OPTIMUM FUTURES HEDGES WITH JUMP RISK AND STOCHASTIC BASIS

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INTRODUCTION

The most commonly used asset price process in the continuous-time financial theory is the lognormal diffusion process. However, asset prices do experience price and volatility jumps upon the arrival of important new information (surprises). The information arrival frequency of these surprises may also be random with discrete changes. The discontinuity of the asset price has been shown empirically in Ball and Torous (1985). Other evidence finds jump components in the market portfolio and exchange rates. Jarrow and Rosenfeld (1984), with the use of daily returns on a value-weighted portfolio of all stocks on the NYSE and AMEX from July 1962 to December 1978, reject the null hypothesis that a market portfolio has a continuous sample path. Schwert (1989) reports actual jumps in the history of major stock indexes from February, 1985 to the end of 1988. Jorion (1988) finds the existence of jumps in the sample

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path of exchange rates and of a stock portfolio even after allowing for conditional heteroskedasticity in the diffusions. These studies imply the need to model asset prices using jump-diffusion processes.

Notwithstanding the pressure of jump risks, the traditional approach of calculating optimum futures hedges [Johnson (1960), Stein (1961), Ederington (1979), etc.] views investing in futures as a way of offsetting only potential diffusion risk associated with a given spot position. Strict stationarity assumptions are imposed upon the spot and futures price processes to circumvent the presence of jump and carrying risk in the markets. Such simplification allows the optimum hedge ratio to be estimated using the simple OLS adjusted for autocorrelation and other data problems, and thus has gained wide popularity in practice. Empirical evidence has shown, however, that the performance of this approach is sometimes unsatisfactory. For example in hedging the risk of stock portfolios, Figlewski (1985), Junkus and Lee (1985), Bailey (1989), and Yau, Savanayana, and Schneeweis (1990) have all found that the hedges are often ineffective, and are more so when the market is volatile. Both Figlewski and Bailey have found that the standard deviations of hedged portfolio returns are often at an annual rate of 10–15% or higher, with the mean returns not higher than comparable riskless rates. They have also found in many cases that ex post hedging effectiveness measures are much lower than the corresponding ex ante hedging effectiveness measures.

These hedging inefficiencies likely originate from the stationarity assumptions that ignore nonstationary price changes and basis behavior. For example, when jumps occur, not only does the spot and futures price volatility increase, but their return correlation may also change because of jumps in liquidity and changes in arbitrage execution risk. A hedge ratio based solely on diffusion risk is not optimum, and consequently a futures contract's hedging effectiveness is suboptimal. The degree of ineffectiveness accentuates as the jump frequency intensifies. On the other hand, as shown in Brennan and Schwartz (1990), stochastic basis affects the optimum strategy of an arbitrageur. To the extent that basis risk is significant, the spot and the futures markets are not fully integrated, as one may expect. There is a significant literature demonstrating how information flow may hit these two markets differently, and that the futures market is the main source of market information [e.g., Chan (1992)]. Therefore, the spot-futures price difference, that is, the basis, may vary over time stochastically. This is why practitioners often view hedging as speculating on the basis behavior. Therefore, a priori, a hedging model that considers both price jumps and stochastic basis is expected to be more effective than one that does not. This study is an attempt to develop and test such a model. Specifically, an optimum hedge ratio in an intertemporal setting is derived that takes jump risk and stochastic basis as given.[1]

Several authors have investigated intertemporal futures hedging following Merton's (1971, 1973) intertemporal portfolio optimization methodology.[2] They have, however, ignored basis behavior and jump risk. This study incorporates jump risk first in a simple setting where the futures price process is exogenous and the correlation between spot and futures price changes is constant. It is found that under the assumption that investors have unbiased estimates of the true spot and futures price processes, the derived jump-diffusion optimum hedge ratio is actually not different from the traditional pure diffusion optimum hedge ratio. That simple model is uninteresting because it does not allow feedback between the spot and futures markets. There is a large body of empirical literature showing that futures and spot prices tend to revert back toward each other because of the pressure of arbitrage trading [see, for example, MacKinlay and Ramaswamy (1988) or Chung (1991)]. Thus, the failure of the model to admit sufficiently rich, realistic behavior for the spot/futures basis may substantially weaken the hedging performance. Therefore, the model is extended by permitting a mean-reverting stochastic basis with time-decay volatility. The inclusion of this realistic behavior endogenizes the futures price process, and leads to an optimum hedge ratio that exhibits a much richer set of hedging properties when compared to the traditional diffusion optimum hedge ratio. Finally the model is extended to permit time-varying jump arrival intensity. Specifically, the arrival intensity value is set high to reflect lumpiness upon the arrival of surprises, and reverts back to a lower value after the news release. This extension results in a path-dependent hedge ratio that is dictated by user inputs—the hedger's ability to accurately forecast the time-varying pattern of the jump intensity.

[1] Merton (1976) is the first to investigate the hedging of jump risk using options contracts.

[2] Stulz (1984) studied optimum currency hedge with the use of forward contracts in an intertemporal framework with constant interest rates. Ho (1984) dealt with hedging a producer's production exposure facing both output and price uncertainty in an intertemporal framework with constant interest rates. Breeden (1984) examined the hedging and allocation role of futures markets in an intertemporal incomplete market setting with constant interest rates. Adler and Detemple (1988) explored the intertemporal hedging structure of an optimum hedging decision with constant interest rates when investors hold a portfolio of freely shortable traded assets in addition to a nontradable spot position. Chang and Fang (1990) explicitly developed intertemporal optimum hedge ratios and hedging effectiveness measures when the instantaneous risk-free interest rate follows a mean-reverting square-root process and marking to market is present. Myers (1991) used GARCH to estimate time-varying hedge ratios. He found, however, that the GARCH approach only leads to marginal hedging improvement when compared to the traditional approach.

The rest of the article is organized as follows. In the next section the simple jump-diffusion optimum hedge ratio is derived. In the third section the model is extended to subsume stochastic basis and time-varying jump arrival intensity. Numerical analyses are conducted to illustrate the hedging properties of these new hedge ratios. In the fourth section out-of-sample hedging tests are conducted on a stock index, a long-term T-bond index, gold, oil, and wheat. The purpose is to find out if the incorporation of stochastic basis enhances the hedging effectiveness of futures contracts even in an ex ante test. In a majority of the cases, significant hedging enhancements are found, especially in the case of higher basis risk. The fifth section includes concluding remarks and suggestions for future research.

A SIMPLE MODEL FOR JUMP-DIFFUSION OPTIMUM FUTURES HEDGE

This simple model assumes that the futures price process is exogenous and the correlation between spot and futures price changes is constant. It is assumed that the movement of spot price, S, follows a jump-diffusion process of the form

$$dS/S = \mu_s\,dt + \sigma_s\,dZ_s + (X - 1)\,d\pi \tag{1}$$

where μ_s and σ_s are, respectively, the spot contract's instantaneous expected and standard-deviation rate of return conditional on no arrivals of important new information; dZ is the standard Gauss-Wiener process; $\pi(t)$ is the standard homogeneous Poisson process representing the arrival process of important new information. The arrival frequency parameter is j, and $d\pi$ and dZ_s are assumed to be independent; X is the gross jump size of the asset price triggered by the Poisson information arrival. It is assumed that ln $X \sim N(\mu_x, \sigma_x)$ and that X is independent of dZ_s and $d\pi$.

In other words, the asset price dynamics can be written as a combination of two types of changes:

1. The first type is the *normal vibration* in price due to, for example, a temporary imbalance between supply and demand, changes in the economic look, or other new information that causes marginal changes in the asset value. This component of change in return is modeled by a standard diffusion process.
2. The second type of change is the *abnormal vibration* in price, which is caused by the arrival of important new information that has more than a marginal effect on the asset value. This component of change

is modeled by a homogeneous jump process with an inherently non-continuous sample path reflecting the nonmarginal impact of information. The expected information arrival rate is assumed to be time unvarying.

Similarly, the futures price, F, follows a correlated jump-diffusion process:

$$dF/F = \mu_f\,dt + \sigma_f\,dZ_f + (Y - 1)\,d\pi \tag{2}$$

where μ_f and σ_f are, respectively, the futures contract's instantaneous expected and standard-deviation rate of return conditional on no arrivals of important new information; dZ_f is the standard Gauss-Wiener process; $\pi(t)$ is a Poisson arrival process as in eq. (1). It is assumed that both the spot and the futures abnormal price changes are caused by the same information arrival process; the jump size of the futures price is represented by a lognormal distribution Y, that is, $\ln Y \sim N(\mu_y, \sigma_y)$; the instantaneous covariance between dZ_s and dZ_f is denoted as σ_{sf} and that between $\ln X$ and $\ln Y$ is denoted as σ_{xy}. Both covariances are assumed to be constant.

It is assumed that the hedger holds a long-spot position and intends to short futures such that the variance of the hedged position over a certain hedge horizon, $T - t$, is minimized. The hedge ratio, h, is defined as the dollar amount of futures short per dollar of the long-spot position. With the assumptions of 100% margin on spot trading, 0% margin on futures trading, and no marking to market [see Chang and Fang (1990), for the hedging impact of marking to market], the price dynamics of the hedged portfolio position is simply

$$\begin{aligned} dP/P = dS/S - h\,dF/F &= (\mu_s - h\mu_f)dt \\ &+ \sigma_s\,dZ_s - h\sigma_f\,dZ_f + [(X - 1) - h(Y - 1)]\,d\pi \end{aligned} \tag{3}$$

The investor's hedging decision is to choose an optimum hedge ratio, h_1, that minimizes the variance of the portfolio return measured using log relative over $T - t$; that is,

$$\begin{aligned} \operatorname*{Min}_h \operatorname{Var}(\ln R_p) &= \sum_{n=0}^{\infty} \Gamma(n)[(\sigma_s^2 + h^2\sigma_f^2 - 2h\sigma_{sf})(T - t) \\ &\quad + n(\sigma_x^2 + h^2\sigma_y^2 - 2h\sigma_{xy})], \\ &= (\sigma_s^2 + h^2\sigma_f^2 - 2h\sigma_{sf})(T - t) \\ &\quad + j(T - t)(\sigma_x^2 + h^2\sigma_y^2 - 2h\sigma_{xy}), \end{aligned} \tag{4}$$

where $\Gamma(n) = e^{-j(T-t)}[j(T - t)]^n/n!$ is the Poisson probability mass function.

The first-order condition leads to the following optimum hedge ratio:

$$h_1 = (\sigma_{sf} + j\sigma_{xy})/(\sigma_s^2 + j\sigma_f^2) \tag{5}$$

The second-order condition is

$$2\sigma_f^2(T - t) + j(T - t)(2\sigma_y^2) > 0$$

which ensures minimization.

When $j = 0$, that is, no jumps are expected to occur over $T - t$, h_1 collapses to the traditional one-period optimum hedge ratio of σ_{sf}/σ_f^2.

The hedging effectiveness measure, HE, defined as 1 minus the quotient of the variance of the hedged portfolio return, $\mathrm{Var}(\ln R_p)$, divided by the variance of the spot portfolio return, $\mathrm{Var}(\ln R_s)$, is

$$\begin{aligned} \mathrm{HE} = 1 - [(\sigma_s^2 + h^2\sigma_f^2 - 2h\sigma_{sf}) \\ + j(\sigma_x^2 + h^2\sigma_y^2 - 2h\sigma_{xy})]/(\sigma_s^2 + j\sigma_x^2) \end{aligned} \tag{6}$$

When $j = 0$, HE reduces to the traditional one-period HE measure.

Now suppose that an investor uses the traditional approach to hedge a portfolio by assuming that the spot and futures price changes follow pure diffusions even though the true spot and futures price processes are of the jump-diffusion type. In other words, the investor ignores the hedging of jump risk. How does jump arrival affect the investor's degree of mishedge and bias his or her measure of hedging effectiveness?

As in Merton (1976), it is assumed that this traditional investor has available a long enough price history so that unbiased estimates of the true covariance and variances of the processes are known, namely,

$$\mathrm{Var}(\ln R_f) = \sigma_f^2(T - t) + j(T - t)\sigma_f^2$$

$$\mathrm{Cov}(\ln R_s, \ln R_f) = \sigma_{sf}(T - t) + j(T - t)\sigma_{xy}$$

With the use of these estimates, the investor constructs the optimum hedge ratio with the traditional approach:

$$h^\circ = \mathrm{Cov}(\ln R_s, \ln R_f)/\mathrm{Var}(\ln R_f) = (\sigma_{sf} + j\sigma_{xy})/(\sigma_f^2 + j\sigma_y^2)$$

It turns out that h° is not different from h_1, the jump-diffusion optimum hedge ratio. Thus, given the unbiasedness assumption, the jump-diffusion approach does not offer any actual improvement over the traditional approach. The intuition is that the basis for the jumps are twofold: the expected on information release days, and the unexpected because of random jump arrival. The unbiasedness assumption stipulates the former, and therefore, the traditional hedge may well be adjusted to capture an alternative jump correlation level. The resulting optimum hedge ratio thus becomes identical to the jump-diffusion optimum hedge ratio.

In general, the model is uninteresting because it uses a very simple setup that does not permit feedback between the spot and futures markets. There is a large body of empirical literature that shows that futures and spot prices tend to revert back toward each other because of the pressure of arbitrage trading [see, for example, MacKinlay and Ramaswamy (1988) or Chung (1991)]. Thus, the failure of the model to admit sufficiently rich, realistic behavior for the spot/futures basis may substantially weaken the hedging performance. The following section extends the model by incorporating basis behavior.

OPTIMUM FUTURES HEDGE WITH STOCHASTIC BASIS

Brennan and Schwartz (1990) have investigated optimum index arbitrage, with the use of the stochastic evolution of the basis as a given. The same is done here. First, the carrying rate process is constructed. The instantaneous rate of the net cost of carry, i, is assumed to be the factor that determines the basis and follows a mean-reverting Ornstein-Uhlenbeck process. A square-root process is not employed because that process would not permit a negative net cost of carry. So, the rate process is

$$di = q(m - i)\,dt + v\,dZ_i \tag{7}$$

where q denotes the speed of adjustment, m the long-run mean rate, and v^2 the instantaneous variance. These parameter values are dictated by the nature of the cost of carry, and so are asset specific.[3]

B, the relative basis, is defined as $B \equiv S/F$, then B denotes the present value of \$1 net cost of carry at the futures' expiration. Because in this one-factor setting B is a function of i and the futures maturity, $\tau - t$ [with the use of a result in Vasicek (1977, eqs. (27) and (28))], the solution of B is derived as

[3]For example, for commodity futures contracts, the net cost of carry includes domestic interest rates, warehouse costs, and a convenience yield on inventory; for currency futures contracts, the net cost of carry includes domestic and foreign interest rates, et cetera. A recent reference explaining basis variation with the use of the net cost of carry is Brennan (1991). Bailey and Chan (1993) have taken an alternative approach to explain basis variation with the use of a set of common macroeconomic information variables.

$$B(\tau - t) = e^{k(\Phi - \tau) - (v\Phi/2)^2/q}\, e^{-i\Phi}, \tag{8a}$$

where

$$\Phi = (1 - e^{-q(\tau - t)})/q,$$
$$k = m + v\phi/q - (v/q)^2/2,$$
$$\phi = (q/v)\left[R(\infty) - m + \frac{1}{2} v^2 q^2\right].$$

$R(\infty)$ is the basis's limiting yield, and ϕ, the market price of carrying risk, is assumed to be constant. Then, by Ito's lemma,

$$dB/B = \mu_b(t)\, dt + \sigma_b(t)\, dZ_i \tag{8b}$$

where $\mu_b(t) = i + (1 - e^{-q(\tau - t)})[R(\infty) - m + 1/2\, v^2/q^2]$, $\sigma_b(t) = v\Phi$, $\tau - t$ is the futures maturity, and $R(\infty)$ is the basis's limiting yield. The process has the desired property that the absolute basis, $S - F$, tends to return to zero and is zero at maturity with probability 1, that is, when the relative basis B approaches 1; $\mu_b(t)$ is stochastic because of the carry risk; and $\sigma_b(t)$ is time varying to exhibit the time decay of the basis volatility.

A jump component is added to incorporate the facts that, when a price jump arrives, the basis may also jump due to liquidity and the execution risk may also jump.

$$dB/B = \mu_b(t)\, dt + \sigma_b(t)\, dZ_i + (W - 1)\, d\pi, \tag{9}$$

where W is the gross jump size such that $\ln W \sim N[0, \sigma_w \sqrt{(\tau - t)}]$. The assumed zero mean is to ensure that, a priori, the basis jump is neutral, and the time-varying jump standard deviation is consistent with the time-decaying behavior of the basis volatility.

Because $F = S/B$, by the generalized Ito's lemma, the futures price process becomes

$$dF/F = \mu_f(t)\, dt + \sigma_f(t)\, dZ_f + (Y - 1)\, d\pi \tag{10}$$

where

$$\mu_f(t) = \mu_s - \mu_b(t) + \sigma_b^2(t) - \delta_{si}\sigma_s\sigma_b(t),$$
$$\sigma_f(t)^2 = \sigma_s^2 + \sigma_b(t)^2 + 2\,\delta_{si}\sigma_s\sigma_b(t),$$
$$\delta_{si}\, dt = dZ_s dZ_i,$$

and

$$\ln(Y) - N(\mu_y, \sigma_y), \mu_y = \mu_x \quad \text{and}$$
$$\sigma_y = [\sigma_x^2 + \sigma_u^2(\tau - t) - 2\sigma_{xu}\sqrt{(\tau - t)}]^{1/2}$$

The price dynamics of the hedged portfolio is now

$$dP/P = dS/S - h\,dF/F = (\mu_s - h\mu_f)\,dt + \sigma_s dZ_s - h\sigma_f\,dZ_f \quad \text{(11a)}$$
$$+ [(X - 1) - h(Y - 1)]\,d\pi$$
$$= \{(1 - h)\mu_s - h[-\mu_b(t) + \sigma_b^2(t) - \delta_{si}\sigma_s\sigma_b(t)]\}\,dt$$
$$+ [(1 - h)^2\sigma_s^2 + h^2\sigma_b^2(t)$$
$$- 2h(1 - h)\,\delta_{si}\sigma_s\sigma_b(t)]^{1/2}\,dZ_p + (V - 1)\,d\pi \quad \text{(11b)}$$

where

$$\ln V - N(\mu_v, \sigma_v), \mu_v = (1 - h)\mu_x, \quad \text{and}$$
$$\sigma_v = [(1 - h)^2\sigma_x^2 + h^2\sigma_u^2(\tau - t) - 2h(1 - h)\delta_{xu}\sigma_x\sigma_u\sqrt{\tau - t}]^{1/2}.$$

The investor's hedging decision is to choose a sequence of optimum intertemporal hedge ratio $h_2(t)$, which minimizes the variance of the portfolio return that is measured with the use of a log price relative over $T - t$,

$$\min_{h(t)} \operatorname{Var}(\ln R_p) = I(t), \quad (12)$$

where

$$\operatorname{Var}(\ln R_p) = \int_t^T \Theta(s)\,ds$$

and $\Theta(s)$, the instantaneous portfolio variance of return, equals

$$\{[(1 - h(s))^2\sigma_s^2 + h(s)^2\sigma_b^2(s) - 2h(s)(1 - h(s))\,\delta_{si}\sigma_s\sigma_b(s)]$$
$$+ j[(1 - h(s))^2\sigma_x^2 + h(s)^2\sigma_u^2(\tau - s)$$
$$- 2h(s)(1 - h(s))\delta_{xu}\sigma_x\sigma_u\sqrt{\tau - s}]\}.$$

With the use of the techniques of stochastic dynamic programming, the following Bellman optimality equation is derived:

$$0 = \operatorname{Min}\{\Theta(t) + dI(T)/dt\} \quad (13)$$

The first-order condition for a regular interior minimum is

$$d\Theta(t)/dh = 0$$

which yields the following intertemporal optimum hedge ratio:

TABLE I

A Numerical Illustration of $h_2(t)$, Optimum Risk-Minimization Hedge Ratio When the Basis Follows a Mean-Reverting jump-Diffusion Process:

$$h_2(t) = [\sigma_s^2 + \delta_{si}\sigma_s\sigma_b(t) + j(\sigma_x^2 + \delta_{xu}\sigma_x\sigma_u\sqrt{\tau - t})]/(\sigma_f^2 + j\sigma_y^2)$$

$\tau - t$	$h_2(t)$	j	$h_2(t)$	v	$h_2(t)$
0.25	0.9232	0	0.9970	0	0.9723
0.20	0.9343	20	0.9782	0.01	0.9712
0.15	0.9461	40	0.9712	0.02	0.9700
0.10	0.9591	60	0.9674	0.03	0.9689
0.05	0.9740	80	0.9651	0.04	0.9677
0.00	1	100	0.9636	0.05	0.9666

Notes: $\sigma_s = 0.1$, $\sigma_x = 0.02$, $q = 0.3$, $\sigma_u = 0.01$, $\delta_{si} = 0.2$, $\delta_{xu} = -0.1$. When determining $h_2(t)$ as a function of $\tau - t$, $j = 40$ and $v = 0.01$; as a function of j, $\tau - t = 0.05$ and $v = 0.01$; as a function of v, $\tau - t = 0.05$ and $j = 40$. $\tau - t$ denotes futures maturity as a fraction of a year.

$$h_2(t) = [\sigma_s^2 + \delta_{si}\sigma_s\sigma_b(t) + j(\sigma_x^2 + \delta_{xu}\sigma_x\sigma_u\sqrt{\tau - t})]/(\sigma_f(t)^2 + j\sigma_y(t)^2) \tag{14}$$

where

$$\sigma_b(t) = vp(t) = v(1 - e^{-q(\tau - t)})/q,$$
$$\sigma_f(t)^2 = \sigma_s^2 + \sigma_b(t)^2 + 2\delta_{si}\sigma_s\sigma_b(t),$$

and

$$\sigma_y^2 = [\sigma_x^2 + \sigma_u^2(\tau - t) - 2\delta_{xu}\sigma_x\sigma_u\sqrt{(\tau - t)}]$$

The second-order condition is

$$(\sigma_f^2 + j\sigma_y^2) > 0$$

which ensures minimization.

This intertemporal hedge ratio differs from the traditional one-period constant hedge ratio by incorporating the basis behavior and the spot price and basis jumps. To examine its properties, a numerical analysis is conducted. The following set of parameter values is used: $\sigma_s = 0.1$, $\sigma_x = 0.02$, $q = 0.3$, $v = 0.00$–0.05, $\sigma_u = 0.01$, $\delta_{si} = 0.2$, $\delta_{xu} = -0.1$, $j = 0$–100, and $\tau - t = 0$–0.25 years. The results are reported in Table I. The findings are

1. Except at maturity, all the hedge ratios are smaller than 1 because of the imperfect correlations between spot and futures price changes, that is, basis risk, and the assumption that the correlation between the spot price change and the carrying rate is positive. This assumption makes the futures volatility larger than the spot volatility. When the correlation is negative, it is possible for the hedge ratio to be larger than 1,

2. As the maturity is shortened, the hedge ratios increase to approach 1. This is due to the time decay of the basis volatility,
3. As the jump arrival intensity increases, the hedge ratio decreases. This is because the absolute jump correlation is assumed to be smaller than the absolute diffusion correlation.
4. As the basis volatility increases, the hedge ratio decreases. This is because, given the positive correlation between the spot price change and the carrying rate, increasing basis volatility causes increasing futures volatility, ceteris paribus.

Therefore, as expected, the incorporation of basis behavior enriches the hedging analysis. The resulting intertemporal optimum hedge ratio is capable of explaining more realistic hedging situations than is allowed when using the traditional constant optimum hedge ratio.

Next, the analysis is extended to permit time-varying jump arrival intensity. Intensity will be high at the time of the arrival of important news. After the release of the news, intensity reverts back to a lower or more normal level. For ease of exposition, it is assumed that there are only two states, high and low intensities, denoted as j_u and j_d, respectively. The analysis, however, generalizes naturally to the case of multiple states. The optimum hedge ratio can be easily shown to be

$$h_3(t) = [\sigma_s^2 + \delta_{sj}\sigma_s\sigma_b(t) + j(t)(\sigma_x^2 + \delta_{xu}\sigma_x\sigma_u\sqrt{\tau - t})]/(\sigma_f^2 + j(t)\sigma_x^2) \tag{15}$$

where $j(t)$ equals either j_u or j_d.

A numerical example is used to illustrate its implementation. The following parameter values are used: $\sigma_s = 0.1$, $\sigma_x = 0.02$, $q = 0.3$, $v = 0.01$, $\sigma_u = 0.01$, $\delta_{sj} = 0.2$, $\delta_{xu} = -0.1$, $j_u = 54$, $j_d = 28$, and $\tau - t = 0.00 - 0.25$ years. Assume that, at the initialization of the hedge, the intensity is low. The results are

$$\begin{array}{lccccc} & & 0.9359 & 0.9475 & 0.9602 & 0.9747 & \\ h_3: 0.9086 < & & & & & & > 1 \\ & & 0.9214 & 0.9353 & 0.9506 & 0.9684 & \\ \tau - t: 0.25 & & 0.20 & 0.15 & 0.10 & 0.05 & 0 \end{array}$$

Because of the intensity shift, the hedge ratio process is binomial during each time period with two possible states until the last period, when the hedge ratios converge to 1. Initially, the hedge ratio when there is 0.25 year remaining is 0.9086. At the end of the period, when there is 0.20 year remaining, the intensity either remains low or shifts to the high state if information release is expected. The hedge ratio for the next period given the low/high state is 0.9214/0.9359, et cetera. For each period, the larger (smaller) hedge ratio corresponds to the high (low) state for the intensity in the next period. Any hedge ratio path starting at 0.9086 and ending at 1 is possible and is dictated by the hedger's knowledge of the time-varying pattern of the jump intensity.

When $j = 0$, that is no jumps are expected to occur over $T - t$, the ratio collapses to

$$h_4(t) = [\sigma_s^2 + \delta_{si}\sigma_s\sigma_b(t)]/\sigma_f^2 \tag{16}$$

The difference between this intertemporal hedge ratio and the traditional hedge ratio is the incorporation of the stochastic basis behavior.

EMPIRICAL TESTS

The hedging performance of a hedging model that takes stochastic basis as a given (stochastic basis hedging model hereafter), that is, eq. (16), is compared to one that ignores stochastic basis, that is, the traditional approach. An ex ante analysis (out of sample) is employed in lieu of an ex post test (in the sample), because the latter assumes perfect foresight in determining the optimum hedge ratios. In an ex ante, test the actual hedging effectiveness for a sample period is determined with the use of actual returns of unhedged and hedged portfolios with the prescribed hedge ratios determined with the use of the prior period's data. This is a more reliable methodology because there may be little correspondence between expected and actual risk-reduction performance because of basis risk. Because the hedging of basis risk is considered in neither model, a priori, one would expect the impact of basis risk to be significant.

The test is applied to the SP-500 stock index, the Lehman Brothers long T-bond index, gold, oil, and wheat, with corresponding June '93 (on the SP-500 index, the T-bond index, gold, and oil) and May '93 (on wheat) futures contracts, for the sample periods, 6/22/92–6/17/93, 6/22/92–6/17/93, 6/22/92–6/17/93, 5/18/92–5/19/93, and 5/18/92–5/19/93, respectively. The daily futures settlement prices are taken from a data disk included with Kolb's investments textbook (Kolb publishing, 1994), and the daily closing spot prices are taken from the *Wall Street Journal*. The 3-month T-bill rate, collected from the *Wall Street Journal*, is used as a proxy for the carrying rate because of the data availability problem. Although the interest rate risk may be the important component of the carrying risk, there are other sources of carrying risk that may also be important. These other sources may include seasonals in agricultural and oil futures, the quality option in T-bond futures, dividend uncertainty in index futures, and other possible realistic features. Therefore, this test only serves as a weak test of the hedging effectiveness of the stochastic basis hedging model.

Because tracking errors (caused by nonsynchronized trading among the instruments) become relatively more pronounced when their values are measured on a daily basis, the test uses weekly returns. Weekly returns are computed using logarithmic price relatives. The ex post hedging effectiveness is computed as the actual percent variance of return reduction on the unhedged position during the hedge duration via hedging. During the hedge duration, the daily value of the hedged portfolio is computed as the corresponding unhedged value plus/minus gain/loss from marking to market the futures position, established based upon the optimum hedge ratio determined in the prior period and the initial spot value. Following the convention, margin rates for trading the spot and the futures are assumed to be 100 and 0%, respectively.

For each spot contract, three consecutive quarterly hedges are constructed with weekly updates of the hedge ratios based on a quarterly window. The first quarterly data in the sample period are the first estimation period. One-year futures contracts with maturities matching the sample periods are used to avoid rolling-over risk. For the stochastic basis hedging model, the hedge ratio is updated weekly with the use of eq. (16) and a quarterly moving-window scheme to allow consideration of weekly basis behavior. The moving window is implemented by dropping the first trading week in the quarterly window and adding the subsequent trading week until the entire sample is employed. Thus, the futures position is rebalanced weekly to reflect a new hedge ratio and a new corresponding spot value of the stock index. For the conventional model, a dynamic hedge is implemented with the hedge ratio updated weekly by reestimating the hedge ratio with the use of rolling regression with the same moving window.

To compute the stochastic basis hedge ratios, the parameters of the carrying rate processes need to be estimated. For empirical purposes, the mean-reverting process of stochastic interest rates is replaced by the following discrete approximation:

$$i_t - i_{t-1} = q(m - i_{t-1}) + \varepsilon_t \tag{17}$$

TABLE II
Estimates of the Parameters of the Mean-Reverting Carry Rate Process:
$$i_t - i_{t-1} = q(m - i_{t-1}) + \varepsilon_t$$

Quarter (92–93)	m	q	v
Q1	0.0303 (0.0013)	0.3253 (0.1617)	0.0352
Q2	0.0318 (0.0029)	0.1327 (0.0625)	0.0336
Q3	0.0294 (0.0003)	0.3011 (0.1058)	0.0145
Q4	0.0295 (0.0007)	0.2466 (0.1135)	0.0225

Notes: (a) m denotes the long-run mean interest rate, q the speed of adjustment, and v the standard deviation of the changes of interest rates. (b) The numbers in the parentheses are standard errors.

The maximum-likelihood estimators for the parameters m, q, and v are reported in Table II. The parameters of m and q are significant in all cases, and as expected, the long-run mean rate (m) is stable over time.

In Table III the weekly optimum hedge ratios for both approaches are reported. As expected, the stochastic basis hedge ratios are much more stable and revert back to 1 as the futures contracts approach their expirations. In contrast, the conventional hedge ratios often significantly deviate from 1 even at the expiration. This is especially conspicuous for oil and wheat. In these two cases, the conventional hedge ratios are very unstable and significantly over- and underhedged relative to the corresponding stable stochastic basis hedge ratios.

In Table IV the ex ante hedging effectiveness of the futures contracts under the two models are reported. The findings are

1. In 73% of the cases (11 out of 15) the incorporation of stochastic basis significantly improves the futures contracts' hedging effectiveness. In the four cases where the conventional approach prevails, the differences are very small.
2. The stochastic basis hedge performs better for commodities than for financial investments. For oil and wheat especially, the stochastic basis hedges significantly outperform the conventional hedge by as high as 10%. Also, in these cases, futures contracts are not very effective hedging vehicles. The HEs range from 63.7% to 92.9% with one outlier at 26.8%, even though the corresponding stochastic basis hedge ratios are stable.

These results seem to suggest that the larger the basis risk, the better the performance of the stochastic basis hedge, and for oil and wheat the hedging of basis risk itself is warranted.

TABLE III
Weekly Optimum Hedge Ratios: The Conventional Model versus the Stochastic Basis Model

S&P		*Gold*		*Oil*		*Wheat*		*T-Bond*	
h_c	h_s	h_c	h_s	h_c	h_s	h_c	h_s	h_c	h_s
0.929	0.962	0.991	0.950	1.552	0.936	1.513	0.984	0.931	1.105
0.934	0.974	0.984	0.954	1.555	0.939	1.536	0.986	0.936	1.112
0.932	0.953	0.990	0.941	1.527	0.937	1.480	0.980	0.933	1.121
0.947	1.048	0.985	0.927	1.259	0.940	1.551	0.982	0.907	1.108
0.957	1.026	0.980	0.958	1.283	0.948	1.469	0.979	0.903	1.107
0.959	1.025	0.978	0.955	1.319	0.951	1.425	0.968	0.888	1.096
0.953	1.021	0.978	0.962	1.265	0.972	1.378	0.963	0.921	1.152
0.945	1.027	0.971	0.963	1.059	0.975	1.425	0.975	0.927	1.154
0.927	1.024	0.972	0.974	1.051	0.984	1.420	1.015	0.895	1.163
0.921	1.020	0.996	0.976	1.056	0.964	1.348	1.007	0.894	1.115
0.920	0.988	1.004	0.998	1.326	0.981	1.353	1.007	0.879	1.107
0.906	0.987	1.004	0.995	1.340	0.989	1.305	1.006	0.865	1.115
0.924	1.014	0.998	0.990	1.306	0.988	1.119	1.011	0.826	1.066
0.917	1.012	0.988	0.983	1.316	0.838	1.166	1.007	0.858	1.077
0.916	1.015	0.989	0.976	1.330	0.986	1.072	1.008	0.836	1.102
0.913	1.019	0.989	0.977	1.273	0.995	1.118	1.012	0.820	1.062
0.901	0.955	0.980	1.000	1.407	0.996	0.853	1.012	0.795	1.012
0.902	0.987	0.987	0.978	1.431	0.988	0.848	1.025	0.782	1.028
0.915	0.982	1.009	0.977	1.327	0.997	0.880	0.975	0.818	1.029
0.913	0.987	1.009	0.977	1.353	0.998	0.921	1.008	0.826	1.024
0.929	0.970	1.001	0.976	1.318	0.988	0.875	0.992	0.795	1.013
0.958	0.996	0.987	0.963	1.340	0.989	0.751	0.965	0.793	1.012
0.956	0.997	0.964	0.975	1.362	0.996	0.926	0.990	0.892	1.006
0.980	0.998	0.975	0.977	1.366	1.001	1.048	0.990	0.884	1.002
0.981	0.996	0.975	0.990	1.361	1.000	1.032	0.991	0.888	0.997
0.970	0.996	0.966	0.993	1.368	1.001	1.067	0.992	0.905	0.996
0.976	0.995	0.975	0.997	1.346	0.999	1.029	0.994	0.906	0.997
0.976	0.995	0.975	0.999	1.200	0.999	1.084	0.996	0.913	0.998
1.044	0.999	0.982	1.011	1.205	0.998	1.080	0.999	0.921	0.999
0.990	1.002	0.981	1.013	1.145	0.996	1.077	1.000	0.959	1.004
0.996	1.000	0.983	1.013	1.147	0.997	1.127	0.999	0.969	1.003
0.979	1.000	1.014	0.990	1.161	0.996	1.043	1.000	0.971	1.008
0.968	1.002	1.022	0.992	1.147	0.996	1.050	1.000	0.970	1.007
0.978	1.002	1.014	0.993	1.123	0.998	1.061	1.000	0.978	1.005
0.971	1.002	1.026	0.994	1.060	0.999	0.972	1.000	0.978	1.005
0.973	1.000	1.013	0.996	1.083	0.999	0.667	1.000	0.964	1.005
0.947	1.000	1.006	0.998	1.063	0.998	0.561	1.001	0.957	1.002
0.959	1.000	1.013	1.000	1.059	0.999	0.653	1.001	0.957	1.000
				1.041	1.000	0.730	1.000		

Note h_c and h_s denote the optimum hedge ratio derived with the use of the dynamic conventional hedge and the stochastic basis hedge, respectively

TABLE IV

Ex Ante Hedging Effectiveness: The Conventional Model versus the Stochastic Basis Model

Quarter (92–93)	*S&P*		*Gold*		*Oil*		*Wheat*		*T-Bond*	
	HE_c	HE_s	HE_c	HE_s	HE_c	HE_s	HE_c	HE_s	HE_c	HE_s
Q2	0.9892	0.9773	0.9640	0.9690	0.5995	0.6370	0.6743	0.7687	0.8972	0.9062
Q3	0.9901	0.9916	0.9855	0.9856	0.9385	0.9293	0.7829	0.7893	0.9477	0.9465
Q4	0.9904	0.9908	0.9984	0.9992	0.7889	0.8011	0.1631	0.2681	0.9755	0.9748

Note: HE_c and HE_s denote the ex ante hedging effectiveness of the dynamic conventional hedge and the stochastic basis hedge, respectively.

CONCLUDING REMARKS

An intertemporal futures hedging model is derived with mean-reverting basis, time-decay volatility, and price and basis jumps. The resulting intertemporal optimum hedge ratio is capable of explaining a much richer set of realistic hedging behaviors than the conventional approach. The results of out-of-sample hedging tests based on a version of the model that focuses on the stochastic basis behavior is quite positive. In the majority of the cases, the model outperforms the conventional approach. The more significant the basis behavior, the better the performance. However, the full model is not tested and only stochastic interest rates are considered, so it is impossible to say how closely the full model describes actual hedging behavior. Resolution of this issue awaits future research.

It is found, however, that in the cases of wheat and oil, the futures contracts' hedging effectiveness is unsatisfactory. It ranges from 63.7% to 92.9% with one outlier at 26.8%, even though the corresponding hedge ratios are stable. The reasons are probably twofold: 1. If the T-bill rate is used as a proxy for the carrying rate, the model's implementation does not capture seasonals in the commodities. 2. The basis risks are so significant that they also need to be hedged. For the latter, the technique of using the yield curve as suggested by Jarrow and Madan (1992) may be employed.

On the theoretical front, because the model assumes deterministic jump arrival shifts and zero costs of hedging, further improvements may be made by allowing for random jump arrival and by incorporating the costs of setting up and managing a hedge.

BIBLIOGRAPHY

Adler, M., and Detemple, J. B. (1988): "On the Optimum Hedge of a Non Traded Cash Position," *Journal of Finance*, 43:143–153.

Bailey, W. (1989): "The Market for Japanese Stock Futures: Some Preliminary Evidence," *The Journal of Futures Markets*, 9:283–295.

Bailey, W., and Chan, K. C. (1993): "Macroeconomic Influences and the Variability of the Commodity Futures Basis," *Journal of Finance*, 48:555–573.

Ball, C. A., and Torous, W. A. (1985): "On Jumps in Common Stock Price and Their Impact on Call Option Pricing," *Journal of Finance*, 40:155–173.

Breeden, D. T. (1984): "Futures Markets and Commodity Options: Hedging and Optimality in Incomplete Markets," *Journal of Economic Theory*, 32:275–300.

Brennan, M. J. (1991): "The Price of Convenience and the Valuation of Commodity Contingent Claims," in *Stochastic Models and Option Values*, Lund, D., and Oksendal, B. (eds.). Amsterdam: Elsevier.

Brennan, M. J., and Schwartz, E. S. (1990): "Arbitrage of Stock Index Futures," *Journal of Business*, 63:S7–S31.

Chang, J. S. K., and Fang, H. (1990): "An Intertemporal Measure of Hedging Effectiveness," *The Journal of Futures Markets*, 10:307–321.

Chan, K. C. (1992): "A Further Analysis of the Lead-Lag Relationship between the Cash Market and the Stock Index Futures Market," *The Review of Financial Studies*, 5:123–152.

Chung, Y. P., (1991): "A Transactions Data Test of Stock Index Futures Market Efficiency and Index Arbitrage Profitability," *Journal of Finance*, 46:1791–1809.

Ederington, L. (1979): "The Hedging Performance of the New Futures Markets," *Journal of Finance*, 34:157–170.

Figlewski, S. (1985): "Hedging with Stock Index Futures: Theory and Application in a New Market," *The Journal of Futures Market*, 2:183–199.

Ho, T. S. Y. (1984): "Intertemporal Commodity Futures Hedging and the Production Decision," *Journal of Finance*, 34:351–376.

Jarrow, R., and Madan, D., (1992): "Option Pricing Using the Term Structure of Interest Rates to Hedge Systematic Continuities in Asset Returns," *Mathematical Finance*, 1:31–44.

Jarrow, R., and Rosenfeld, E. (1984): "Jump Risks and the Intertemporal Capital Asset Pricing Model," *Journal of Business*, 57:337–351.

Johnson, L. L. (1960): "The Theory of Hedging and Speculation in Commodity Futures," *Review of Economic Studies*, 27:139–151.

Jorion, P. (1988): "On Jump Processes in the Foreign Exchange and Stock Markets," *Review of Financial Studies*, 1:427–446.

Junkus, J., and Lee, C. F., (1985): "Use of Three Stock Index Futures in Hedging Decisions," *The Journal of Futures Markets*, 5:201–222.

Kolb, R. W. (1994): *Futures, Options, and Swaps*. Miami: Kolb Publishing Company.

Mackinlay, A. C., and Ramaswamy, K. (1988): "Index-Futures Arbitrage and the Behavior of Stock Index Futures Prices," *Review of Financial Studies*, 1:137–158.

Merton, R. C. (1971): "Optimum Consumption and Portfolio Rules in a Continuous Time Model," *Journal of Economic Theory*, 3:373–413.

Merton, R. C. (1973): "An Intertemporal Capital Asset Pricing Model," *Econometrica*, 41:867–888.

Merton, R. C. (1976): "The Impact of Option Pricing of Specification Errors in the Underlying Stock Price Returns," *Journal of Finance*, 31:333–350.

Myers, R. J. (1991): "Estimating Time-Varying Optimal Hedge Ratios on Futures Markets," *The Journal of Futures Markets*, 11:39–53.

Schwert, G. W. (1989): "Why Does Stock Market Volatility Change Over Time?" *The Journal of Finance*, 44, 1115–1153.

Stein, J. (1961): "The Theory of Hedging and Speculation in Commodity Futures," *American Economic Review*, 9:347–372.

Stulz, R. M. (1984): "Optimum Hedging Policies," *Journal of Financial and Quantitative Analysis*, 19:127–140.

Vasicek, O. A. (1977): "An Equilibrium Characterization of the Term Structure," *Journal of Financial Economics*, 5:177–188.

Yau, J., Savanayana, U., and Schneeweis, T. (1990): "An Analysis of the Effectiveness of the Nikkei 225 Futures Contracts in Risk-Return Management," *Global Finance Journal*, 1:79–91.