Time-Dependent Barrier Option Values

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INTRODUCTION

European barrier options are path-dependent options in which the existence of the European options depends on whether the underlying asset price has touched a barrier level during the option life. They have emerged as significant products for hedging and investment in foreign exchange, equity, and commodity markets since the late 1980s, largely in the over-the-counter (OTC) markets. The only barrier option that is traded on the options exchanges is the capped index spread on the S&P 100 and S&P 500. The cap is automatically exercised with a fixed profit when the underlying price closes beyond the barrier level. Thus the option is canceled with a fixed return at the barrier level. Its pricing and hedging are discussed by Chance (1994).

An example of a barrier option is an up-and-out put. An investor could buy an up-and-out U.S. dollar put (Japanese yen call) instead of an ordinary U.S. dollar put to hedge the value of U.S. dollar versus Japanese yen. The put option disappears as the market exchange rate moves favorably to the Japanese yen exposure. If it is knocked out, the exposure can be covered or rehedged more favorably. A standard spot exchange rate order can be placed at the barrier to cover the amount to be hedged. When the option is knocked out, the spot order will be automatically filled. The knock-out feature makes the up-and-out put cheaper than the ordinary put. Another example is a down-and-out put, which is also called a reverse knock-out put. The option is extinguished when the market rate moves unfavorably to the Japanese yen exposure. Such hedges imply a

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strong view that the market rate will not be traded at or through the barrier during the option life. Other such options include down-and-out calls and up-and-out calls. There are also "in" options, which actually commence when the underlying price touches the barrier. Barrier options are cheaper than ordinary European options because of these extinguishing and activating features. Thus they are attractive to investors who are averse to paying high premiums. In addition, sellers of barrier options may be able to limit their downside risk.

Merton (1973) was the first to derive a closed-form solution for a down-and-out European call option. Other closed-form pricing formulas of single-barrier options are published by Cox and Rubinstein (1985) and Rubinstein and Reiner (1991a, 1991b). Rich (1994) provided a unified and intuitive mathematical framework for pricing the single-barrier options. The valuation method for double-barrier options using the probabilistic approach is discussed in Kunitomo and Ikeda (1991). The values of the double-barrier options can also be obtained by solving the Black–Scholes partial differential equation [Black and Scholes (1973)] with the corresponding boundary conditions using the Fourier series method. Analytical solutions of one-touch double-barrier binary options in which a fixed pay off is determined by whether it is touching the barrier are discussed by Hui (1996). Their derivations assume the underlying asset price follows a log-normal diffusion process and continuous knock-out and knock-in. However, some barrier options, for example, the exchange-traded capped index spread, may only be knocked out or in at daily closing times. This barrier discontinuity poses a difficult problem for pricing barrier options and is discussed in Benson and Daniel (1992) and Hudson (1992).

Over the last few years the foreign exchange market has witnessed periods of dramatic volatility, including the devaluation of sterling in the fall of 1992, the EMS currency crisis that followed, and a highly unstable yen/U.S. dollar in 1995. This has prompted more used exotic options, especially the barrier options, for hedging and investing in the foreign exchange markets. Because the foreign currency barrier options are knocked out and in at any time in the OTC market, the assumption of continuous knock-out and knock-in is valid in the foreign currency barrier options. The closed-form solutions assuming continuous knock-out and knock-in are commonly used by dealers for pricing the foreign currency barrier options.

Regular barrier options subject investors to barrier exposure throughout the option life. Time-dependent barrier options are hybrids between the regular barrier options and ordinary options. They let an option purchaser choose a barrier period covering a segment of time either at the beginning (front end) or the end (rear end) of the option life. This feature makes them more flexible than the regular barrier options for an investor having a particular view on an underlying asset in a certain period of time. The time-dependent barrier options are used for hedging and investment in the foreign exchange OTC market because the investors can diversify the risk of the option being knocked out or in during the option life. If an option purchaser expects an underlying asset price to hold in the short term, adding front-end knock-out barriers can reduce the option premium. On the other hand, selling rear-end knock-out barrier options lets investors receive more premium than would be realized by selling regular knock-out options. Selling front-end knock-in options receives option premiums for taking a shorter-term risk of knocking in than selling a regular knock-in option. If an investor expects some information and events to occur in a certain period of time, he can hedge the underlying asset with a time-dependent knock-in option at a cheaper cost than by using a regular option or knock-in option.

The time-dependent barrier options have a one-time barrier discontinuity, which makes their pricing formulas different from the regular barrier option-pricing formulas. The valuation formulas of time-dependent knock-out options with single and double barriers are developed in the following section. In Section III, the delta, gamma, and vega risks of time-dependent up-and-out calls are discussed and compared with the risk parameters of ordinary options. Applications of the time-dependent barrier options are proposed in Section IV. Conclusions are summarized in Section V.

OPTION VALUATION

The objective is to value the option in a Black–Scholes environment [Black and Scholes (1973)]. The underlying asset price is assumed to follow a log-normal diffusion process. The interest rate and dividend are taken to be continuous and constant over the option life. The valuation method is a risk-neutral valuation approach. A time-dependent knock-in option can be priced by buying an ordinary option (with same strike and maturity) and selling a time-dependent knock-out option (with the same barrier, knock-out period, strike, and maturity). Two types of time-dependent knock-out options are derived: (a) front-end knock-out options with barrier period from option start time to time t, and (b) rear-end knock-out option with barrier period from time t to option maturity T.

Both single-barrier and double-barrier option pricing formulas are obtained in the derivation.

The derivation is from the Black–Scholes equation:

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + (r - q)S\frac{\partial f}{\partial S} - rf = 0 \tag{1}$$

where f is the option value, S is the underlying price, t is the time, σ is the volatility, r is the risk-free interest rate, and q is the dividend if the underlying price is a stock price. If the underlying price is a foreign exchange, r is the domestic currency interest rate and q is the foreign currency interest rate. These definitions for a currency option follow Garman and Kohlhagen (1983). The Black–Scholes equation is transformed into a diffusion equation (heat equation) for analytical derivation of the option values in Appendix A.

The option life of the time-dependent knock-out option is divided into two time segments. The barrier (or double barrier) exists in one of the time segments. This implies that the boundary condition changes at time t within the option life. The option value is obtained by solving eq. (1) from the option maturity back to the option start time according to the boundary conditions. The front-end knock-out option value at time t is an ordinary option value. The value is then used as the initial condition in the time segment with the barrier boundary condition to solve eq. (1) and get the front-end knock-out option value. Similarly, a knock-out option value is used as the initial condition in the time segment without the barrier to solve eq. (1) and get the rear-end knock-out option value. Different volatilities and interest rates can be used in the two time segments, but for simplicity they are established at the same level in the following derivation.

The Pricing of a Front-End Barrier Call Option

Single-barrier Up-and-Out Call Option

The barrier H of the front-end up-and-out call option exists from the option start time to time t. The option behaves as a regular up-and-out call within the barrier period, and the extinguishing feature reduces the option premium. The option then becomes an ordinary call after the barrier period. If a longer barrier period is put into the option, the premium is reduced more. Because the barrier period does not last to the maturity T, the option payoff is not limited by the barrier level and is the same as an ordinary call. This flexible feature does not apply to the regular up-and-out call in which the option payoff is capped by the barrier. The front-end up-and-out call consists of both characteristics of the regular up-and-out call and ordinary call. Its delta, gamma, and vega risks are discussed in Section III.

The barrier discontinuity at time t divides the option life into two time segments. The Black–Scholes equation is solved in these two-time segments according to the respective boundary conditions. The details of the derivation are in Appendix B1. The front-end up-and-out call option value is obtained as

$$\begin{aligned} C_{1uo} = {} & Se^{-qT}\, N_2[-b'(t), a'(T), -\rho] - Ke^{-rT} \\ & N_2[-b(t), a(T), -\rho] - \left(\frac{H}{S}\right)^{k_1+1} Se^{-qT}\, N_2[-c'(t), d'(T), -\rho] \\ & + \left(\frac{H}{S}\right)^{k_1-1} Ke^{-rT}\, N_2[-c(t), d(T), -\rho] \end{aligned} \tag{2}$$

where N_2 is a bivariate cumulative normal distribution function, $\rho = \sqrt{t/T}$,

$$a = \frac{\ln(S/K) + (r-q)t}{\sigma\sqrt{t}} - \frac{\sigma\sqrt{t}}{2}, \qquad a' = a + \sigma\sqrt{t}$$

$$b = \frac{\ln(S/H) + (r-q)t}{\sigma\sqrt{t}} - \frac{\sigma\sqrt{t}}{2}, \qquad b' = b + \sigma\sqrt{t}$$

$$c = \frac{\ln(H/S) + (r-q)t}{\sigma\sqrt{t}} - \frac{\sigma\sqrt{t}}{2}, \qquad c' = c + \sigma\sqrt{t}$$

$$d = \frac{\ln(H^2/SK) + (r-q)t}{\sigma\sqrt{t}} - \frac{\sigma\sqrt{t}}{2}, \qquad d' = d + \sigma\sqrt{t},$$

and k_1 is defined in eq. (A1). The pricing formulas of other front-end knock-out options are derived in Appendix B1.

Double-Barrier Knock-out Call Option

A double-barrier front-end knock-out call option is similar to the single-barrier knock-out call except that it has one more barrier level. The additional barrier makes the option premium less than the single-barrier, front-end, knock-out call option. The method of separation of variables is used to solve the option value, with the double barrier as the boundary condition. The details of the derivation are in Appendix B2. The front-end double-barrier knock-out option value is obtained as

$$C_f = \sum_{n=1}^{\infty} b_n K \left(\frac{S}{K}\right)^{-(1/2)(k_1-1)} \sin\left(\frac{n\pi}{L} \ln \frac{S}{H_1}\right) e^{-(1/2)(n\pi/L)^2\sigma^2 t + (1/2)\beta\sigma^2 T} \quad (3)$$

where

$$\begin{aligned} b_n = \frac{2}{L} \int_{\ln(H_1/K)}^{\ln(H_2/K)} & \left[e^{(1/2)(k_1+1)x + (1/8)(k_1+1)^2\sigma^2(T-t)} \right. \\ & N_1\left(\frac{x}{\sigma\sqrt{(T-t)}} + \frac{1}{2}(k_1+1)\sigma\sqrt{(T-t)}\right) \\ & - e^{(1/2)(k_1-1)x + (1/8)(k_1-1)^2\sigma^2(T-t)} N_1\left(\frac{x}{\sigma\sqrt{(T-t)}}\right. \\ & \left.\left. + \frac{1}{2}(k_1-1)\sigma\sqrt{(T-t)}\right)\right] \sin\left[\frac{n\pi}{L}\left(x - \ln\frac{H_1}{K}\right)\right] dx \end{aligned} \quad (4)$$

$L = \ln(H_2/H_1)$, and β is defined in eq. (A2). b_n is integrated numerically. The series in eq. (3) is an exact solution. The sine factors are all bounded as $n \to \infty$; therefore this series behaves similarly to the series, $\sum_{n=1}^{\infty} e^{-n^2 t}$. The convergence of the series is estimated as

$$\sum_{n=1}^{\infty} e^{-n^2 t} = \sum_{n=1}^{\infty} \left(\frac{1}{1 + n^2 t + \frac{n^4 t^2}{2} + \cdots}\right) \leq \sum_{n=1}^{\infty} \frac{1}{n^2 t}$$

The factor, $e^{-(1/2)(n\pi/L)^2\sigma^2 t}$, with $t > 0$ makes the series converge similarly to the right-hand side of the inequality. A satisfactory approximation to the solution can be obtained by using a few terms in the series. The double-barrier knock-out put value is obtained by multiplying -1 in the argument of the cumulative normal distribution function and interchanging $(k_1 + 1)$ and $(k_1 - 1)$ in eq. (4).

The Pricing of a Rear-End Barrier Call Option

Single-Barrier up-and-out Call Option

The barrier, H, of the rear-end up-and-out call option exists from time t to the option maturity T. The option behaves as an ordinary call before the barrier period. The option then becomes an up-and-out call within the barrier period. Because the barrier period lasts to the maturity, the option payoff is limited by the barrier level like the regular up-and-out call. Its delta, gamma, and vega risks are discussed in Section III. The barrier discontinuity at time t divides the option life into two time segments. The value of the rear-end up-and-out call is solved in these two time segments according to the respective boundary conditions. The details of the derivation are in Appendix C1. The rear-end up-and-out call option value is

$$\begin{aligned} C_{2uo} = {} & Se^{-qT}\{N_2[-b'(t), a'(T), -\rho] - N_2[-b'(t), b'(T), -\rho]\} \\ & - Ke^{-rT}\{N_2[-b(t), a(T), -\rho] - N_2[-b(t), b(T), -\rho]\} \\ & - \left(\frac{H}{S}\right)^{k_1+1} Se^{-qT}\{N_2[c'(t), d'(T), \rho] - N_2[c'(t), c'(T), \rho]\} \\ & + \left(\frac{H}{S}\right)^{k_1-1} Ke^{-rT}\{N_2[c(t), d(T), \rho] - N_2[c(t), c(T), \rho]\} \end{aligned} \tag{5}$$

The pricing formulas of other rear-end knock-out options are in Appendix C1.

Double-Barrier Knock-out Call Option

The valuation method is similar to the approach in the previous section. The initial condition in the time segment without the barrier is a double-barrier knock-out call option value derived by using the Fourier series method. It is put into eq. (C3) and the rear-end double-barrier knock-out call option value with $K > H_1$ is obtained as

$$\begin{aligned} C_r = {} & \sum_{n=1}^{\infty} K\left(\frac{S}{K}\right)^{-(1/2)(k_1-1)} \left\{\frac{2}{L\left[\frac{1}{4}(k_1+1)^2 + \left(\frac{n\pi}{L}\right)^2\right]}\right. \\ & \left[\frac{1}{2}(k_1+1)\sin\left(\frac{n\pi}{L}\ln\frac{H_1}{K}\right) + \frac{n\pi}{L}\cos\left(\frac{n\pi}{L}\ln\frac{H_1}{K}\right) - \frac{n\pi}{L}(-1)^n\right. \\ & \left.\left(\frac{H_2}{K}\right)^{(1/2)(k_1+1)}\right] - \frac{2}{L\left[\frac{1}{4}(k_1-1)^2 + \left(\frac{n\pi}{L}\right)^2\right]} \\ & \left[\frac{1}{2}(k_1-1)\sin\left(\frac{n\pi}{L}\ln\frac{H_1}{K}\right) + \frac{n\pi}{L}\cos\left(\frac{n\pi}{L}\ln\frac{H_1}{K}\right)\right. \\ & \left.\left.-\frac{n\pi}{L}(-1)^n\left(\frac{H_2}{K}\right)^{(1/2)(k_1-1)}\right]\right\}\left[b_{1n}\sin\left(\frac{n\pi}{L}\ln\frac{S}{H_1}\right)\right. \\ & \left.+ b_{2n}\cos\left(\frac{n\pi}{L}\ln\frac{S}{H_1}\right)\right] e^{-(1/2)(n\pi/L)^2\sigma^2(T-t)+(1/2)\beta\sigma^2 T} \end{aligned} \tag{6}$$

where

$$b_{1n} = \int_{a_1}^{a_2} \cos\left(\frac{n\pi x}{L}\sqrt{\sigma^2 t}\right) e^{-(1/2)x^2}\, dx,$$

$$b_{2n} = \int_{a_1}^{a_2} \sin\left(\frac{n\pi x}{L}\sqrt{\sigma^2 t}\right) e^{-(1/2)x^2}\, dx,$$

$$a_1 = \frac{\ln(H_1/S)}{\sqrt{\sigma^2 t}},\ a_2 = \frac{\ln(H_2/S)}{\sqrt{\sigma^2 t}} \tag{7}$$

The rear-end double-barrier knock-out call option value with $K < H_1$ is

$$\begin{aligned} C_r = \sum_{n=1}^{\infty} K\left(\frac{S}{K}\right)^{-(1/2)(k_1-1)} & \left\{\frac{2n\pi}{L^2\left[\frac{1}{4}(k_1+1)^2 + \left(\frac{n\pi}{L}\right)^2\right]}\right. \\ & \left[\left(\frac{H_1}{K}\right)^{(1/2)(k_1+1)} - (-1)^n\left(\frac{H_2}{K}\right)^{(1/2)(k_1+1)}\right] \\ & - \frac{2n\pi}{L^2\left[\frac{1}{4}(k_1-1)^2 + \left(\frac{n\pi}{L}\right)^2\right]}\left[\left(\frac{H_1}{K}\right)^{(1/2)(k_1-1)} - (-1)^n\right. \\ & \left.\left.\left(\frac{H_2}{K}\right)^{(1/2)(k_1-1)}\right]\right\}\left[b_{1n}\sin\left(\frac{n\pi}{L}\ln\frac{S}{H_1}\right)\right. \\ & \left. + b_{2n}\cos\left(\frac{n\pi}{L}\ln\frac{S}{H_1}\right)\right] e^{-(1/2)(n\pi/L)^2\sigma^2(T-t)+(1/2)\beta\sigma^2 T} \end{aligned} \tag{8}$$

The series in eqs. (7) and (8) converge similarly to the series, $\Sigma_{n=1}^{\infty} n^{-1}e^{-n^2(T-t)}$. All terms after the first few terms are almost negligible when t is sufficiently large. The pricing formulas of a rear-end double-barrier knock-out put option are in Appendix C2.

HEDGE PARAMETERS OF TIME-DEPENDENT KNOCK-OUT OPTIONS

When dealers trade any derivative instruments, they often need to hedge their positions. It is important to determine the hedge parameters of the derivative instruments. The following discussion is focused on the deltas, gammas, and vegas (the effect of the volatility on the option values) of the time-dependent knock-out options. Examples are a front-end and a rear-end up-and-out U.S. dollar (USD) call against Japanese yen (JPY)

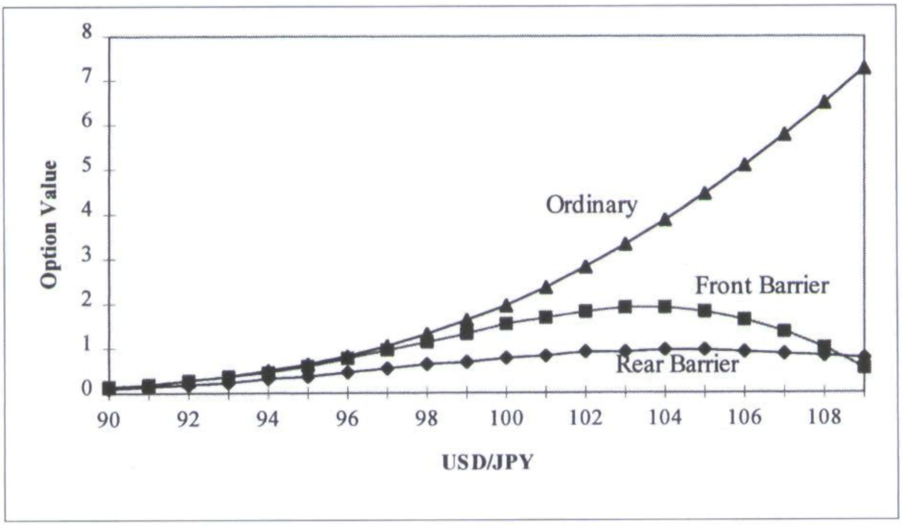

FIGURE 1
Variation of option value with the underlying price.

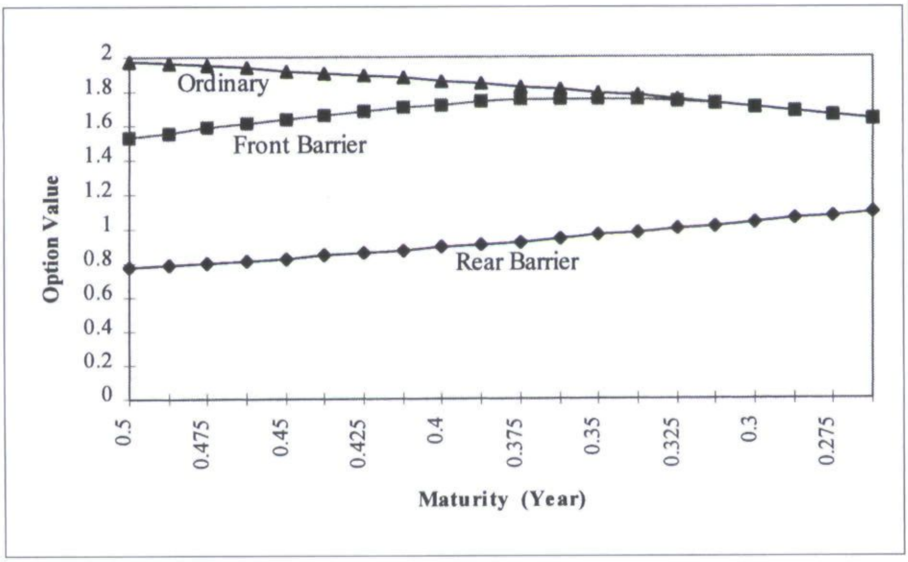

FIGURE 2
Variation of option value with the option maturity.

put option with strike = 100, barrier = 110, time to maturity = 0.5 year, barrier period = 0.25 year, USD interest rate = 6%, JPY interest rate = 1%, volatility = 11%. The variations of the option values and hedge parameters of the examples with the underlying price and time to maturity are illustrated in Figures 1–8. They are compared with the var-

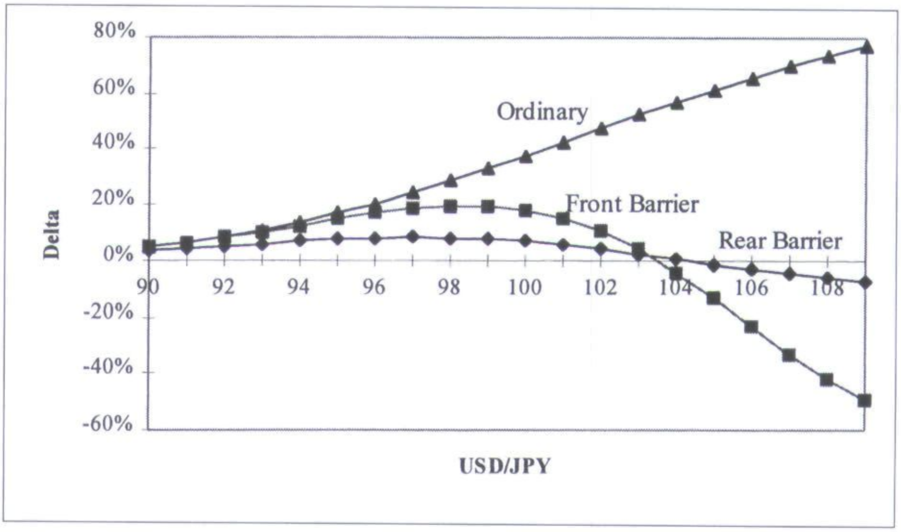

FIGURE 3
Variation of delta with the underlying price.

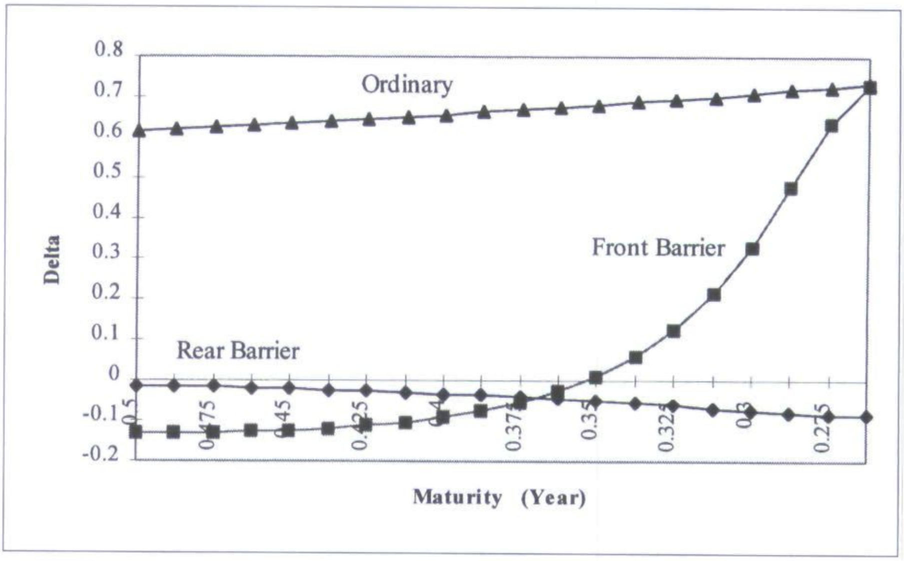

FIGURE 4
Variation of delta with the option maturity.

iations of the option values and hedge parameters of an ordinary call option. The values of the hedge parameters are measured as finite-difference approximations to their continuous-time equivalents.

Figure 1 illustrates the option value variations with different USD/JPY of the time-dependent barrier options and ordinary option. The time-

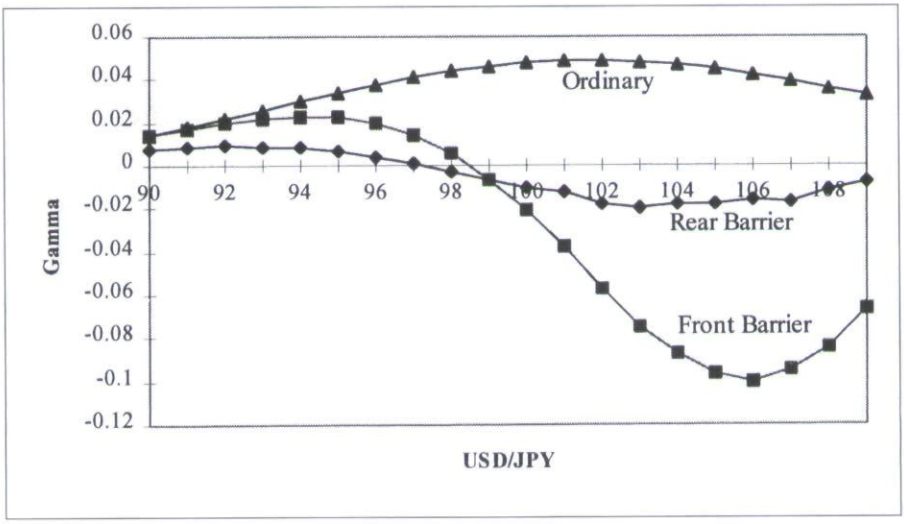

FIGURE 5
Variation of gamma with the underlying price.

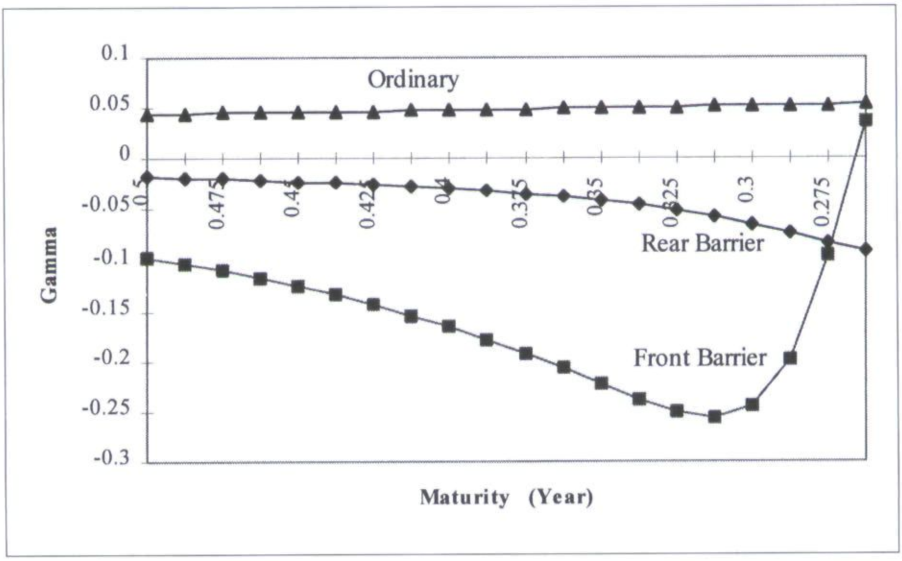

FIGURE 6
Variation of gamma with the option maturity.

dependent barrier options, especially the rear-end knock-out option, cost less than the ordinary option because of the knock-out feature of the barrier options. The front-end knock-out option value decreases from 1.92 with USD/JPY = 104 to 0.53 with USD/JPY = 109, which is close to the barrier. The risk of the option being knocked out increases as the

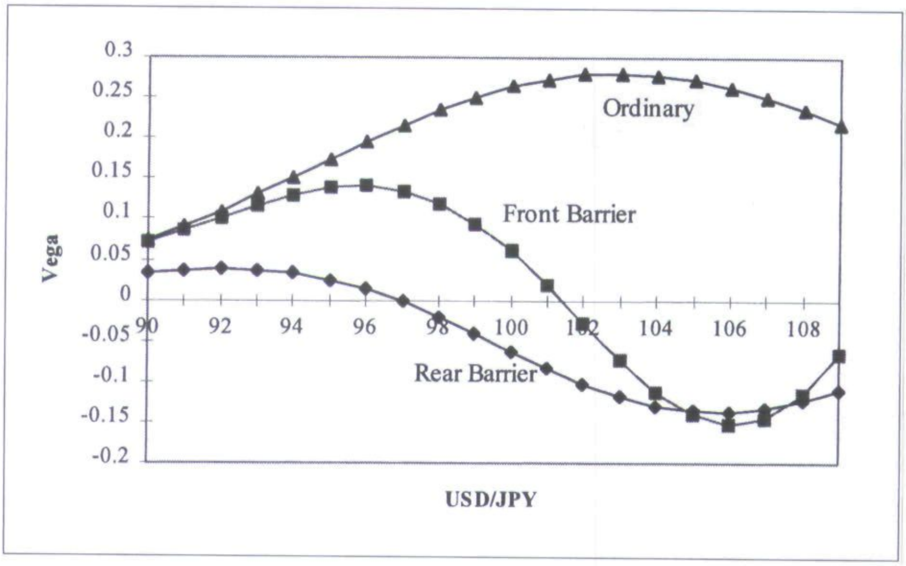

FIGURE 7
Variation of vega with the underlying price.

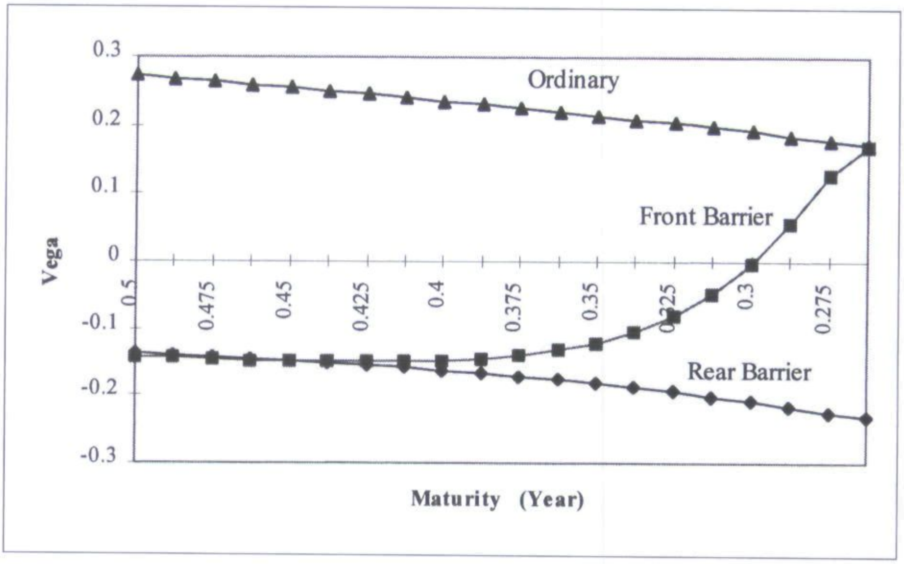

FIGURE 8
Variation of vega with the option maturity.

underlying price approaches the barrier. The option values are reduced even though the options are more in the money. This is behavior typical of a regular up-and-out call. Because the barrier period has not started in the rear-end up-and-out call, its risk of being knocked out and the effect of being in the money cancel each other such that the variation of the option values with the underlying price is small. The effect of time to maturity on the option values is illustrated in Figure 2 with USD/JPY = 100. The ordinary call shows a normal time decay in the option value. Its value goes down from 1.98 to 1.64 when the time to maturity decreases from 0.5 to 0.25. On the other hand, the values of the front-end and the rear-end up-and-out calls increase from 1.53 to 1.64 and from 0.78 to 1.09, respectively, with the shorter time to maturity. The shorter time to maturity reduces the probability of being knocked out and enhances the values of the time-dependent up-and-out calls. Because the front-end up-and-out call becomes an ordinary call after the barrier period, the front-end up-and-out call value approaches the ordinary call value with a shorter barrier period.

Dealers need accurate values for the delta and the delta behavior to hedge their option portfolios effectively. The delta variations with USD/JPY of both front-end and rear-end knock-out options show two features that are typical of the up-and-out option in Figure 3. First, their deltas are smaller than that of the ordinary option because of the cheaper knock-out option values. Second, the deltas are negative when the underlying price is close to the knock-out barrier. This is because the option values decrease when the underlying price moves higher and closer to the barrier. In Figure 4, the underlying USD/JPY is set at 105 to illustrate the delta variations with time to maturity. Although the deltas of the ordinary call and rear-end up-and-out call are relatively stable with time, the delta of the front-end up-and-out call increases with the shorter time to maturity (shorter barrier period) and approaches the delta of the ordinary option. The steep slope in the delta of the front-end up-and-out call shows a high gamma, especially near the end of the front barrier period. The gammas are shown in Figures 5 and 6.

Figure 5 illustrates the variations of gamma with the underlying price. The gamma is defined as the change in delta per unit price of the underlying asset. The front-end up-and-out call shows a significant negative gamma -0.1 when the underlying price is 106 which is close to the barrier. On the other hand, the gammas of the ordinary call and the rear-end up-and-out call are only up to 0.048 and -0.02, respectively, and are relatively stable. Figure 6 shows that the gamma of the front-end up-and-out call increases in magnitude with the shorter barrier period.

TABLE I

Gammas of up-and-out Call and Ordinary Call

Time to Maturity	*Gamma of Up-and-Out Call*	*Gamma of Call*
0.25	−0.065845	0.025750
0.2375	−0.071058	0.024809
0.225	−0.076943	0.023765
0.2125	−0.083625	0.022608
0.2	−0.091670	0.021330
0.1875	−0.100479	0.019922
0.175	−0.110695	0.018378
0.1625	−0.114564	0.016695
0.15	−0.136275	0.014873
0.1375	−0.153118	0.012921
0.125	−0.173539	0.010860
0.1125	−0.198736	0.008730
0.1	−0.230524	0.006597
0.0875	−0.271809	0.004558
0.075	−0.327576	0.002749
0.0625	−0.407179	0.001331
0.05	−0.530310	0.000437
0.0375	−0.745593	0.000065
0.025	−1.207086	0.000001
0.0125	−2.701574	0.000000

The variation of the gamma is large when the barrier period is close to the end. These two figures indicate that the gamma of the front-end up-and-out call with the short barrier period and the underlying price near the barrier are unstable and high in magnitude. The behavior of the gamma is caused by the front-end knock-out option having either zero or an ordinary option value at the end of the barrier period. This characteristic looks like a binary option. The binary option has a discontinuous payoff, zero, or a fixed amount, at maturity. The delta should be relatively stable for effective hedging, which would be indicated by a low gamma. The instability in the deltas indicates that hedging the front-end up-and-out call is no simple task and may be nearly possible on the practical level.

The rear-end up-and-out call becomes a regular up-and-out call during the rear-end barrier period. The variations of gamma with time to maturity of a regular in-the-money up-and-out call and an in-the-money ordinary call are illustrated in Table I with USD/JPY = 109, barrier = 110, and time to maturity = 0.25. The gamma of the up-and-out call is large in magnitude (up to −2.70) when the underlying price is close to the barrier and the time to maturity is short. It is difficult for dealers to hedge barrier options. Therefore, both in-the-money front-end and rear-end up-and-out calls have high gammas at the end of the barrier period.

The vega variations in Figures 7 and 8 are the difference of the option value with 1% increase in the volatility. In Figure 7, the vega of the time-dependent up-and-out calls turns from positive to negative when the underlying price is close to the barrier. It is a feature typical of the up-and-out call. The higher volatility makes it easier for the up-and-out call to be knocked out near the barrier. In Figure 8, the vega of the front-end up-and-out call increases with the shorter time to maturity (shorter barrier period). It approaches the vega of the ordinary call because the front-end up-and-out call becomes an ordinary call after the barrier period.

APPLICATIONS

Some potential investment and corporate applications of the time-dependent barrier options are discussed in this section. These time-dependent barrier options are used for hedging, like the regular barrier options. An investor could buy a front-end down-and-out U.S. dollar put (Japanese yen call) instead of an ordinary U.S. dollar put to hedge the value of U.S. dollar versus Japanese yen. The option is extinguished when the Japanese yen is strong against the U.S. dollar. The regular down-and-out U.S. dollar put is subject to the risk that the option will be knocked out during the option's life. The front-end barrier period that does not last to the option maturity reduces the risk of the option being knocked out. Although its option premium is more expensive than the regular down-and-out put premium, it is still cheaper than the ordinary put, which is attractive to the investor. The length of the barrier period can be fit to a particular view on the USD/JPY exchange rate in a certain period of time.

Cox and Rubinstein (1985, pp. 408–411) suggested that a down-and-out call is used to value a bond with a safety covenant. Hudson (1992) pointed out that Nikkei-linked bonds with embedded short European up-and-out puts were marketed with great success to Japanese investors. In some issued bonds, the embedded knock-out options are forward start. Rich (1994) suggested that a forward-start barrier option could accommodate the protection-period feature common to many callable fixed-income securities. The strike and barrier values in the forward-start barrier option-pricing formulas are the forward underlying asset price multiplied by prespecified constants. They are not fixed values. However, the strikes and barriers of the embedded knockout options in those bonds are prespecified, fixed values rather than relative values to the forward underlying asset. The rear-end barrier options can resolve this problem because their strike and barrier are fixed values in the option-pricing formulas. They can also be applied to a convertible bond in which a forward-start forced conversion feature (converting the bond into a fixed number of shares of a stock) consists of knock-out options starting at some time during the bond life.

CONCLUSION

Barrier options have been used extensively for hedging and investment in the OTC foreign exchange, equity, and commodity markets since late 1980s. Time-dependent barrier options are hybrids of regular barrier options and ordinary options. Because the barrier period covers a segment of time either at the front end or rear end of the option life, the risk of the option being knocked out/in is diversified during the option life. The analytical pricing formulas of front- and rear-end knock-out options are derived in the conventional Black–Scholes option-pricing environment. The delta, gamma, and vega risks of the front-end and rear-end up-and-out call are discussed. The high gammas at the end of the barrier period of both in-the-money front-end and rear-end up-and-out calls introduce the instability in the deltas. Hedging under these conditions is not simple and is not practical.

APPENDIX A

To make the Black–Scholes equation (1) dimensionless (Wilmott, DeWynne, and Howison 1993), the following transformations are made:

$$S = Ke^{x}, \qquad t = \frac{2\tau}{\sigma^2}, \qquad f = Ke^{\alpha x + \beta\tau}\, u(x,\tau), \qquad k_1 = \frac{2(r - q)}{\sigma^2} \tag{A1}$$

where

$$\alpha = -\frac{1}{2}(k_1 - 1), \qquad \beta = -\frac{1}{4}(k_1 - 1)^2 - \frac{2r}{\sigma^2} \tag{A2}$$

The diffusion equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \tag{A3}$$

is then obtained.

APPENDIX B

The Pricing of Front-End Knock-out Options

Single-Barrier Options

The barrier, H, of the front-end up-and-out call option exists from the option start time to time t' (where the prime denotes variables in the dimensionless equation). Its boundary condition is

$$u\left(h = \ln\frac{H}{K}, \tau\right) = 0, \qquad 0 \leq \tau \leq t' \leq T' \tag{B1}$$

The initial condition at time t' is an ordinary call option value, u_0, with maturity $(T' - t')$. It is

$$\begin{aligned} u_0(x, T' - t') = {} & e^{(1/2)(k_1+1)x + (1/4)(K_1+1)^2(T'-t')}\, N_1 \\ & \left[\frac{x}{\sqrt{2(T' - t')}} + \frac{1}{2}(k_1 + 1)\sqrt{2(T' - t')}\right] \\ & - e^{(1/2)(k_1-1)x + (1/4)(K_1+1)^2(T'-t')}\, N_1 \\ & \left[\frac{x}{\sqrt{2(T' - t')}} + \frac{1}{2}(k_1 - 1)\sqrt{2(T' - t')}\right] \end{aligned} \tag{B2}$$

where N_1 is a cumulative normal distribution function.

The coordination is shifted to $x = 0$ at the barrier, H, to make the algebra simpler in solving the diffusion equation, eq. (A3). The problem in the time segment with the barrier is the same as heat diffusion in a semiinfinite solid, $(-\infty < x \leq 0)$, which is kept at temperature zero at $x = 0$ with an initial temperature distribution, $u_0(x, T' - t')$, in eq. (B2). The solution of heat diffusion in a semiinfinite solid is given by Churchill and Brown (1985) as

$$\begin{aligned} u(x, \tau) = {} & \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-(x/\sqrt{2\tau})} u_0 - (x - \zeta + y\sqrt{2t'}, T' - t') e^{-(y^2/2)} dy \\ & - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(x/\sqrt{2\tau})} u_0 (-x - \zeta + y\sqrt{2t'}, T' - t') e^{-(y^2/2)} dy \end{aligned} \tag{B3}$$

where $\zeta = \ln(K/H)$.

If the transformations in eq. (A1) are reinserted, the solution of the front-end up-and-out call option value is obtained in eq. (2).

With the use of the same approach, the front-end down-and-out call value is

$$
\begin{aligned}
C_{1do} = {} & Se^{-qT} N_2 [b'(t), a'(T), \rho] - Ke^{-rT} N_2[b(t), a(T), \rho] \\
& - \left(\frac{H}{S}\right)^{k_1+1} Se^{-qT} N_2 [c'(t), d'(T), \rho] \\
& + \left(\frac{H}{S}\right)^{k_1-1} Ke^{-rT} N_2[c(t), d(T), \rho] \qquad \text{(B4)}
\end{aligned}
$$

The front-end up-and-out put value is

$$
\begin{aligned}
P_{1uo} = {} & Ke^{-rT} N_2[-b(t), -a(T), \rho] - Se^{-qT} N_2 [-b'(t), \\
& - a'(T), \rho] - \left(\frac{H}{S}\right)^{k_1-1} Ke^{-rT} N_2[-c(t), -d(T), \rho] \\
& + \left(\frac{H}{S}\right)^{k_1+1} Se^{-qT} N_2[-c'(t), -d'(T), \rho] \qquad \text{(B5)}
\end{aligned}
$$

The front-end down-and-out put value is

$$
\begin{aligned}
P_{1do} = {} & Ke^{-rT} N_2[b(t), -a(T), \rho] - Se^{-qT} N_2 [b'(t), - a'(T), \\
& -\rho] - \left(\frac{H}{S}\right)^{k_1-1} Ke^{-rT} N_2[c(t), -d(T), \rho] \\
& + \left(\frac{H}{S}\right)^{k_1-1} Se^{-qT} N_2[c'(t), -d'(T), -\rho] \qquad \text{(B6)}
\end{aligned}
$$

Double-barrier Knock-out Options

The valuation method is similar to the derivation of the single-barrier option value in the previous section. The boundary conditions in the time segment with barriers, H_1 and H_2 ($H_1 < H_2$), are

$$
u\left(\ln\frac{H_1}{K}, \tau\right) = 0, \qquad u\left(\ln\frac{H_2}{K}, \tau\right) = 0 \qquad \text{(B7)}
$$

for $0 \leq \tau \leq t' \leq T$ and the initial condition is $u_0(x, T' - t')$ in Eq. (6).

The method of separation of variables is used to solve the diffusion equation, eq. (A3) with homogeneous boundary conditions in eq. (B7). The function, $u(x, \tau)$, can be expressed as a series of eigenfunctions,

$$u(x, \tau) = \sum_{n=1}^{\infty} b_n(\tau) \sin\left[\frac{n\pi}{L}\left(x - \ln\frac{H_1}{K}\right)\right] \tag{B8}$$

where $b_n(\tau) = b_n\, e^{-(n\pi/L)^2\tau}$ and $L = \ln(H_2/H_1)$.

Because the function of the initial condition, eq. (B2), and its first derivative are piecewise continuous, it is represented by its Fourier sine series, where

$$b_n = \frac{2}{L}\int_{\ln(H_1/K)}^{\ln(H_2/K)} u_0(x, T' - t') \sin\left[\frac{n\pi}{L}\left(x - \ln\frac{H_1}{K}\right)\right] dx \tag{B9}$$

The front-end double-barrier knock-out option value is obtained by combining the above results in eqs. (B8) and (B9) and reinserting the transformations, eq. (A1), into the option value. The value is expressed in eq. (3).

APPENDIX C

The Pricing of Rear End Knock-out Options

Single Barrier Up-and-Out Call Option

The barrier H, of the rear-end up-and-out call option exists from time t' to option maturity T'. The value at time t' is an up-and-out call option value with maturity $T' - t'$. It is obtained by using the method of image to solve the diffusion equation and is expressed as

$$\begin{aligned} u_2(x, T' - t') = \frac{1}{\sqrt{2\pi}}\Bigg[&\int_{-x/\sqrt{2(T'-t')}}^{\infty} u_1(x + y\sqrt{2(T' - t')})\, e^{-(y^2/2)}\, dy \\ &- \int_{-[(x-h)/\sqrt{2(T'-t')}]}^{\infty} u_1(x + y\sqrt{2(T' - t')})\, e^{-(y^2/2)}\, dy \\ &- \int_{[(x-h)/\sqrt{2(T'-t')}]}^{\infty} u_1(-x + 2h + y\sqrt{2(T' - t')})\, e^{-(y^2/2)}\, dy \\ &+ \int_{[(x-h)/\sqrt{2(T'-t')}]}^{\infty} u_1(-x + 2h + y\sqrt{2(T' - t')})\, e^{-(y^2/2)}\, dy \Bigg], \quad -\infty < x \le h, \\ u_2(x, T' - t') = 0,\ h < x \end{aligned} \tag{C1}$$

where

$$u_1(x) = \max\left[e^{(1/2)(k_1+1)x} - e^{(1/2)(k_1-1)x}, 0\right], \qquad -\infty < x < h \tag{C2}$$

The expression in eq. (C1) is the initial condition of the diffusion equation in the time segment without the barrier. The problem in the time segment without the barrier is the same as heat diffusion in an unlimited solid with an initial temperature distribution, $u_2(x, T' - t')$. The well-known solution of the diffusion equation in an unlimited solid is

$$u(x, \tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_2(x + y\sqrt{2\tau})\, e^{-(y^2/2)}\, dy \tag{C3}$$

By re-inserting the transformations in eq. (A1), the solution of eq. (C3) is the rear-end up-and-out call option value in eq. (5). The pricing formulas of other rear-end knock-out options are as follows:

The rear-end down-and-out call value is

$$\begin{aligned} C_{2do} = {} & Se^{-qT}\, N_2[b'(t), a'(T), \rho] - Ke^{-rT}\, N_2[b(t), a(T), \rho] \\ & - \left(\frac{H}{S}\right)^{k_1+1} Se^{-qT}\, N_2[-c'(t), d'(T), -\rho] \\ & + \left(\frac{H}{S}\right)^{k_1-1} Ke^{-rT}\, N_2[-c(t), d(T), -\rho] \end{aligned} \tag{C4}$$

The rear-end up-and-out put value is

$$\begin{aligned} P_{2uo} = {} & Ke^{-rT}\, N_2[-b(t), -a(T), \rho] - Se^{-qT}\, N_2[-b'(t), \\ & -a'(T), \rho] - \left(\frac{H}{S}\right)^{k_1-1} Ke^{-rT}\, N_2[c(t), -d(T), -\rho] \\ & + \left(\frac{H}{S}\right)^{k_1-1} Se^{-qT}\, N_2[c'(t), -d'(T), -\rho] \end{aligned} \tag{C5}$$

The rear-end down-and-out put value is

$$\begin{aligned} P_{2do} = {} & Ke^{-rT}\, \{N_2[b(t), -a(T), -\rho] - N_2[b(t), -b(T), -\rho]\} \\ & - Se^{-qT}\, \{N_2[b'(t), -a'(T), -\rho] - N_2[b'(t), -b'(T), -\rho]\} \\ & - \left(\frac{H}{S}\right)^{k_1-1} Ke^{-rT}\, \{N_2[-c(t), -d(T), \rho] \\ & - N_2[-c(t), -c(T), \rho]\} + \left(\frac{H}{S}\right)^{k_1+1} Se^{-qT} \\ & \{N_2[-c'(t), -d'(T), \rho] - N_2[-c'(t), -c'(T), \rho]\} \end{aligned} \tag{C6}$$

Double-Barrier Knock-Out Call Option

The rear-end double-barrier knock-out put value (with $K < H_2$) is

$$P_r = \sum_{n=1}^{\infty} K\left(\frac{S}{K}\right)^{-(1/2)(k_1-1)} \left\{\frac{2}{L\left[\frac{1}{4}(k_1+1)^2 + \left(\frac{n\pi}{L}\right)^2\right]}\right.$$
$$\left[\frac{1}{2}(k_1+1)\sin\left(\frac{n\pi}{L}\ln\frac{H_1}{K}\right) + \frac{n\pi}{L}\cos\left(\frac{n\pi}{L}\ln\frac{H_1}{K}\right)\right.$$
$$\left. - \frac{n\pi}{L}\left(\frac{H_1}{K}\right)^{(1/2)(k_1+1)}\right] - \frac{2}{L\left[\frac{1}{4}(k_1-1)^2 + \left(\frac{n\pi}{L}\right)^2\right]}$$
$$\left[\frac{1}{2}(k_1-1)\sin\left(\frac{n\pi}{L}\ln\frac{H_1}{K}\right) + \frac{n\pi}{L}\cos\left(\frac{n\pi}{L}\ln\frac{H_1}{K}\right)\right.$$
$$\left.\left. - \frac{n\pi}{L}\left(\frac{H_1}{K}\right)^{(1/2)(k_1-1)}\right]\right\}\left[b_{1n}\sin\left(\frac{n\pi}{L}\ln\frac{S}{H_1}\right)\right.$$
$$\left. + b_{2n}\cos\left(\frac{n\pi}{L}\ln\frac{S}{H_1}\right)\right] e^{-(1/2)(n\pi/L)^2\,\sigma^2(T-t)+(1/2)\beta\sigma^2 T} \qquad \text{(C7)}$$

The rear-end double-barrier knock-out put value (with $K > H_2$) is

$$P_r = \sum_{n=1}^{\infty} K\left(\frac{S}{K}\right)^{-(1/2)(k_1-1)} \left\{\frac{2n\pi}{L^2\left[\frac{1}{4}(k_1+1)^2 + \left(\frac{n\pi}{L}\right)^2\right]}\right.$$
$$\left[(-1)^n\left(\frac{H_2}{K}\right)^{(1/2)(k_1+1)} - \left(\frac{H_1}{K}\right)^{(1/2)(k_1+1)}\right]$$
$$- \frac{2n\pi}{L^2\left[\frac{1}{4}(k_1-1)^2 + \left(\frac{n\pi}{L}\right)^2\right]}\left[(-1)^n\left(\frac{H_2}{K}\right)^{(1/2)(k_1-1)}\right.$$
$$\left.\left. - \left(\frac{H_1}{K}\right)^{(1/2)(k_1-1)}\right]\right\}\left[b_{1n}\sin\left(\frac{n\pi}{L}\ln\frac{S}{H_1}\right)\right.$$
$$\left. + b_{2n}\cos\left(\frac{n\pi}{L}\ln\frac{S}{H_1}\right)\right] e^{-(1/2)(n\pi/L)^2\sigma^2(T-t)+(1/2)\beta\sigma^2 T} \qquad \text{(C8)}$$

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