
A NOTE ON MODIFIED LATTICE APPROACHES TO OPTION PRICING

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INTRODUCTION

Various numerical procedures have been developed for valuing options. These include Monte Carlo simulation, lattice procedures, and finite difference methods. These procedures provide solutions that are asymptotically equivalent to those provided by closed-form solutions, and they are also capable of handling complex cases where there is no closed-form solution.

In a recent issue of *The Journal of Futures Markets*, Tian (1993) investigated the numerical efficiency of various lattice models. Numerical efficiency was measured as the minimum number of steps required to achieve a given level of model accuracy. For example, if an approximation error of less than 5¢ was required, and if it was found that Model 1 (Model 2) required 20 (30) steps to achieve that level of accuracy, then Model 1 was deemed to be more efficient than Model 2.

Tian found, with respect to call and American put options, that his alternative binomial model has greater numerical efficiency than the Cox, Ross, and Rubinstein (CRR) (1979) model. Conversely, he found, with respect to European put options, that the CRR model has greater efficiency than his alternative model. However, no theoretical argument was presented as to why the two models might provide different levels of

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numerical efficiency. Further, numerical efficiency was examined by using parameter values corresponding to only three different option values.

The purpose of this note is to document, both analytically and by using a wider range of parameter values, that Tian's results are not robust. No evidence is found to suggest that Tian's model is more numerically efficient than the well-known CRR model. Tian also examined three trinomial models. His conclusions regarding these models were found to be robust across a range of parameter values, and therefore they are not considered in this article.

BINOMIAL LATTICE MODELS AND NUMERICAL EFFICIENCY

Both of the binomial lattice models examined by Tian assume that the stock price follows, in a risk-neutral world, the following stochastic process:

$$\frac{dS_t}{S_t} = rdt + \sigma dz \quad (1)$$

where r is the constant risk-free rate of interest, and σ is the standard deviation of the underlying stock return. A logarithmic transformation simplifies the above process to:

$$d \log S_t = (r - \sigma^2/2) dt + \sigma dz \quad (2)$$

A binomial approximation for this stochastic process may be developed by assuming that during a short time interval, Δt , stock prices move from an initial value, S , to one of two new values, Su or Sd . The probability of moving to Su is assumed to be p , so that the probability of moving to Sd is $1 - p$. These parameters uniquely determine the evolution of stock prices, which, in turn, determine a unique value of the option on the stock.

The parameters p , u , and d cannot be chosen arbitrarily. They must give correct values according to the continuous-time process for the mean and variance of the change in the stock price during the time interval, Δt . This imposes the following two conditions on p , u , and d :

$$pu + (1 - p)d = e^{r\Delta t} \quad (3)$$

and

$$pu^2 + (1 - p)d^2 = e^{(2r + \sigma^2)\Delta t} \quad (4)$$

For reasons of modeling simplicity, CRR (1979) also imposed the following condition:

$$u = 1/d \quad (5)$$

Given these conditions, it may be shown that as Δt approaches zero:

$$p = (a - d)/(u - d) \quad (6)$$

$$u = e^{\sigma\sqrt{\Delta t}} \quad (7)$$

and

$$d = e^{-\sigma\sqrt{\Delta t}} \quad (8)$$

where $a = e^{r\Delta t}$. Equations (6), (7), and (8) constitute Model C in this article. It should be noted that Model C only satisfies eq. (4) in the limit as the time interval approaches zero.

Instead of imposing condition (5), Tian imposed the condition that the third moment of the change in the stock price during the time interval, Δt , is also correct according to the continuous-time process. This requires that:

$$pu^3 + (1 - p)d^3 = a^3V^3 \quad (9)$$

where $V = e^{\sigma^2\Delta t}$. Given these conditions, it may be shown that eq. (6) holds and that:

$$u = (aV/2)[(V + 1) + \sqrt{V^2 + 2V - 3}] \quad (10)$$

and

$$d = (aV/2)[(V + 1) - \sqrt{V^2 + 2V - 3}] \quad (11)$$

Equations (6), (10), and (11) constitute Model T in this article.

Tian (p. 565) suggests that since the binomial distribution is skewed, ensuring that the third moment of the discrete-time process is correct according to the continuous-time process, it "might be more sensible and result in a more accurate binomial procedure." But given the constraint that stock price changes are uncorrelated, the central limit theorem ensures that the final distribution is log normal irrespective of the third moment of the discrete-time process. Therefore, it is not clear why the

imposition of this condition might be more sensible, nor why it might result in a more accurate procedure.

To investigate the numerical efficiency of the lattice procedures, and to enable comparisons to be made with Tian's analysis, the same evaluation criterion is used as that employed in his article. Tian used the minimum convergence step, N_ϵ , as the measure of the efficiency of a model, where N_ϵ was defined as:

$$N_\epsilon = \text{Min}[N: |(x(n) - x^*)/x^*| < \epsilon, \text{ for all } n \geq N] \quad (12)$$

where $\chi(n)$ is the price obtained from an n -step approximation procedure, ϵ is the precision level, and x^* is the accurate price of the option. This definition of efficiency emphasizes the importance of the stability of convergence. It identifies the minimum number of steps that are needed to ensure that the relative approximation error is less than the required precision level for all subsequent steps.¹

COMPARISON OF THE MODELS

Tian concluded that Model T is more numerically efficient than Model C for call options and American put options, while Model C is more numerically efficient than Model T for European put options. The European put-call parity relationship may be used to demonstrate analytically that, for at-the-money options, i.e., where the stock price is equal to the present value of the exercise price, this finding is incorrect. European put-call parity holds for prices found using the Black-Scholes model and for prices found using Models C and T. That is:

$$P^* + S = C^* + Xe^{-rt} \quad (13)$$

and

$$P(n) + S = C(n) + Xe^{-rt} \quad (14)$$

where, in addition to those variables previously defined, P^* and C^* are, respectively, the put and call option prices found using the Black-Scholes model; $P(n)$ and $C(n)$ are, respectively, the put and call option prices found using a numerical procedure with n steps; and X is the exercise price. Therefore, the absolute pricing errors, $P(n) - P^*$ and $C(n) - C^*$,

¹This concept may be illustrated by considering two sequences: $(a_n = 1 - 1/2^n)$ and $(b_n = 1 - 1/n)$. Both sequences converge to 1 but the sequence (a_n) converges to 1 faster than the sequence (b_n) . At the 0.05 precision level, the minimum convergence step for a_n is 5, while the minimum convergence step for b_n is 21.

generated by Models C and T must be identical for European put and call options. For at-the-money options, $P^* = C^*$ and, therefore, the relative pricing errors, $(P(n) - P^*)/P^*$ and $(C(n) - C^*)/C^*$, must also be identical. Therefore, from eq. (12) it may be seen that for at-the-money options, Models C and T must yield the same minimum convergence step when applied to European put options as it does when applied to call options. Therefore, Model C cannot be superior to Model T when applied to European put options, *and* inferior to Model T when applied to call options.

Tian examined the numerical efficiency of the two models using a very limited range of option parameter values. Specifically, the current stock price was set equal to \$100, the time to maturity equal to 4 months, the risk-free rate of interest equal to 5% per annum, and the standard deviation of the underlying stock return equal to 30% per annum. Three values of the exercise price were used, namely, \$90, \$100, and \$110. Therefore, parameter values corresponding to only three different option values were used. Given that it can be shown analytically that Tian's findings are incorrect for at-the-money European options, this suggests the need to examine the numerical efficiency of the two binomial models across a wider range of parameter values.

Three values of the time to maturity are used, namely 1, 4, and 7 months; three values of the risk-free rate of interest are used, namely 3%, 5%, and 7% per annum; and three values of the standard deviation of the underlying stock return are used, namely 20%, 30%, and 40% per annum. The current exercise price is set equal to \$100. To ensure an equal number of in-the-money and out-of-the-money options, the stock price is set equal to kXe^{-rt} , where k is a constant. Three values are used for the constant, k , namely 0.9, 1.0, and 1/0.9. Therefore, parameter values corresponding to 81 different option values are used.²

Tian examined the efficiency of the models at three precision levels, namely 5%, 1%, and 0.5%. These precision levels are used here also. The numerical efficiency of the two models is examined with respect to call options, European put options, and American put options. For call options and European put options, the accurate price of the option is found using the Black-Scholes model. Since no closed-form solution is available for American put options, the "accurate" price is calculated using a 1500-step procedure for Model T.

RESULTS

Table I provides a comparison of the numerical efficiency of the two models when applied to call options. For each set of parameter values,

²Consistent with Tian, the underlying stock is assumed to pay no dividend.

TABLE I
Comparison of Call Option Models

Interest Rate (Per Annum) Time to Maturity (in Months)		3%			5%			7%		
		1	4	7	1	4	7	1	4	7
Panel A 5% Precision Level										
σ	k									
0.2	0.9	0.825	0.588	0.875	0.855	0.667	1.000	0.758	0.667	0.583
	1.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	2.000
	1/0.9	1.000	2.000	1.000	1.000	2.000	1.000	1.000	2.000	1.000
0.3	0.9	0.722	1.800	0.571	0.565	1.000	0.800	0.565	1.000	0.364
	1.0	0.667	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	1/0.9	1.000	0.500	1.500	1.000	0.500	1.500	1.000	0.500	1.500
0.4	0.9	0.833	0.857	1.143	0.833	0.857	1.143	1.000	1.200	1.143
	1.0	0.667	1.000	0.500	0.667	1.000	0.500	1.000	1.000	0.500
	1/0.9	2.000	1.500	1.500	2.000	1.500	1.500	2.000	1.500	1.500
Panel B 1% Precision Level										
σ	k									
0.2	0.9	0.793	0.720	0.684	0.770	0.648	0.609	0.791	0.590	0.534
	1.0	1.000	0.778	1.357	1.100	1.167	0.826	1.375	0.609	0.905
	1/0.9	1.000	0.714	1.000	1.000	1.667	1.000	1.000	1.000	0.778
0.3	0.9	0.756	1.225	1.000	0.780	0.942	1.129	0.696	1.021	0.854
	1.0	0.750	1.045	1.150	0.818	1.278	1.643	0.900	1.643	1.095
	1/0.9	1.500	1.143	0.909	1.500	1.143	0.769	1.500	0.889	0.769
0.4	0.9	0.787	0.921	1.063	0.808	1.000	1.214	0.831	1.061	1.417
	1.0	0.583	1.045	0.773	0.583	1.150	1.063	0.636	1.438	1.214
	1/0.9	0.833	0.909	0.733	0.833	0.909	0.733	1.667	0.909	0.846
Panel C 0.5% Precision Level										
σ	k									
0.2	0.9	0.881	0.753	0.754	0.833	0.723	0.725	0.821	0.696	0.590
	1.0	0.905	1.536	1.195	1.118	0.860	1.140	1.462	1.049	1.690
	1/0.9	1.000	0.714	0.875	1.000	1.000	0.778	1.000	1.000	1.273
0.3	0.9	0.776	0.821	0.955	0.801	0.920	1.105	0.820	1.000	0.923
	1.0	0.609	1.184	1.094	0.700	1.731	0.778	0.824	0.957	0.778
	1/0.9	0.750	1.500	0.769	0.750	1.500	1.176	0.750	1.350	0.952
0.4	0.9	0.914	1.200	0.952	0.795	1.273	1.053	0.815	1.024	0.822
	1.0	0.896	0.833	1.184	0.977	1.094	1.731	1.075	1.458	0.957
	1/0.9	0.833	0.769	1.696	1.250	0.769	1.444	1.250	1.176	1.345

The number shown in the table is the ratio of the minimum convergence step for Model T divided by the minimum convergence step for Model C. Model C is the Cox, Ross, and Rubinstein (1979) model. Model T is a binomial model where the third moment of the change in the stock price during the time interval, Δt , is correct. A ratio of less (greater) than one suggests the superiority of Model T (Model C). In all cases, the exercise price is \$100, σ is the standard deviation of the underlying stock return, and k is a constant where the stock price is set equal to kXe^{-rt} .

TABLE II
Comparison of European Put Option Models

Interest Rate (Per Annum)		3%			5%			7%		
Time to Maturity (in Months)										
		1	4	7	1	4	7	1	4	7
Panel A 5% Precision Level										
σ	k									
0.2	0.9	1.000	0.500	2.000	1.000	0.500	1.000	1.000	0.500	1.000
	1.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	2.000
	1/0.9	0.889	0.714	2.000	1.200	1.000	2.000	0.980	1.000	1.556
0.3	0.9	1.000	2.000	1.000	1.000	2.000	0.500	1.000	2.000	0.667
	1.0	0.667	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	1/0.9	1.278	1.143	1.000	1.150	1.143	1.000	1.150	1.143	1.286
0.4	0.9	0.500	1.000	2.000	0.500	1.000	2.000	0.500	0.667	2.000
	1.0	0.667	1.000	0.500	0.667	1.000	0.500	1.000	1.000	0.500
	1/0.9	0.833	1.286	1.000	0.833	1.000	1.250	1.250	1.000	1.250
Panel B 1% Precision Level										
σ	k									
0.2	0.9	1.000	0.300	0.875	1.000	0.600	1.000	1.000	0.375	1.400
	1.0	1.000	0.778	1.357	1.100	1.167	0.826	1.375	0.609	0.905
	1/0.9	1.158	1.083	1.400	1.294	1.279	1.628	1.252	1.099	1.373
0.3	0.9	1.000	1.125	0.733	0.667	1.000	0.846	0.667	1.000	1.000
	1.0	0.750	1.045	1.150	0.818	1.278	1.643	0.900	1.643	1.095
	1/0.9	1.301	1.135	0.972	1.252	1.024	0.875	1.229	0.933	1.400
0.4	0.9	0.600	1.222	1.455	0.600	1.571	1.455	0.600	1.571	1.778
	1.0	0.583	1.045	0.773	0.583	1.150	1.063	0.636	1.438	1.214
	1/0.9	1.219	1.688	1.560	1.182	1.500	1.345	1.182	1.421	1.182
Panel C 0.5% Precision Level										
σ	k									
0.2	0.9	1.000	0.471	1.125	1.000	0.533	0.857	1.000	0.533	0.947
	1.0	0.905	1.536	1.195	1.118	0.860	1.140	1.462	1.049	1.690
	1/0.9	1.079	1.102	1.064	1.098	1.174	1.149	1.179	1.296	1.234
0.3	0.9	2.000	0.909	1.389	0.727	1.000	0.893	0.727	1.111	1.042
	1.0	0.609	1.184	1.094	0.700	1.731	0.778	0.824	0.957	0.778
	1/0.9	1.126	1.230	1.465	1.226	1.115	1.253	1.182	1.337	1.576
0.4	0.9	0.667	1.350	1.000	0.667	1.350	1.077	0.800	1.500	1.167
	1.0	0.896	0.833	1.184	0.977	1.094	1.731	1.075	1.458	0.957
	1/0.9	1.052	1.215	1.403	1.025	1.113	1.208	1.200	1.000	1.933

The number shown in the table is the ratio of the minimum convergence step for Model T divided by the minimum convergence step for Model C. Model C is the Cox, Ross, and Rubinstein (1979) model. Model T is a binomial model where the third moment of the change in the stock price during the time interval, Δt , is correct. A ratio of less (greater) than one suggests the superiority of Model T (Model C). In all cases, the exercise price is \$100, σ is the standard deviation of the underlying stock return, and k is a constant where the stock price is set equal to kXe^{-rt} .

TABLE III
Comparison of American Put Option Models

Interest Rate (Per Annum) Time to Maturity (in Months)		3%			5%			7%		
		1	4	7	1	4	7	1	4	7
Panel A 5% Precision Level										
σ	k									
0.2	0.9	1.000	1.000	0.500	1.000	1.000	1.000	1.000	1.000	1.000
	1.0	1.000	1.000	1.000	1.500	1.000	2.000	1.000	2.000	1.000
	1/0.9	1.021	1.250	1.000	1.200	1.000	0.778	1.143	1.000	0.778
0.3	0.9	1.000	2.000	0.667	1.000	2.000	0.667	1.000	0.500	1.000
	1.0	0.667	1.000	1.000	0.667	1.000	0.500	1.000	1.000	1.500
	1/0.9	1.769	1.143	0.556	1.150	1.000	0.714	1.150	0.778	1.000
0.4	0.9	0.500	0.667	1.000	0.500	0.667	0.667	0.500	0.667	0.667
	1.0	0.667	0.667	0.500	0.667	1.000	0.500	0.667	1.000	0.500
	1/0.9	0.833	0.556	1.250	1.250	0.556	1.250	1.250	0.556	0.667
Panel B 1% Precision Level										
σ	k									
0.2	0.9	1.000	0.750	0.833	1.000	0.667	0.800	1.000	1.000	1.000
	1.0	0.917	1.000	0.857	1.100	0.842	0.526	1.375	0.632	0.615
	1/0.9	1.186	1.393	1.104	1.154	1.286	0.884	1.308	1.189	1.188
0.3	0.9	2.000	0.900	0.917	0.667	0.875	0.900	0.667	0.833	0.800
	1.0	0.769	0.955	1.278	0.833	1.063	1.267	1.000	0.588	0.474
	1/0.9	1.136	1.135	0.972	1.300	1.024	0.875	1.272	0.933	1.296
0.4	0.9	1.000	1.444	1.273	1.000	0.917	1.333	1.200	0.900	1.143
	1.0	0.615	0.958	0.950	0.667	1.150	1.063	0.727	1.313	1.000
	1/0.9	1.219	1.094	1.560	1.219	0.972	1.345	1.500	0.921	0.727
Panel C 0.5% Precision Level										
σ	k									
0.2	0.9	1.000	1.333	1.500	1.000	0.600	1.000	1.000	1.000	1.000
	1.0	0.952	1.207	1.000	1.250	0.512	0.771	1.026	0.606	0.667
	1/0.9	1.102	1.158	1.094	1.202	1.223	1.224	1.238	1.330	1.476
0.3	0.9	1.000	1.667	1.278	1.000	1.600	1.357	1.333	0.900	1.083
	1.0	0.667	1.237	1.167	0.762	1.154	0.767	0.889	0.911	0.730
	1/0.9	1.026	1.230	1.055	1.118	1.167	1.426	1.215	1.025	1.167
0.4	0.9	0.909	1.136	0.929	0.727	1.389	0.917	0.889	1.313	0.800
	1.0	0.938	0.833	1.167	0.935	1.094	0.697	1.075	0.892	0.488
	1/0.9	1.276	1.215	0.969	1.061	1.082	0.886	1.030	1.026	1.319

The number shown in the table is the ratio of the minimum convergence step for Model T divided by the minimum convergence step for Model C. Model C is the Cox, Ross, and Rubinstein (1979) model. Model T is a binomial model where the third moment of the change in the stock price during the time interval, Δt , is correct. A ratio of less (greater) than one suggests the superiority of Model T (Model C). In all cases, the exercise price is \$100, σ is the standard deviation of the underlying stock return, and k is a constant where the stock price is set equal to kXe^{-rt} .

the number shown in the table is the ratio of the minimum convergence step for Model T divided by the minimum convergence step for Model C. Therefore, a ratio of less (greater) than one suggests the superiority of Model T (Model C).

From Table I it may be seen that no model is dominant in terms of convergence to the continuous-time values. The reported ratio is less than one in 114 cases, equal to one in 48 cases, and greater than one in 81 cases. While these numbers might suggest that Model T is more numerically efficient than Model C, the average of the ratios is 1.013. This suggests that Model C is the superior model. Therefore, it is not possible to conclude that Model T is more numerically efficient than Model C.

Table II shows the results for European put options. Again, no model is dominant in terms of convergence to the continuous-time values. The reported ratio is less than one in 69 cases, equal to one in 53 cases, and greater than one in 121 cases.³ The average of the ratios is 1.089. While it is possible that the apparent superiority of Model C might prove illusory across an even wider set of parameter values; from the evidence provided in Table II, one may conclude that Model T is not more numerically efficient than Model C.

Table III shows the results for American put options. Again, no model is dominant in terms of convergence to the continuous-time values. The reported ratio is less than one in 96 cases, equal to one in 50 cases, and greater than one in 97 cases. The average ratio is 1.007. Again, it is not possible to conclude that Model T is more numerically efficient than Model C.

SUMMARY

Tian (1993) investigated the numerical efficiency of two binomial option pricing models. Model C was the binomial model developed by Cox, Ross, and Rubinstein, while Model T was a binomial model where the third moment of the change in stock price during the time interval, Δt , is correct. Using parameter values corresponding to only three different option values, Tian concluded that Model T was more efficient than Model C. This article demonstrates, both analytically and by using a wider range of parameter values, that Tian's model is not more numerically efficient than the Cox, Ross, and Rubinstein model. The results suggest that if a binomial procedure is to be used, then there is no reason not to use the well-known Cox, Ross, and Rubinstein model.

³For at-the-money options, the results in Table II are identical to those in Table I. This is consistent with the argument advanced earlier in the article.

BIBLIOGRAPHY

- Cox, J. C., Ross, S., and Rubinstein, M. (1979): "Option Pricing: A Simplified Approach," *Journal of Financial Economics*, 7:229-264.
- Cox, J. C., and Rubinstein, M. (1985): *Options Markets*. Englewood Cliffs: Prentice-Hall.
- Tian, Y. (1993): "A Modified Lattice Approach to Option Pricing," *The Journal of Futures Markets*, 13(5):563-577.

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