infty}^{x_D} H(x)dx$$

$$\int_{-\infty}^{x_D} xf(x)dx > (S_0 - \alpha'' C_0)(1 + r)F(x_D) + \alpha'' \int_K^{x_D} u(x - K)f(x)dx$$

Thus:

$$(S_0 - \alpha'' C_0)(1 + r)F(x_D) + \alpha'' E_{x_D}(\tilde{C}_T) < E_{x_D}(\tilde{S}_T)$$

and

$$\frac{S_0(1 + r)F(x_D) + \alpha'' E_{x_D}(\tilde{C}_T) - E_{x_D}(\tilde{S}_T)}{\alpha''(1 + r)F(x_D)} < C_0$$

With α' such that there is one intersection point, x_D becomes infinite, and this last equation becomes, for every α':

$$\frac{S_0(1 + r) + \alpha' E(\tilde{C}_T) - E(\tilde{S}_T)}{\alpha'(1 + r)} < C_0$$

From these last two results comes eq. (10).

With α''' such that there are three intersection points, C, D, and E, when $x < x_C$, then Final Value(s3) > Final Value(s1) and $H(x) < F(x)$, whereas for $x_D < x < x_E$, $H(x) < F(x)$, and for $x_C < x < x_D$ and $x > x_E$, $H(x) > F(x)$. Using lemma 3 to prevent stochastic dominance of s3, the following inequalities are obtained for any α''':

$$\int_{-\infty}^{x_E} F(x)dx < \int_{-\infty}^{x_E} H(x)dx$$

or

$$\int_{-\infty}^{\infty} F(x)dx < \int_{-\infty}^{\infty} H(x)dx$$

Thus, the following relation holds for any α''':

$$C_0 > \mathrm{Min}\left[\frac{S_0\ (1 + r)F(x_D) + \alpha'''\ E_{x_D}\ (\tilde{C}_t) - E_{x_D}\ (\tilde{S}_T)}{\alpha'''\ (1 + r)F(x_D)};\right.$$

$$\left.\frac{S_0\ (1 + r) + \alpha'''\ E(\tilde{C}_T) - E(\tilde{S}_T)}{\alpha'''\ (1 + r)}\right]$$

From this comes eq. (11).

Proof of Theorem 2

Comparison 3

Depending on $u(\cdot)$, there are one or two intersection points. If there is only one intersection point, a condition to prevent the dominance of s1′ on s2′ can be obtained. As there is only one intersection point, even by inverting the strategies, F' lies under the distribution function of s2′ for the most negative values of the strategies at T, and then above it, starting at point A. Thus, if $E(\mathrm{s}1') \geq E(\mathrm{s}2')$, then s1′ DS2 s2′: hence a contradiction. It follows that $E(\mathrm{s}1') < E(\mathrm{s}2')$. This yields:

$$-E(\tilde{S}_T) < -E\left(\frac{S_0}{C_0}\ u(\tilde{S}_T - K)\right)$$

and

$$C_0 > S_0\ \frac{E(\tilde{C}_T)}{E(\tilde{S}_T)}$$

If there are two intersection points, lemma 1 can be used again for the value of $-x_A$, which is the higher value of intersection. Thus:

$$\int_{-\infty}^{-x_A} (1 - F(-x))dx < \int_{-\infty}^{-x_A} (1 - G(-x))dx$$

which implies:

$$\int_{x_A}^{\infty} (1 - F(x))dx < \int_{x_A}^{\infty} (1 - G(x))dx$$

Thus:

$$\int_{-\infty}^{\infty} xf(x)dx - \int_{-\infty}^{x_A} xf(x)dx < \frac{S_0}{C_0}\left[\int_{-\infty}^{\infty} u(x - K)f(x)dx - \int_{-\infty}^{x_A} u(x - K)f(x)dx\right]$$

and finally:

$$C_0 < S_0 \frac{E^{x_A}(\tilde{C}_T)}{E^{x_A}(\tilde{S}_T)}$$

Comparison 4

In comparison 4, there are three cases. *The first case* is when α is such that there is one intersection point, which happens exactly for the same values of α' and S_T as in comparison 2 (only the signs of the final values of the strategies are inverted). However, this time, the dominance of s3′ on s1′ must be prevented. This implies, for every α':

$$-\alpha' E(\tilde{C}_T) - (S_0 - \alpha' C_0)(1 + r) < -E(\tilde{S}_T)$$

Thus:

$$C_0 < \frac{S_0(1 + r) + \alpha' E(\tilde{C}_T) - E(\tilde{S}_T)}{\alpha' (1 + r)}$$

The second case is when α is such that there are two intersection points, which happens exactly for the same values of α'' and S_T as in comparison 2. However, the signs of the possible final values of the strategies are inverted. When $x > -x_C$ or $< -x_D$, then Final Value(s3′) < Final Value(s1′). Moreover, $-x_D < -x_C$, which is important for the application of the lemma. When $-x_D < x < -x_C$, the opposite relation holds. Because there are two intersection points, the stochastic dominance of s1′ must be prevented again. Lemma 2 can be used for the common final values $-x_D$ and $-x_C$. However, the less negative gain is for $-x_C$. The following relation is necessarily obtained for every α'':

$$\int_{x_C}^{\infty} (1 - F(x))dx > \int_{x_C}^{\infty} (1 - H(x))dx$$

Thus:

$$E\ (s3') - E_{x_C}\ (s3') < E\ (s1') - E_{x_C}\ (s1')$$

$$\int_{-\infty}^{\infty} xf(x)dx - \int_{-\infty}^{x_C} xf(x)dx > (S_0 - \alpha''\ C_0)(1 + r)(1 - F(x_C))$$

$$+ \alpha''\left(\int_K^{\infty} u(x - K)f(x)dx - \int_K^{x_C} u(x - K)f(x)dx\right)$$

$$\frac{S_0(1 + r) + \alpha''\ E^{x_C*}\ (\tilde{C}_T) - E^{x_C*}\ (\tilde{S}_T)}{\alpha''\ (1 + r)} < C_0$$

The third case is obtained for the same strategies but in taking α''' in order to have three intersection points. In this case, when $x < -x_E$ or $-x_D < x < -x_C$, then Final Value(s3′) > Final Value(s1′) and the opposite relation holds for $x > -x_C$ or $-x_D > x > -x_E$. This time the stochastic dominance of s3′ must be prevented. For every α''', the following relation holds:

$$C_0 < \text{Max}\left[\frac{S_0(1 + r) + \alpha'''\ E^{x_D*}\ (\tilde{C}_T) - E^{x_D*}\ (\tilde{S}_T)}{\alpha'''\ (1 + r)};\right.$$

$$\left.\frac{S_0\ (1 + r) + \alpha'''\ E(\tilde{C}_T) - E(\tilde{S}_T)}{\alpha'''(1 + r)}\right]$$

APPENDIX C: PROOFS OF THEOREMS OBTAINED FOR CONCAVE PUTS

Proof of Lemma 4

At point S_T, s4 has the value $(\delta P_0 + S_0)(1 + r) - \delta u(K)$. The derivative of this expression is $P_0(1 + r) - u(K)$, which is negative.

Proof of Theorem 3

Comparison 5

If γ verifies eq. (16), s1 and s4 intersect each other at point $M(x_M, F(x_M))$ and for $x < x_M$, $F(x)$ is above the cumulative distribution function of s4. To prevent s4 from dominating stochastically, it is necessary that

$$E(\tilde{S}_T) + \gamma E(\tilde{P}_T) - \gamma P_0(1 + r) < E(\tilde{S}_T)$$

Then:

$$P_0 > \frac{E(\tilde{P}_T)}{(1 + r)}$$

Comparison 6

With $\delta'' > \delta_0$, the strategies have two intersection points, $N(x_N, F(x_N))$ and $P(x_P, F(x_P))$; lemma 2 can be applied. To prevent s1 from dominating s5, it is necessary that

$$\int_0^{x_P} xf(x)dx < \int_0^{x_P} [(S_0 + \delta'' P_0)(1 + r) - \delta'' u(K - x)]f(x)dx \Rightarrow E_{x_P}(\tilde{S}_T)$$

$$< -\delta'' E_{x_P}(\tilde{P}_T) + (S_0 + \delta'' P_0)(1 + r)F(x_P)$$

which gives:

$$P_0 > \frac{E_{x_p}(\tilde{S}_T) + \delta'' E_{x_p}(\tilde{P}_T) - S_0(1 + r)F(x_P)}{\delta''(1 + r)F(x_P)}$$

and thus:

$$P_0 > \frac{E^*_{x_p}(\tilde{S}_T) + \delta'' E^*_{x_p}(\tilde{P}_T) - S_0(1 + r)}{\delta''(1 + r)}$$

With $\delta' < \delta_0$, if the strategies have one intersection point, $N(x_N, F(x_N))$, lemma 1 can be applied. To prevent s5 from dominating s1, gives:

$$P_0 < \frac{E(\tilde{S}_T) + \delta' E(\tilde{P}_T) - S_0(1 + r)}{\delta' (1 + r)}$$

With $\delta' < \delta_0$, if the strategies have three intersection points, $N(x_N, F(x_N))$, $P(x_P, F(x_P))$, and $Q(x_Q, F(x_Q))$, lemma 3 can be applied. To prevent s5 from dominating s1, it is necessary that

$$E(\tilde{S}_T) > -\delta' E(\tilde{P}_T) + (S_0 + \delta' P_0)(1 + r)$$

or

$$E_{x_P}(\tilde{S}_T) > -\delta' E_{x_P}(\tilde{P}_T) + (S_0 + \delta' P_0)(1 + r)F(x_P)$$

Then:

$$P_0 < \frac{E(\tilde{S}_T) + \delta' E(\tilde{P}_T) - S_0(1 + r)}{\delta'(1 + r)}$$

or

$$P_0 < \frac{E_{x_P} (\tilde{S}_T) + \delta' E_{x_P}(\tilde{P}_T) - S_0(1 + r)F(x_P)}{\delta'(1 + r)F(x_P)}$$

Proof of Theorem 4

Comparison 7

With $\delta' < \delta_0$, if the strategies have one intersection point, $N(-x_N, F'(-x_N))$, lemma 1 can be applied. To prevent s5′ from dominating s1′, it is necessary that

$$\int_0^\infty -[(S_0 + \delta' P_0)(1 + r) - \delta' u(K - x)]f(x)dx < -\int_0^\infty xf(x)dx$$

which gives:

$$-\delta' E(\tilde{P}_T) + (S_0 + \delta' P_0)(1 + r)$$

$$> E(\tilde{S}_T) \Rightarrow P_0 > \frac{E(\tilde{S}_T) + \delta' E(\tilde{P}_T) - S_0(1 + r)}{\delta' (1 + r)}$$

With $\delta' < \delta_0$, if the strategies have three intersection points, $N(-x_N, F'(-x_N))$, $P(-x_P, F'(-x_P))$, and $Q(-x_Q, F'(-x_Q))$, lemma 3 can be applied. To prevent s5′ from dominating s1′, it is necessary that

$$-\delta' E(\tilde{P}_T) + (S_0 + \delta' P_0)(1 + r) > E(\tilde{S}_T)$$

which implies:

$$P_0 > \frac{E (\tilde{S}_T) + \delta' E(\tilde{P}_T) - S_0(1 + r)}{\delta' (1 + r)}$$

or with L the distribution function of s5:

$$\int_{-\infty}^{-x_P} (1 - L(-x))dx > \int_{-\infty}^{-x_P} (1 - F(-x))dx$$

$$\Rightarrow \int_{x_P}^{\infty} (1 - L(x))dx > \int_{x_P}^{\infty} (1 - F(x))dx$$

which implies:

$$E(s_5) - E_{x_P}(s_5) > E(s_1) - E_{x_p}(s_1)$$

$$(S_0 + \delta' P_0)(1 + r)(1 - F(x_P)) - \delta' (E (\tilde{P}_T) - E_{x_P}(\tilde{P}_T))$$

$$> E (\tilde{S}_T) - E_{x_P}(\tilde{S}_T)$$

and thus:

$$P_0 > \frac{E^{x_P*} (\tilde{S}_T) + \delta' E^{x_P*} (\tilde{P}_T) - S_0(1 + r)}{\delta' (1 + r)}$$

With $\delta'' > \delta_0$, the strategies have two intersection points, $N(-x_N, F'(-x_N))$ and $P(-x_P, F'(-x_P))$, lemma 2 can be applied. To prevent s5′ from dominating s1′, the same condition as in the above case is obtained, but at the point x_N, instead of x_P.

Comparison 8

The strategies have one or two intersection points, $U(-x_U, F'(-x_U))$ and $V(-x_V, F'(-x_V))$. If there are two, lemma 2 can be applied. To prevent s1′ from dominating s6′, it is necessary that

$$\int_{-\infty}^{-x_U} (1 - I(-x))dx < \int_{-\infty}^{-x_U} (1 - F(-x))dx$$

$$\Rightarrow \int_{x_U}^{\infty} (1 - I(x))dx < \int_{x_U}^{\infty} (1 - F(x))dx$$

Thus:

$$E(\tilde{S}_T) - E_{x_U}(\tilde{S}_T) > \gamma(E(\tilde{S}_T) - E_{x_U}(\tilde{S}_T)) - (E (\tilde{P}_T) - E_{x_U}(\tilde{P}_T)).$$

and

$$P_0 < S_0 \frac{E^{x_U} (\tilde{P}_T)}{E^{x_U} (\tilde{S}_T)}$$

which becomes, if there is only one intersection point:

$$P_0 < S_0 \frac{E(\tilde{P}_T)}{E(\tilde{S}_T)}$$

APPENDIX D: PROOFS OF PUT/CALL PARITY

Proof of Theorem 5

With k′ such that there is only one intersection point, the dominance of s7 over s1 must be prevented. Thus:

$$k'\ E(\check{C}_T) + (k'\ P_0 - k'\ C_0 + S_0)(1 + r) - k'\ E(\check{P}_T)$$

$$< k'\ E(S_T) \Rightarrow P_0 - C_0 < \frac{E(\check{P}_T) - E(\check{C}_T) + E(\check{S}_T)}{(1 + r)} - \frac{S_0}{k'}$$

With k″ such that there are two intersection points, the dominance of s7 over s1 must be prevented. The condition is expressed at the second point, Z. Thus:

$$k''\ E_{x_z}(\check{C}_T) + (k''\ P_0 - k''\ C_0 + S_0)(1 + r)F(x_Z) - k''\ E_{x_z}(\check{P}_T)$$

$$> k''\ E_{x_z}(S_T) \Rightarrow P_0 - C_0 > \frac{E_{x_Z}(\check{P}_T) - E_{x_Z}\ (\check{C}_T) + E_{x_Z}\ (\check{S}_0)}{(1 + r)F(x_z)} - \frac{S_0}{k''}$$

Proof of Theorem 6

The proof of theorem 6 relies on the same relations for bearish strategies.

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