A Note on the Valuation of an Exotic Timing Option

MONDHER BELLALAH
JEAN-LUC PRIGENT

This note compares the valuation of a "lookback" put option with that of an option which, at payoff, gives its holder the difference between the maximum value recorded during the option's life and an initial value based on underlying asset price at the time of initiation. This latter instrument is called an exotic timing option.

Although the payoff on the exotic option corresponds to a lookback option, a major advantage is that, in contrast to the lookback option, the holder does not run the risk of zero value when the maximum value is achieved at maturity.

This note derives the value of an exotic option which gives its holder the difference between the maximum value of the underlying asset during the option's life and the initial asset value. Since the highest price of the asset during the option's life cannot be less than the initial price, this difference cannot be less than zero. It is shown that the value of this exotic timing option can be decomposed into two parts: a "lookback" put option and a forward contract on the underlying asset. The lookback option, which has a known solution, gives its holder the difference between

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■ *Mondher Bellalah is a Professor in the Department of Economics at the University of Cergy-Pontoise.*

■ *Jean-Luc Prigent is a Professor in the Department of Economics at the University of Cergy-Pontoise.*

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the maximum value of the underlying asset during the option's life and the asset value at option expiration. Similarly, this option cannot have a value less than zero. The lookback put then is like a conventional put with an exercise price equal to the maximum value of the underlying asset during the option's life. The key difference between the exotic timing and lookback put options is that the value of the maximum difference is based on initial asset price for the former and a terminal price for the latter.

The exotic timing option has some appeal to investors, especially those who believe that they have special skills at market timing, since the option is at-the-money when it is purchased and will finish in-the-money if the asset price moves higher any time during the life of the option.

The exotic timing option can be created as an independent instrument or can be embedded in bond issues or equity issues where the underlying assets may be as diverse as an index price, a share of stock, a currency, an interest rate, a commodity or any other tradable asset.

ANALYSIS AND VALUATION

The main difference between lookbacks and the exotic timing option can be simply illustrated using a 3-month option on the CAC 40 index. Assume the CAC 40 is at 1800 and over a subsequent 90-day period it reaches 2200. This latter value corresponds to the maximum. The net change is +400 points multiplied by the size of the contract, 200 FF, which gives a total gain of 80,000 FF. An investor who buys an exotic option contract will have 80,000 FF. Now, assume that the investor who believes he has superior market timing skills buys the lookback option on day 30 and also buys an exotic timing option on the same day when the index is at 1400. At maturity, the payoff from the lookback is zero. The payoff from the exotic timing option is given by the net change (+800 points) multiplied by the size of the contract, 200 FF, which gives a total gain of 160,000 FF.[1]

Assume 0 denotes the initial time; T denotes the contract's maturity date; and, τ denotes the present time. The remaining time to maturity is $t = T - \tau$. S represents the underlying asset price; r, the riskless interest rate; and σ, the instantaneous standard deviation.

The maximum value of the underlying asset in a given time interval is $M(\tau) = \max S(\delta)_{0 \le \delta \le \tau}$.

[1]Market timing and the choice of the date when to enter the contract are the principal characteristics of this derivative asset. These two features make the option particularly desirable. Hence, if investors want to improve the timing of their market entry.

When pricing this exotic timing option, the value of an asset paying the realized maximum over a given period must be determined. Then, the current value of the underlying asset is substructed from it. In a risk-neutral world, the option value is given by:[2]

$$C_{\max}(\tau) = e^{-r\tau}E_Q\left[(M(T) - S(T)) + (S(T) - S(0))|F_\tau\right] \qquad (1)$$

where the conditional expectation E_Q is taken with respect to the risk-neutral probability Q and the available information $F\tau$. The payoff corresponds to that of a lookback put initiated at time 0 and a forward contract.

The exotic timing option value is:

$$C_{\max}(\tau) = P(\tau) + [S(\tau) - e^{-r\tau}S(0)] \qquad (2)$$

where $P(\tau)$ is the lookback put given by eq. (14) in Conze and Viswanathan (1991). At $\tau = 0$, eq. (2) is rewritten as:[3]

$$C_{\max}(0) = S(0)(1 - e^{-rT}) - S(0)N(-d1) + e^{-rT}S(0)N$$
$$(-d1 + \sigma\sqrt{T}) + S(0)e^{-rT}(\sigma^2/2r)[e^{rT}N(d1) - N(d1 - (2r/\sigma)\sqrt{T})] \qquad (3)$$

where:

$$d1 = ((r + 1/2\sigma^2)/\sigma)\sqrt{T}$$

and $N(\cdot)$ is the cumulative normal density function.

SIMULATIONS AND OPTION CHARACTERISTICS

It is useful to compare the exotic timing option sensitivities with those of lookbacks and standard puts. For the sake of clarity, the exotic timing option is rewritten as:

$$C_{\max}(0) = P(0) + S(0)(1 - e^{-rT}),$$

This expression demonstrates that the exotic option is more expensive than lookback puts and standard puts with a strike price $S(0)$.

When the time to maturity tends to ∞, $C_{\max}(0) - P(0) = S(0)$. This result demonstrates that a long position in a perpetual exotic timing op-

[2]We thank the editor and the referee for suggesting this decomposition.

[3]Note that the traditional lookback strategies can be implemented using exotic timing options with a minor modification. Also, the exotic timing option can be valued in terms of binary options and down and out options.

TABLE I

Comparison of Exotic Timing Option Values, $C_{\max}(0)$ with Lookback Put Option Values, $P(0)$

$T = 0.25$	$\sigma = 0.2$		$r = 0.1$	$\sigma = 0.2$		$r = 0.1$	$T = 0.25$	
r	$C_{max}(0)$	$P(0)$	T	$C_{max}(0)$	$P(0)$	σ	$C_{max}(0)$	$P(0)$
0.10	8.49	6.02	0.25	8.49	6.02	0.20	8.49	6.02
0.15	8.82	5.13	0.40	10.98	7.058	0.25	10.54	8.071
0.20	9.29	4.41	0.50	12.48	7.607	0.30	12.68	10.211
0.25	9.89	3.83	0.75	15.88	8.654	0.35	14.88	12.411
0.30	10.59	3.36	1.00	18.98	9.463	0.40	17.06	14.591

tion and a short position in a perpetual lookback put is equivalent to a position in the initial underlying asset.

The option's delta is greater than that of a lookback put and a standard put:

$$\Delta_{C_{\max}(0)} = \Delta_{P(0)} + (1 - e^{-rT}).$$

The gammas of the option and the lookback put are nil in this context.

The option's derivative with respect to the volatility parameter is positive and equals that of a lookback put. It is given by:

$$\Lambda_{C_{\max}(0)} = (S(0)e^{-rT}\sigma/r)[e^{rT}N(d1) - (N(d1 - (2r/\sigma)\sqrt{T}))].$$

The option's derivative with respect to the interest rate parameter is greater than that of a lookback put:

$$\rho_{C_{\max}(0)} = \rho_{P(0)} + S(0)Te^{-rT}.$$

The option's derivative with respect to the time to maturity is positive and greater than that of a lookback put:

$$\Theta_{C_{\max}(0)} = \Theta_{P(0)} + rS(0)e^{-rT}.$$

For illustrative purposes, the formula (3) is used to simulate the exotic timing option values. When the underlying asset price, S, is 100, the results for different levels of r, σ, and T are reported in Table I. Simulations show that exotic option values are increasing functions of the underlying asset price, the interest rate, the volatility parameter, and time to maturity.

CONCLUSION

Using standard arbitrage arguments, this note provides a simple analytic formula for the valuation of an exotic timing option. This option has some specificities when compared to lookback options. The option price sensitivities are given and options prices are simulated. The well-known lookback strategies can be easily implemented using the exotic timing option.

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