Marking-to-Market and the Demand for Interest Rate Futures Contracts

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INTRODUCTION

The effect of marking-to-market on the pricing of derivative assets is well documented.[1] When interest rates are deterministic, the price of a futures contract is identical to that of a forward contract written on the same underlying asset and with the same maturity. However, when interest rates are stochastic, marking-to-market generally causes the settlement price of the futures contract to be different from the price of the forward contract.

An important issue, when hedging a non-traded position, is the effect of marking-to-market on the demand for futures contracts as compared to the demand for forward contracts. This question has practical relevance as well as theoretical interest.

Figlewski–Landskroner–Silber (1991) address this issue. In their framework, interest rates are deterministic and the derivative assets are written on the asset in the non-traded position. The investor wishes to

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Two anonymous referees provided numerous insightful comments which greatly improve the presentation of this article. I have also benefited from discussions with Rafael Eldor. Miryam Krausz provided excellent editorial assistance. I am solely responsible for any remaining error.
[1]See Duffie–Stanton (1992).

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minimize the risk of his portfolio which is composed of the non-traded position and the derivative asset (either the forward or the futures contract) position. The forward contract hedging position is equal in size to the non-traded position. The number of futures contracts held by the investor in his portfolio is equal in size to the product of the non-traded position multiplied by a tailing factor. The tailing factor is brought about by the marking-to-market of the futures position. At each time t, it is equal to the value of a discount bond which matures at the same time as the futures contract. Thus, while the position in forwards contracts is not time dependent, the futures position should be continuously rebalanced until the point in time when the hedge is lifted.

The purpose of this article is to address this issue when the underlying asset of the derivative contract is a discount bond. From a theoretical point of view, this issue is important since it is well known that if significant differences between forward and futures contracts prices exist, the most likely place to find them is in interest-rate sensitive asset-based contracts.[2] Figlewski–Landskroner–Silber (1991) show that a tailing factor is present even when the forward and the futures contract prices are identical. Moreover, this tailing factor is only a function of interest rates and is independent of the particular asset underlying the derivative contracts. What happens when the underlying asset is an interest risk sensitive asset? How does the difference between the two prices affect the tailing factor? These questions are addressed in this article. Note that from a practical point of view, this issue is also important since interest rate derivatives are widely used by financial institutions for hedging purposes [see Gorton–Rosen (1995)].

The discussion is extended by considering the effect of marking-to-market on an investor who wishes not only to minimize his risk but also to maximize his expected utility, i.e., includes a speculative component in his demand for derivatives (either forward or futures contracts). The straightforward solution calls for a comparison between the optimal futures strategy and the optimal forward strategy. However, if interest rates introduce a new source of uncertainty into the derivative assets' prices, then the derivative assets are no longer perfect substitutes for the underlying asset and the investor will not be able to reach the first best solution. Furthermore, since the investor will be trading on an incomplete derivative assets market, the optimum when trading futures contracts will be different from the optimum when trading forward contracts. In such

[2]See Flesaker (1993) for a derivation of the theoretical prices for interest rates futures and forward contracts. See Meulbroek (1992) for an empirical investigation of the difference between the two contracts prices.

cases, one would naturally expect a difference in demand for futures contracts and forward contracts. Thus, when isolating the effect of marking-to-market, more interesting results can be established by insuring that the investor reaches the same optimum when hedging with either of the two derivatives.

The framework used in this article includes an investor who has a non-traded position which is interest rate sensitive. He can use interest rate derivatives (i.e., forward or futures contracts) written on the asset in the non-traded position. Within this framework the investor who maximizes his expected utility reaches a first best solution when hedging either with forward or with futures contracts. It is shown that the tailing factor is not affected by the investor's decision of whether to minimize the risk of his portfolio or to maximize the expected utility from his terminal wealth.

THE FRAMEWORK

Agents trade continuously on a frictionless financial market until time τ which is the horizon of the economy. Two assets are traded on this market—a locally riskless asset and a risky asset. The risky asset is a zero coupon bond which pays \$1 at its maturity date, $T_D < \tau$. The locally riskless asset is a savings account which yields at each time t an instantaneous interest rate.

The price of the discount bond is

$$P(t, T_D) = \exp\left\{-\int_t^{T_D} f(t, T)\, dT\right\} \tag{1}$$

where $f(t,T)$ is the instantaneous forward rate at t for time, T. Thus, the spot rate, $r(t)$, is $r(t) = f(t,t)$ and the locally riskless asset is such that:

$$B(t) = \exp\left\{\int_0^t r(s)\, ds\right\} \tag{2}$$

To give the model some structure, the instantaneous forward rate is assumed to solve the following stochastic differential equation[3]

$$f(t,T) = f(0,T) + \int_0^t \mu(s,T)\, ds + \sigma W(t) \tag{3}$$

[3]The suggested model for the instantaneous forward rate allows this economic variable to take negative values. Unfortunately, this is one of the few polar cases in Financial Economics which allows for explicit results. For a discussion of this issue, see Subrahmanyam (1995).

where $f(0,T)$ is the instantaneous forward rate prevailing at 0, $\sigma > 0$ is the instantaneous volatility of the forward rate and $W(t)$ is a standard Brownian motion defined on a complete probability space, (Ω, F, Q), where Ω is the state space, F is the σ-algebra representing measurable events, and Q is the historical probability measure. The forward rate is adapted to the augmented filtration generated by the Brownian motion. This filtration is assumed to satisfy the usual conditions[4] and is denoted by $F = \{F_t\}_{t\in[0,\tau]} \cdot \mu(t,T)$ is the drift of the forward rate. It is assumed to satisfy conditions such that (3) has a unique solution.[5]

Each agent follows a portfolio strategy which consists of choosing the number of units of the locally riskless asset and of the discount bond contained in his portfolio. These strategies are assumed to be admissible strategies.[6]

There is no arbitrage opportunity on this financial market. This is equivalent[7] to the existence of a probability measure equivalent to Q with respect to a given numeraire such that the prices expressed in terms of the numeraire are martingales. Following Heath–Jarrow–Morton (1992),[8] when the numeraire is the riskless asset, this is also equivalent to the existence of a bounded stochastic process, $\phi(t)$, satisfying[9] $E^Q\,[\exp\{\int_0^{T_D}\phi^2(s)ds\}] < \infty$ and such that:

$$\mu(t,T) = \sigma^2\,(T - t) - \sigma\phi(t) \tag{4}$$

for $t \leq T_D$. This is the risk premium process associated with the instantaneous forward rate. The martingale measure, denoted Q_P, is constructed as follows:

$$\left.\frac{dQ_p}{dQ}\right|_{F_{T_D}} \equiv \eta(T_D) = \exp\left\{\int_0^{T_D} \phi(s)dW(s) - \frac{1}{2}\int_0^{T_D} \phi^2\,(s)ds\right\} \tag{5}$$

Q_P is unique since the market is complete.

[4]The σ-algebra contains the events whose probability with respect to Q is null. See Karatzas–Shreve (1991), p. 89.
[5]See conditions C.1 p. 80 and C.2 p. 81 of Heath–Jarrow–Morton (1992).
[6]To save space, the properties of the admissible strategies are not specified. See Harrison–Kreps (1979); Harrison–Pliska (1981); Cox–Huang (1989); and Heath–Jarrow–Morton (1992). The complete definition given by Cox–Huang (1989) is the one adopted here.
[7]In fact, this equivalence result holds only for the simple strategies, i.e., strategies which need a portfolio reallocation only a finite number of times [see Harrison–Kreps (1979) and Harrison–Pliska (1981)]. Property 5 of Cox–Huang's (1989) definition of admissible strategies excludes arbitrage opportunities for non-simple strategies as proved by Dybvig–Huang (1989).
[8]Proposition 3 p. 86 of Heath–Jarrow–Morton (1992).
[9]This is a technical constraint which ensures that the Radon-Derivative of the martingale measure with respect to the primitive probability is a martingale. When concerned only with the pricing of derivative assets, there is no need for this constraint. As the purpose is to solve for portfolio choice, there is need to give the risk premium process some structure. See Karatzas–Shreve (1991), p. 198.

Substituting the drift term (4) in (3), the forward rate is such that

$$f(t,T) = f(0,T) + \sigma^2 t\left(T - \frac{t}{2}\right) - \sigma \int_0^t \phi(s)\, ds + \sigma W(t) \qquad (6)$$

and the riskless rate is then

$$r(t) = f(0,t) + \sigma^2 \frac{t^2}{2} - \sigma \int_0^t \phi(s)\, ds + \sigma W(t) \qquad (7)$$

A straightforward calculation leads to

$$P(t,T_D) = \exp\left\{ - \int_t^{T_D} f(0,T)dT \right.$$

$$\left. + \sigma(T_D - t) \int_0^t \phi(s)\, ds - \frac{\sigma^2 t T_D}{2}(T_D - t) - \sigma(T_D - t)\, W(t) \right\} \qquad (8)$$

and, applying Ito's lemma,

$$dP(t,T_D) = P(t,T_D)[r(t) + b(t,T_D)]dt - \sigma(T_D - t)P(t,T_D)dW(t) \qquad (9)$$

where one need not specify $b(t,T_D)$ for the remainder of the analysis.

In the following section the investor's problems are addressed and their solutions are given.

THE INVESTOR'S PROBLEMS

The investor is endowed with a discount bond portfolio that he cannot trade until time, T_I, where T_I is the horizon of his investment. His non-traded position consists of $\pi > 0$ units of the discount bond traded in the financial market. He can use either a forward or a futures contract to hedge his non-traded position. The two contracts have the same maturity T_F, $T_I < T_F < T_D$. However, the futures position is assumed to be continuously marked-to-market while the forward position is not. Their arbitrage free prices are[10] as follows:

$$G(t) = \frac{P(t,T_D)}{P(t,T_F)} \qquad (10)$$

[10]The derivation is trivial for the forward contract price but not for the futures contract price. In the Appendix a simple proof of this formula is given. Relevant references include Duffie–Stanton (1992) (for continuous resettlement) and Flesaker (1993) (who assumes discrete marking to market).

$$H(t) = \frac{P(t,T_D)}{P(t,T_F)} \exp\left\{-\frac{\sigma^2}{2}(T_D - T_F)(T_F - t)^2\right\} \quad (11)$$

where $G(t)$ is the forward price and $H(t)$ is the futures contract settlement price.

The difference between the prices of the futures contract and the forward contract is clearly due to the marking-to-market effect which is captured by the second term on the right hand side of (11). The forward contract price is greater than the futures contract price. This result could have been expected since the underlying asset of the derivatives contracts and a discount bond maturing at the derivative assets' maturity date are perfectly correlated. Another interesting aspect of the two derivative assets is that their price processes are still perfectly correlated with the underlying asset price process.

The investor chooses a portfolio strategy, i.e., the number of units of the forward (futures) $\beta_G(t)$ ($\beta_H(t)$) and the number of units of the riskless asset $\alpha_G(t)$ ($\alpha_H(t)$) contained in his portfolio.

The portfolio strategies followed by the investor are assumed to be admissible. When he hedges his non-traded position using the forward contract, his wealth at each time t is

$$V_G(t) = \pi P(t, T_D) + \int_0^t \beta_G(s) P(s, T_F)\, dG(s) + \alpha_G(t) B(t) \quad (12)$$

This wealth has three components: the value at time t of the non-traded position, the present value at time t of the cumulated gains or losses from the forward position and the locally riskless asset position.[11]

If the futures contract is used for hedging purposes, then the margin account value at time t is

$$X(t) = \int_0^t \exp\left\{\int_s^t r(u)du\right\} \beta_H(s) dH(s) \quad (13)$$

The initial value of the margin account is implicitly assumed to be null.[12] The investor's wealth at each time t is then given by

$$V_H(t) = \pi P(t,T_D) + X(t) + \alpha_H(t) B(t) \quad (14)$$

[11]This wealth can be written in a cash balance form. Following Adler–Detemple (1988a,b), it is well known that the two formulations are equivalent.

[12]This is done without loss of generality. Furthermore, when necessary, one can easily take into account an initial amount placed by the investor in the account. The investor's wealth, when using forwards to hedge, should also be adjusted.

The investor maximizes his expected utility from terminal wealth and has a logarithmic[13] utility function such that

$$u(V(T_I,\omega)) = \text{Ln}\ (V(T_I,\omega)),\ \omega \in \Omega \tag{15}$$

The hedging problem is such that

$$\begin{cases} \max\limits_{\alpha,\beta} E^{Q}[\text{Ln}(V(T_1,\omega))] \\ \text{s.t.} \\ \text{The portfolio strategy is an admissible strategy} \\ \text{which satisfies either (12) or (14).} \end{cases}$$

THE OPTIMAL DYNAMIC HEDGING STRATEGIES

This section begins with the investor's forward contract strategy and then continues with his futures contract strategy. This order of presentation is necessary to show the effect of marking-to-market. The forward strategy followed by the investor is given as follows:

Proposition 1. *The investor's optimal dynamic forward contract strategy is*

$$\hat{\beta}_G\ (t) = \frac{\hat{V}_G\ (t)}{P\ (t,\ T_D)} \frac{\phi(t)}{\sigma(T_D - T_F)} - \frac{T_D - t}{T_D - T_F}\pi \tag{16}$$

where $\hat{V}_G(t)$ is the investor's wealth at time **t**.

The above forward strategy allows the investor to reach a first best solution to his hedging problem. This is due to the perfect correlation that exists between the forward price and the underlying asset price process.

The above given forward strategy is composed of the two usual components, namely, a speculative component [the first term on the RHS of (16)] and a minimum-variance hedging component [the second term on the RHS of (16)].[14] Note, that despite perfect correlation, the minimum-variance hedging component, which is a perfect hedge component, is not

[13]The assumption concerning the investor's utility function allows for a simple resolution of the portfolio choice problems assuming a general stochastic market price for risk for the instantaneous forward rate only. The Logarithmic function allows an explicit solution to the problem when there is a general stochastic market price for risk associated with the instantaneous forward rate. Nevertheless, extending the solution to a general utility function and accounting for an intertemporal consumption process is tractable but adds no intuition (when the utility of consumption function is time additive).

[14]See Adler–Detemple (1988b) for a justification of this decomposition.

equal in size to the non-traded position. This component is equal to the product of the non-traded position multiplied by an adjusting factor that accounts for the difference in instantaneous volatilities between the forward contract and its underlying asset. This factor is time dependent, thus, even when the investor seeks only to minimize the instantaneous volatility of his wealth, he must continuously rebalance his forward contract strategy. The source of this time dimension is the time dimension of the underlying asset's instantaneous volatility.

The adjusting factor has intuitive content which can be seen by rewriting it as follows:

$$\frac{T_D - t}{T_D - T_F} = 1 + \frac{T_F - t}{T_D - T_F} \tag{17}$$

The ratio on the LHS of (17) represents the minimum-variance hedge ratio of the instantaneous fluctuations of the price of the discount bond in the non-traded position, using the forward contract. The ratio on the RHS of (17) corresponds to the minimum-variance hedge ratio of the local fluctuations of the price of a discount bond which matures at the same time as the forward contract, when using the forward contract. This latter ratio can be interpreted as the minimum-variance hedge ratio of the local basis risk using the forward contract, where the local basis risk is defined as $dP(t,\tau_D)/P(t,\tau_D) - dG(t)/G(t)$. Hence, to hedge one unit of the non-traded position, the investor shorts one forward contract plus the amount of forwards needed to hedge against basis risk.

The futures contract demand is now derived as follows:

Proposition 2*The investor's optimal dynamic futures contract strategy is*

$$\hat{\beta}_H(t) = \frac{\hat{V}_H(t)}{H(t)} \frac{\phi(t)}{\sigma(T_D - T_F)} - \frac{T_D - t}{T_D - T_F} \frac{P(t,T_D)}{H(t)} \pi \tag{18}$$

where $\hat{V}_H(t)$ is the investor's wealth at time t.

The above futures strategy, as in the case of the forward strategy, allows the investor to reach a first best solution to his hedging problem. Thus, one can perform a meaningful comparison between the two strategies, (16) and (18), which differ only in the marking-to-market effect. This has been made possible by the particular nature of the non-traded position and the derivative assets, namely, that the underlying asset, the basis risk and the marking-to-market are all affected by the same source of uncertainty which drives the term structure.

Turning back to the futures strategy given in Proposition 2, one can see that it also has the usual decomposition into a speculative component and a minimum-variance hedging component. A comparison between the offsetting components in the two strategies will immediately reveal the effect of marking-to-market on the minimum-variance hedge ratio. Note that the offsetting component in (18) is equal to the non-traded position times two terms, one adjusting for the difference in volatilities and the second adjusting for the difference in prices. The first term is identical to the corresponding term in the forward contract hedging strategy (16). This does not come as a surprise since marking-to-market does not affects the instantaneous volatility of the futures contract which remains equal to the instantaneous volatility of the forward contract.

The adjustment which is brought about by differences in prices does not appear in the forward contract strategy in (16). However, as proved in the Appendix, the offsetting component in the case of the forward contract is as follows

$$\frac{T_D - t}{T_D - T_F} = \frac{P(t, T_D)}{P(t, T_F)G(t)} \frac{T_D - t}{T_D - T_F} \tag{19}$$

Hence, even the demand for forward contracts requires an adjustment for the difference in prices. However, while for the futures contract case the adjustment is performed by means of the ratio between the underlying asset price and the futures contract price, in the case of the forward contract the adjustment factor appears in the form of a ratio between the underlying asset price and the present value of the forward contract price. Thus, the tailing factor is derived from this adjustment term and is as follows:

$$TF(t) = P(t,T_F)\exp\left\{\frac{\sigma^2}{2}(T_D - T_F)(T_F - t)^2\right\} \tag{20}$$

The fact that this adjusting factor is affected by the difference in prices is not surprising. The first term on the right-hand side of (20) is identical to the one found by Figlewski–Landskroner–Silber (1991) when interest rates are deterministic.

An interesting extension of these findings is to determine whether the tailing factor remains independent of the investor's characteristics when the investor hedges so as to maximize his expected utility and not only when he minimizes his portfolio's risk. The tailing factor in such a case is shown in the following proposition:

$$\hat{\beta}_H(t) = P(t,T_F)\exp\left\{\frac{\sigma^2}{2}(T_D - T_F)(T_F - t)^2\right\}\hat{\beta}_G(t) \tag{21}$$

The tailing factor in (21) is identical to the tailing factor in (20). Hence, the decision of the investor of whether to minimize the instantaneous volatility of his wealth or to maximize the expected utility from his terminal wealth, does not affect the tailing factor which remains unchanged.

CONCLUDING REMARKS

This paper derives the investor's forward contract and futures contract dynamic hedging strategy when maximizing his expected utility in a stochastic interest rate environment. It suggests an explicit expression for the tailing factor. The investor's characteristics have no effect on the tailing factor which depends only upon the characteristics of the futures contract and its underlying asset.

If the term structure is driven by two or more sources of uncertainty then markets would be incomplete and the findings would be altered. Such an issue must be addressed since it includes the case where the volatility of the instantaneous forward rate is stochastic.

APPENDIX

Computation of the Futures Price

Following Duffie–Stanton (1992), the futures price at time t is equal to the current price of an asset which yields a cash flow at T_F of $P(T_F,T_D)$ $B(T_F)/B(t)$. Thus, the price of the futures contract, denoted $H(t)$, is

$$\frac{H(t)}{B(t)} = E^{Q_P}\left[\frac{1}{B(T_F)}P(T_F,T_D)\frac{B(T_F)}{B(t)}\middle|F_t\right] \tag{a-1}$$

and then

$$H(t) = E^{Q_P}[P(T_F,T_D)|F_t] \tag{a-2}$$

Using (8),

$$H(t) = \exp\left\{- \int_{T_F}^{T_D} f(0,T)\, dT - \frac{\sigma^2 T_F T_D}{2} (T_D - T_F) + \sigma(T_D - T_F) \int_0^t \phi(s)ds - \sigma(T_D - T_F)W(t)\right\} \times E^{Q_P} \left[\exp\left\{\sigma(T_D - T_F) \int_t^{T_F} (dW(s) - \phi(s)ds)\right\} |F_t\right]$$

The first term on the right-hand side can be shown to be equal to

$$\frac{P(t,T_D)}{P(t,T_F)} \exp\left\{- \frac{\sigma^2}{2} (T_D - T_F)(T_F - t)(T_D - t)\right\} \tag{a-3}$$

Girsanov's theorem shows that $dW(s) - \phi(s)ds$ is, in fact, a Brownian motion with respect to the martingale measure. The Laplace transformation gives the following expression:

$$E^{Q_P}\left[\exp\left\{\sigma(T_D - T_F) \int_t^{T_F} (dW(s) - \phi(s)ds)\right\} |F_t\right] = \exp\left\{\frac{\sigma^2}{2} (T_D - T_F)^2 (T_F - t)\right\} \tag{a-4}$$

and the desired result follows.

Proof of Proposition 1. As the market is complete, the martingale approach[15] is used to solve this problem. This is a continuation of the methodology used for the derivative assets. One need not construct a martingale measure on the investor's market (derivative asset and riskless asset) since markets are complete and, thus, the pricing function will be the martingale measure (5).

The investor's program becomes:

$$\begin{cases} \max\limits_{V_G(T_I)} E^{Q}[\mathrm{Ln}(V_G(T_I))] \\ \text{s.t.} \quad E^{Q_P}\left[\dfrac{V_G(T_I)}{B(T_I)}\right] = \pi P(0,T_D) \end{cases}$$

The non-negativity constraint on wealth is ignored since it is not binding for the logarithmic investor. Using Cox–Huang (1991) one can easily check that this program has a unique solution.

[15]For an excellent account see Karatzas (1989) and Duffie (1992). See also the seminal articles by Karatzas–Lehoszky–Shreve (1987) and Cox–Huang (1989), (1991).

It follows from Lagrangian theory and (5) that the solution to the preceding program is such that

$$\frac{1}{\hat{V}_G(T_I)} - \lambda \frac{1}{B(T_I)} \eta(T_I) = 0$$

where the following fact is used: $E^{Q_P}[X(T_I)] = E^{Q}[X(T_I)\eta T_I)]$.

Thus,

$$\hat{V}_G(T_I) = \frac{1}{\lambda} B(T_I)\, \eta^{-1}(T_I) \tag{a-5}$$

where $\lambda > 0$ is the Lagrangian multiplier associated with the budget constraint. Using this constraint leads to

$$\hat{V}_G(T_I) = \pi P(0,T_D) B(T_I) \eta^{-1}(T_I) \tag{a-6}$$

Now, the investor's wealth at each time t is such that

$$\frac{\hat{V}_G(t)}{B(t)} = E^{Q_P}\left[\frac{\hat{V}_G(T_I)}{B(T_I)} \,\middle|\, F_t\right] \tag{a-7}$$

Using Cox–Huang (1989), Lemma 2.5 p. 43 (Bayes rule),

$$\frac{\hat{V}_G(t)}{B(t)} = \eta^{-1}(t)\, E^{Q}\left[\frac{\hat{V}_G(T)_I}{B(T_I)} \eta(T_I) \,\middle|\, \mathbf{F}_t\right] \tag{a-8}$$

and then, using (a-6),

$$\hat{V}_G(t) = \pi P(0,T_D) B(t) \eta^{-1}(t) \tag{a-9}$$

Applying Ito's lemma,

$$d\hat{V}_G(t) = (.)dt - \hat{V}_G(t)\phi(t)dW(t) \tag{a-10}$$

This wealth is generated by an admissible strategy, thus,

$$d\hat{V}_G(t) = \pi dP(t, T_D) + \hat{\beta}_G(t) P(t, T_F) dG(t) + \hat{\alpha}_G(t) dB(t) \tag{a-11}$$

Applying Ito's lemma to (10) yields

$$dG(t) = (.)dt - \sigma(T_D - T_F) G(t) dW(t) \tag{a-12}$$

and, thus,

$$d\hat{V}_G(t) = (\ )dt - \{\pi P(t,T_D)\sigma(T_D - t)$$

$$+ \hat{\beta}_G(t)P(t, T_F)G(t)\sigma(T_D - T_F)\}\, dW(t) \qquad \text{(a-13)}$$

Equating the diffusion coefficients in (a-10) and (a-13) yields the desired result.

Proof of Proposition 2. When allowed to trade futures contracts, the investor's optimal wealth solves

$$\begin{cases} \max\limits_{V_H(T_I)} E^Q[\mathrm{Ln}(V_H(T_I))] \\ \text{s.t.} \quad E^{Q_P}\left[\dfrac{V_H(T_I)}{B(T_I)}\right] = \pi P(0,T_D) \end{cases}$$

Hence, the investor's terminal wealth is given by (a-5) and its dynamic by (a-10). It is also generated by an admissible strategy, thus,

$$dV_H(t) = \pi dP(t,T_D) + dX(t) + \alpha_H(t)dB(t) \qquad \text{(a-14)}$$

Applying Ito's lemma to (13) and (11) yields

$$dX(t) = r(t)X(t)dt + \hat{\beta}_H(t)dH(t) \qquad \text{(a-15)}$$

$$dH(t) = (.)dt - \sigma(T_D - T_F)\, H(t)dW(t) \qquad \text{(a-16)}$$

and, thus,

$$dX(t) = (\ )dt - \sigma(T_D - T_F)\hat{\beta}_H(t)H(t)dW(t) \qquad \text{(a-17)}$$

Using (9) and (a-17), (a-14) becomes

$$d\hat{V}_H(t) = (\ )dt - \{\sigma(T_D - t)\pi P(t,T_D)$$

$$+ \sigma(T_D - T_F)\, \hat{\beta}_H(t)H(t)\}\, dW(t) \qquad \text{(a-18)}$$

Now, equating the two diffusion coefficients in (a-10) and (a-18) yields the desired result.

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