The Effect of the Cointegration Relationship on Futures Hedging: A Note

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In many empirical studies, both spot and futures prices were shown to contain a stochastic trend. Consequently, it is necessary to examine the possible cointegration relationship between the two prices as suggested by the efficient markets hypothesis. The importance of incorporating the cointegration relationship into statistical modeling of spot and futures prices is well documented in the futures market literature. Hedge ratios and hedging performance may change sharply when the cointegrating variable is mistakenly omitted from the statistical model. In Lien and Luo (1994) it was shown that, although GARCH (Generalized Autoregressive Conditional Heteroskedasticity) may characterize the price behavior, the cointegration relationship is the only truly indispensable component when comparing ex post performance of various hedge strategies. Through empirical calculation, Ghosh (1993) concluded that a smaller than optimal futures position is undertaken when the cointegration relationship is unduely ignored. He attributed the underhedge result to model misspecification. This note provides a theoretical analysis to con-

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firm the above conjecture and further explore the impact of the cointegration relationship on futures hedging.

A SIMPLE MODEL

Consider a spot security and a futures contract traded on the basis of the spot security. Let P_t and f_t denote spot and futures prices at time, t, respectively. Because the futures contract is priced off the spot security, $f_t - P_t = 0$ characterizes the long-run equilibrium relationship, that is, the cointegration relationship, between the two prices.[1] Let the cointegrating variable be denoted by $\theta_t = f_t - P_t$. Then, at any given time, θ_t is expected to differ from zero which constructs a "disequilibrium" state. Spot and futures prices generally adjust in response to the disequilibrium. For illustration, consider the following model:

$$\Delta P_t = a\theta_{t-1} + \varepsilon_{1t} \tag{1}$$

$$\Delta f_t = -\beta\theta_{t-1} + \varepsilon_{2t} \tag{2}$$

where Δ is the difference operator, ε_{1t} and ε_{2t} are joint white noise, possibly contemporarily correlated.

A bona fide hedger chooses the hedge ratio to minimize the overall risk, $\mathrm{Var}(P_t - h\Delta f_t)$, based upon the statistical model envisaged and information available, where h is the hedge ratio and Var $(\cdot)$ denotes the variance operator. When the hedger proceeds with a correctly specified model, the minimum risk hedge ratio is:

$$h^* = \mathrm{Cov}(\Delta f_t, \Delta P_t|\theta_{t-1})/\mathrm{Var}(\Delta f_t|\theta_{t-1}) = \rho(\sigma_1/\sigma_2) \tag{3}$$

where σ_i^2 is the variance of ε_{it}, $i = 1, 2$; ρ is the correlation coefficient between ε_{1t} and ε_{2t}. Alternatively, h^* can be derived by regressing ΔP_t on Δf_t and θ_{t-1}. The coefficient for Δf_t is equal to h^* [Myers and Thompson (1989); Viswanath (1993)].

An errant hedger, ignorant of the cointegration relationship, considers both ΔP_t and Δf_t to be white noise. As a consequence, θ_{t-1} is of no use in deriving the minimum risk hedge ratio. Let h° denote the hedge ratio chosen by the errant hedger. Then,

$$h^\circ = \mathrm{Cov}(\Delta f_t, \Delta P_t)/\mathrm{Var}(\Delta f_t) \tag{4}$$

Equation (4) is similar to eq. (3) except that unconditional expectations

[1] If the cointegrating variable is $f_t - \delta P_t$ for some $\delta \neq 1$, an appropriate change in the measurement unit of the spot security leaves the analysis of this note intact.

replace conditional expectations. From eqs. (1) and (2), one derives the following:

$$\begin{aligned}\mathrm{Cov}(\Delta f_t, \Delta P_t) &= \mathrm{Cov}(-\beta\theta_{t-1} + \varepsilon_{2t}, a\theta_{t-1} + \varepsilon_{1t}) \\ &= -a\beta\sigma_\theta^2 + \rho\sigma_1\sigma_2 \qquad (5)\end{aligned}$$

$$\mathrm{Var}(\Delta f_t) = \mathrm{Var}(-\beta\theta_{t-1} + \varepsilon_{2t}) = \beta^2\,\sigma_\theta^2 + \sigma_2^2 \qquad (6)$$

where $\sigma_\theta^2 = \mathrm{Var}\,(\theta_{t-1})$. Consequently,

$$h^\circ = (-a\beta\sigma_\theta^2 + \rho\sigma_1\,\sigma_2)/(\beta^2\sigma_\theta^2 + \sigma_2^2) \qquad (7)$$

is the minimum risk hedge ratio for the errant hedger. If one regresses ΔP_t on Δf_t, then h° corresponds to the coefficient of Δf_t.

COMPARISON OF HEDGE RATIOS

From eqs. (3) and (7), the difference between the two hedge ratios is

$$h^\circ - h^* = -\beta\sigma_\theta^2(a + \beta h^*)\,(\beta^2\sigma_\theta^2 + \sigma_2^2)^{-1} \qquad (8)$$

when $\beta = 0$, $h^\circ = h^*$. That is, if the futures price does not adjust to the deviation from the long-run equilibrium relationship, the errant hedger obtains the correct minimum risk hedge ratio in spite of model misspecification. On the other hand, if $h^* > 0$, which is generally expected, then $h^\circ < h^*$ whenever $a = 0$ and $\beta \neq 0$. Thus, the errant hedger underhedges due to omitting the effect of the cointegrating variable on spot price behavior.

In general, $h^\circ \leq h^*$, if and only if $\beta(a + \beta h^*) \geq 0$. Under the most likely circumstance, spot and futures prices will both respond to the deviation from the long-run equilibrium relationship. In the presence of a positive deviation, either the futures price is too high or the spot price is too low. To correct, the spot price will increase while the futures price will decrease. Similarly, a negative deviation induces the spot price to decrease and the futures price to increase. Both cases indicate that both a and β should be positive. Given that h^* is generally positive, eq. (8) implies $h^\circ \leq h^*$. In other words, the errant hedger who mistakenly omits the cointegration relationship in modeling spot and futures price behavior always undertakes a smaller than optimal futures position. That is the empirical result observed by Ghosh (1993).

Note that $h^\circ \neq h^*$ only if $\beta \neq 0$. The latter condition implies that the futures market is not "informationally" efficient. In fact, Granger

(1986) argued that two price series, each generated by an efficient market, cannot be cointegrated. It is also well recognized that a market may be deemed efficient based upon its own price history but appear to be inefficient when price information from other related markets is incorporated. While there is empirical support for weak-form efficiency in some markets, the methodology is generally restricted to the use of the market's own price history. Thus, the findings of weak-form efficiency and cointegration are not necessarily contradictory. Information inefficiency does not ensure profitable opportunities since transaction costs need to be taken into account.

Given that a cointegration relationship between spot and futures prices is frequently found (for example, in foreign currency markets), the efficiency issue is reduced to the examination of the "simple efficiency hypothesis" or the so-called "unbiasedness hypothesis." Earlier research tested the hypothesis using a simple linear regression model where spot and futures prices are the dependent and independent variables, respectively. This approach is subject to "spurious regression bias" because either price series often contains a unit root. The test results are, therefore, unreliable. Antoniou and Foster (1994) provided a detailed review of recent test methods relying upon cointegration regressions. Brenner and Kroner (1995) compared various tests showing that the unbiasedness hypothesis is upheld when $a = 0$ and $\beta > 0$. They summarized several empirical studies that demonstrate that the two conditions are frequently met.

THE EFFECT OF THE COINTEGRATION RELATIONSHIP

Because h° is smaller than h^{*}, the farther h° is away from h^{*} the poorer is the errant hedger's hedging performance. To evaluate the effect of the cointegration relationship, one may examine the impacts of a and β on $h^{\circ} - h^{*}$. To proceed, $\mathrm{Var}(\theta_{t-1})$ is derived as follows. By subtracting eq. (1) from eq. (2), one obtains

$$\theta_t = (1 - a - \beta)\, \theta_{t-1} + (\varepsilon_{2t} - \varepsilon_{1t}) \tag{9}$$

Because $\{\theta_t\}$ is a stationary time series, it follows that

$$\mathrm{Var}(\theta_t) = \mathrm{Var}(\theta_{t-1}) = \sigma^2/(1 - k^2) \tag{10}$$

where $\sigma^2 = \mathrm{Var}(\varepsilon_{2t} - \varepsilon_{1t}) = \sigma_2^2 - 2\rho\sigma_1\sigma_2 + \sigma_1^2$ and $k = 1 - a - \beta$. If one substitutes eq. (10) into eq. (8) and takes the partial derivative of $(h^{\circ} - h^{*})$ with respect to a and β, the following results are available:

$$\partial(h^\circ - h^*)/\partial a = -M^2\{\beta^3\sigma^2 + \sigma_2^2\beta[1 - k^2 - 2k(a + \beta h^*)]\} \quad (11)$$

$$\partial(h^\circ - h^*)/\partial\beta = M^2\{a\beta^2\sigma^2 - \sigma_2^2[(1 - k^2)(a + 2\beta h^*) - 2k\beta(a + \beta h^*)]\} \quad (12)$$

where $M = \sigma[\beta^2\sigma^2 + \sigma_2^2(1-k^2)]^{-1}$

In the case that $a = 0$, eq. (11) can be rewritten as:

$$\partial(h^\circ - h^*)/\partial a = -M^2[\beta^3\sigma^2 + \sigma_2^2\beta(2\beta(1 - \beta)(1 - h^*) + \beta^2)] \quad (13)$$

which is negative provided $0 < \beta, h \leq 1$. An increasing a leads to a smaller futures position relative to the true optimal level. In other words, when a is small, the cost for the errant hedger is greater (in terms of hedging effectiveness) the more responsive the spot price is to the deviation from the long-run equilibrium relationship. Similarly, when $\beta = 0$, eq. (12) becomes

$$\partial(h^\circ - h^*)/\partial\beta = -aM^2\sigma_2^2(1 - k^2) \quad (14)$$

which is negative because $-1 < k < 1$ is required to ensure the stationarity of $\{\theta_t\}$. Once again, when β is small, the cost for the errant hedger is greater if the futures price is more responsive to the disequilibrium. Both cases demonstrate that the more influential the cointegration relationship is (in terms of the magnitude of a and β), the more costly it is for a hedger to omit the cointegrating variable in statistical modeling of spot and futures prices.

More generally, when $a, \beta \geq 0$ and $a + \beta < 1$, $a\beta^2$ can be ignored.[2] Provided $a + \beta > 2 - 2^{1/2}$, $1 - k^2 - 2k$ is positive. As a consequence, $\partial(h^\circ - h^*)/\partial\beta < 0$. A similar consideration leads to $\partial(h^\circ - h^*)/\partial a < 0$ as long as $a + \beta$ is sufficiently large. Thus, when spot and futures prices altogether respond actively to the deviation from the long-run equilibrium relationship, the more important the cointegration relationship is, the more losses in hedge performance the errant hedger bears.

The above results concern the minimum risk (variance) hedge ratio. This hedge ratio corresponds to the hedge strategy of a bona fide hedger who cares only for reduction in risk. It is also consistent with the "necessity test" required by the Commodity Futures Trading Commissions when a new contract is initiated. In contrast, if the hedger has a mean-variance expected utility function, the resultant optimal futures position will consist of a speculative component and a hedging component. The

[2]These two properties, $a, \beta \geq 0$ and $a + \beta < 1$, are observed in most foreign exchange markets [Kroner and Sultan (1993); Lien and Luo (1993, 1994)].

latter coincides with the minimum risk hedge ratio. Lence (1995) provided a necessary and sufficient condition for the minimum risk hedge strategy to be optimal when there is no a priori knowledge regarding the utility functional form except that it is strictly concave. This condition is not met in the current context. The issue is then reduced to the legitimacy of the mean-variance approach. Justification for this approach is well documented in the literature. A brief summary is provided in Lien (1992).

Finally, in the context of a mean-variance approach, when the hedger chooses a smaller than optimal futures position (for the hedging component), he also forgoes any possible speculative opportunities. The speculative component may be on either the long or short side; however, on average, they are expected to cancel each other out.

THE GENERAL CASE

Engle and Granger (1987) provided a general statistical model for spot and futures prices in the presence of the cointegration relationship:

$$\Delta P_t = a\theta_{t-1} + \sum_{i=1}^{m} a_i \Delta P_{t-i} + \sum_{j=1}^{n} b_j \Delta f_{t-j} + \varepsilon_{1t} \tag{15}$$

$$\Delta f_t = -\beta\theta_{t-1} + \sum_{i=1}^{r} c_i \Delta P_{t-i} + \sum_{j=1}^{s} d_j \Delta f_{t-j} + \varepsilon_2 t \tag{16}$$

A hedger proceeding with the correctly specified model chooses the minimum risk hedge ratio as follows:

$$\begin{aligned} h^* &= \mathrm{Cov}\,(\Delta f_t, \Delta P_t | \Delta f_{t-j}, \Delta P_{t-i}, \theta_{t-1}) / \mathrm{Var}(\Delta f_t | \Delta f_{t-j}, \Delta P_{t-i}, \theta_{t-1}) \\ &= \mathrm{Cov}(\varepsilon_{1t}, \varepsilon_{2t}) / \mathrm{Var}(\varepsilon_{2t}) = \rho(\sigma_1/\sigma_2) \end{aligned} \tag{17}$$

In contrast, an errant hedger ignorant of the cointegration relationship concludes with the following minimum risk hedge ratio:

$$\begin{aligned} h^\circ &= \mathrm{Cov}(\Delta f_t, \Delta P_t | \Delta F_{t-j}, \Delta P_{t-i}) / \mathrm{Var}(\Delta f_t | \Delta f_{t-j}, \Delta P_{t-i}) = (\rho\sigma_1\sigma_2 \\ &\quad - a\beta \mathrm{Var}(\theta_{t-1} | \Delta f_{t-j}, \Delta P_{t-i})) / (\sigma_2^2 + \beta^2 \mathrm{Var}(\theta_{t-1} | \Delta f_{t-j}, \Delta P_{t-i})) \end{aligned} \tag{18}$$

When comparing h° to h^*, some properties established for the simple model can be extended to the more general case: (1) $h^\circ < h^*$ when a, $\beta > 0$; (2) $h^\circ = h^*$ when $\beta = 0$; (3) $h^\circ < h^*$ when $a = 0$. Thus, the errant hedger will undertake a smaller than optimal futures position and incur losses in hedging effectiveness therefrom. Nonetheless, the impacts of changes in α (or β) on hedging effectiveness are indeterminate for the conditional variance of θ_{t-1}, Var(θ_{t-1} |ΔP_{t-i}, Δf_{t-j}) is a function of all underlying parameters.

CONCLUSIONS

Suppose that spot and futures prices are cointegrated. This note shows that a hedger who omits the cointegration relationship will adopt a smaller than optimal futures position, which results in a relatively poor hedge performance. A simple statistical model also suggests that the cost for the errant hedger is greater when spot and futures prices are more responsive to the cointegration relationship. Whether this conclusion applies to some more complicated models is a topic to be addressed in future research.

The conclusions derived in this note also apply to the case that hedger misspecifies spot and futures price behavior by modeling a partial cointegration system, instead of the complete cointegration system. For example, assume there are two spot securities and two corresponding futures contracts. There may be in total of three cointegrating variables. By modeling each pair of markets separately, the hedger will likely misspecify the price behavior resulting in a smaller than optimal futures position provided certain conditions on the coefficients of cointegrating variables are satisfied. However, in this case the conditions are rather imposing.

Finally, the possibility of GARCH effects has no affect on the "underhedge" conclusions. They merely lead to time-varying second-order moments of (ε_{1t}, ε_{2t}), and, thus, time-varying minimum risk hedge ratios. The analysis in this note can be carried out by fixing the time frame. In other words, while minimum risk hedge ratios are time-varying, at any given time, an errant hedger always takes a smaller than optimal futures position.

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