Efficient Option-Implied Volatility Estimators

CHARLES J. CORRADO
THOMAS W. MILLER, Jr.

INTRODUCTION

This article presents an analysis of efficient methods to estimate option-implied volatilities. Currently, there is no generally accepted implied volatility estimation method, despite the widespread use of option-implied volatilities in financial markets research. For example, two implied volatility estimator types are popular. First, there are the simultaneous equations estimators originally suggested by Beckers (1981) and Whaley (1982). Second, there are the weighted average estimators originating with Latané and Rendleman (1976). Both estimator types use weighted data observations, although no general agreement exists regarding a preferred weighting scheme. This article derives efficiency criteria for each estimator type and shows that with appropriate weights, both estimator types attain the same variance lower bound.

This article first reviews existing concerns regarding biases in option-implied volatilities. It is argued that biases induced by time-varying stochastic volatilities are minimized when implied volatilities are estimated with the use of at-the-money options. This argument originated with Feinstein (1989), who pointed out that the Black-Scholes (1973) option

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- *Charles J. Corrado is an Associate Professor of Finance at the University of Missouri.*
- *Thomas W. Miller, Jr. is an Assistant Professor of Finance at the University of Missouri.*

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pricing model recovers virtually unbiased estimates of return standard deviations when return volatility is stochastic. Feinstein's claim is supported by the virtually linear relationship between option prices and return standard deviations for at-the-money European options. This study extends Feinstein's argument to American options, and also shows that implied volatility estimates are nearly indistinguishable across stock-option pricing models for at-the-money options. Thus, although much of the analysis is couched in a Black-Scholes framework, the discussion extends to many other stock option pricing models.

The subsequent econometric analysis exploits the virtually linear relationship between at-the-money option prices and return standard deviations to derive variance lower bounds for option-implied volatility estimators. It is shown that simultaneous equation estimators and weighted average estimators are potentially equally efficient, because with properly selected weights both estimator types attain an identical variance lower bound.

An example derivation of an optimal weighting scheme for an option-implied volatility estimator when stock index and index option price data are nonsynchronous is presented. This illustrates that a simple, equal-weighting scheme yields an efficient implied standard deviation estimator when only at-the-money put and call options data are used.

TIME-VARYING VOLATILITY AND UNBIASED IMPLIED VOLATILITY ESTIMATES

The Black–Scholes (1973) option pricing model specifies the following price for a call option on a non-dividend-paying share of common stock:

$$C = SN(d) - Ke^{-rT}N(d - \sigma\sqrt{T}),$$
$$d = \frac{\ln(S/Ke^{-rT}) + \sigma^2 T/2}{\sigma\sqrt{T}} \tag{1}$$

The stock price, strike price, interest rate, and time until option expiration are denoted by S, K, r, and T, respectively. The underlying stock return standard deviation is represented by σ, and $N(d)$ is the standard normal distribution function with the argument d. An important feature of the Black–Scholes model is that all model parameters, except the return standard deviation, are directly observable from market prices and terms specified by option contracts. This property allows a market-based estimate of return volatility to be obtained by numerically inverting the Black–Scholes formula to yield an implied volatility.

The Black–Scholes model assumes that stock log-prices follow a constant variance diffusion process. The constant variance assumption was tested and rejected in early studies by Beckers (1980), Black and Scholes (1972), Christie (1982), Officer (1973), and Schmalensee and Trippi (1978). Since then an extensive body of research has investigated the time-varying nature of volatility, most notably the conditional heteroskedasticity literature originating with Engle (1982) and Bollerslev (1986), and surveyed by Bollerslev, Chou, and Kroner (1992). Time-varying volatility in stock returns raises concerns regarding estimation biases embedded in Black–Scholes implied volatilities. These concerns and related issues are discussed below.

Deterministic Volatility

Merton (1976b) provides the first treatment of option pricing with time-varying volatility. He shows that when return variance is a deterministic function of time, the variance parameter in the Black–Scholes model corresponds to an average variance over the life of an option. If one lets $\sigma^2(t)$ denote an instantaneous variance at time t, an average variance over the life of an option expiring at time T is defined as

$$\bar{V} = \frac{1}{T}\int_0^T \sigma^2(t)\, dt \tag{2}$$

Day and Lewis (1988), Patell and Wolfson (1979), Resnick, Sheikh, and Song (1993), and others use Merton's extension to interpret their empirical results. However, this interpretation is subject to the criticism that stock return variances are stochastic.

Stochastic Volatility

Extending early work by Garman (1976), Hull and White (1987) analyze the effects of stochastic volatility. They assume a risk-neutral setting and a stochastic variance uncorrelated with stock prices, and their analysis yields this expression for the value of a call option:

$$C = \int C(\bar{V})h(\bar{V})\, d\bar{V} \tag{3}$$

In eq. (3), $\bar{V}$ is a random average variance over the life of an option, $h(\bar{V})$ is its probability density function, and $C(\bar{V})$ is a Black–Scholes call price conditional on a specific realization of $\bar{V}$. Expanding $C(\bar{V})$ in a Taylor series about an expected average variance, $E(\bar{V})$, produces the following expression for a stochastic volatility option price, where $Var(\bar{V})$ and $Skew(\bar{V})$ are the second and third central moments of $\bar{V}$:

$$C = C(E(\bar{V})) + \frac{1}{2}\frac{\partial^2 C}{\partial \bar{V}^2}\bigg|_{E(\bar{V})} Var(\bar{V}) + \frac{1}{6}\frac{\partial^3 C}{\partial \bar{V}^3}\bigg|_{E(\bar{V})} Skew(\bar{V}) + \cdots \quad (4)$$

With the use of this expansion, Hull and White show that substituting an expected average variance into the Black–Scholes formula overvalues at-the-money options and undervalues deep in-the-money and out-of-the-money options. However, Hull and White report that at-the-money overvaluation is slight, and Black–Scholes implied standard deviations are close to actual return standard deviations. Ball and Roma (1994), Eisenberg and Jarrow (1994), Stein and Stein (1991), and Wiggins (1987) also show that Black–Scholes implied standard deviations from at-the-money options are close to long-run average standard deviations in models where volatility follows a mean-reverting diffusion process. Similarly, Heynen, Kemna, and Vorst (1994) verify that Black–Scholes implied standard deviations recover accurate estimates of expected future return standard deviations when return variances are driven by simple GARCH and EGARCH processes.

Hull and White (1987) further show that when stock return variances are positively correlated with stock prices, the Black–Scholes model overvalues in-the-money options and undervalues out-of-the-money options. When the correlation is negative, the pattern of mispricing is reversed. Dothan (1987), Heston (1993), Johnson and Shanno (1987), Hull and White (1988), and Wiggins (1987) corroborate these results. However, these researchers all show that stochastic volatility option values are similar to Black–Scholes values for at-the-money options, and, therefore, each model yields implied standard deviations similar to Black–Scholes values for at-the-money options.

Merton's (1976a) jump-diffusion model provides a basic illustration of stochastic volatility when volatility is uncorrelated with stock prices. In Merton's model, an option expiration date stock log-price has the following conditional and unconditional variances:

$$\begin{aligned} E(\bar{V} \mid n) &= \delta^2 + n\gamma^2/T, \\ E(\bar{V}) &= \delta^2 + \lambda\gamma^2 \end{aligned} \quad (5)$$

The parameter, δ^2, is the variance of a log-price diffusion process, and γ^2 is the variance of a single log-price jump. The Poisson variable, n, represents a random number of jumps, where the expected number is $E(n)$

$= \lambda T$. Because the expected value, variance, and skewness of a Poisson variable are the same; that is, $E(n) = Var(n) = Skew(n)$ [Stuart and Ord (1987, p. 162)], the variance and skewness of an average variance are easily derived:

$$Var(\bar{V}) = \lambda\gamma^4/T, \qquad Skew(\bar{V}) = \lambda\gamma^6/T^2 \tag{6}$$

Substituting these parameters into the Hull and White expansion yields an accurate formula to calculate jump-diffusion option prices. Merton (1976b) shows, however, that the Black–Scholes option pricing formula yields reliable option prices for all but extreme jump processes. Ball and Torous (1985) empirically corroborate this result for option prices of individual stocks.

Bates (1991) develops a significant extension to jump-diffusion option pricing by allowing the embedded jump process to have a nonzero mean. He demonstrates that the Black–Scholes option pricing formula can produce substantial moneyness biases when the underlying jump process has a nonzero mean. However, in an empirical investigation of S&P 500 index option prices over the period January, 1985 through October, 1987, he also shows that implicit total volatilities obtained from a jump-diffusion model are typically indistinguishable from Black–Scholes implied volatilities.

Constant Elasticity of Variance (CEV)

Cox and Ross (1976) develop a constant elasticity of variance (CEV) diffusion model. In their CEV model, the natural logarithm of an instantaneous-return standard deviation is perfectly correlated with a stock log-price according to the log-linear relationship shown below, where $\sigma(t)$ and $S(t)$ denote an instantaneous-return standard deviation and stock price at time, t, respectively, and $\sigma(0)$ and $S(0)$ are initial values of these parameters.

$$\ln \sigma(t) = \ln \sigma(0) + (\psi - 1)\ln S(t)/S(0) \tag{7}$$

The Black–Scholes formula corresponds to the special case, $\psi = 1$. MacBeth and Merville (1980) show that where the stochastic process is a CEV diffusion and $\psi < 1$, the Black–Scholes model undervalues in-the-money calls and overvalues out-of-the-money calls, but yields similar values for at-the-money options. Emanuel and MacBeth (1982) show that where $\psi > 1$ this relative pricing pattern is reversed.

As shown in eq. (7), the return variance of a CEV model is a deterministic function of the stock price. Thus, the CEV model is not a true stochastic variance model. But because a CEV variance is instantaneously perfectly correlated with a stock price, CEV option prices are similar to stochastic variance option prices when return variances are strongly correlated with stock prices. For this reason, the CEV model is used in several examples below.

European Option Prices as Linear Functions of Return Standard Deviations

Feinstein (1989, 1992) points out that the Black–Scholes model recovers virtually unbiased estimates of return standard deviations when return volatility is stochastic. The basis for this assertion is illustrated in Cox and Rubinstein (1985, pp. 218–220), where Black–Scholes option prices are graphed as functions of return standard deviations. Comparative statics for this relationship can be analyzed algebraically through the behavior of an option's vega, defined as the first derivative of an option price with respect to its return standard deviation. A Black–Scholes vega is stated immediately below. The second derivative of an option price with respect to its return standard deviation is stated also.

$$\frac{\partial C}{\partial \sigma} = S\sqrt{T}n(d), \qquad \frac{\partial^2 C}{\partial \sigma^2} = \frac{\partial C}{\partial \sigma} d(d - \sigma\sqrt{T})/\sigma \tag{8}$$

In eq. (8), $n(d)$ is a standard normal density with the argument, d, defined in eq. (1). The second derivative is obtained using the equalities, $n'(d) = -dn(d)$ and $\partial d/\partial\sigma = (\sigma\sqrt{T} - d)/\sigma$.

Exact linearity in the relationship between an option price and a return standard deviation requires that the second derivative in (8) above be equal to zero. The sign of the second derivative is determined by the product, $d(d - \sigma\sqrt{T})$. Because the stock price, $S = K\,exp(-(r + \sigma^2/2)T)$, yields $d = 0$, and the price, $S = K\,exp(-(r - \sigma^2/2)T)$ yields $(d - \sigma\sqrt{T}) = 0$, vega is virtually constant in the neighborhood where $S = K\,exp(-rT)$. In this neighborhood options are said to be at the money. Consequently, at-the-money Black–Scholes option prices are virtually linear functions of underlying stock return standard deviations.

Feinstein's argument is important. When stock return volatility is stochastic and uncorrelated with stock prices, an implied standard deviation from an at-the-money option is a virtually unbiased estimator of an expected future return standard deviation. Furthermore, Harvey and

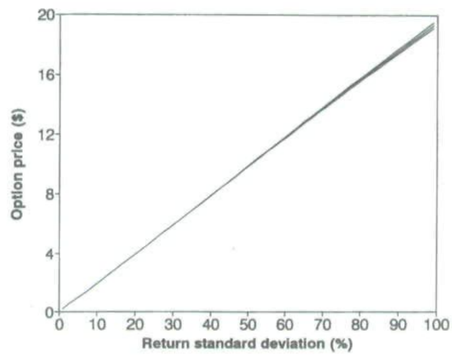

FIGURE 1

At-the-money European call option prices.

Whaley (1991) point out that an option vega is maximized for an at-the-money option, implying that at-the-money options are the most sensitive to changes in volatility, and therefore the most efficient for implied volatility estimation. Feinstein's extension of Merton (1973) is used by Day and Lewis (1992), Lamoureux and Lastrapes (1993), Stein (1989), and others to interpret their empirical results.

Figure 1 illustrates the virtually linear specification of Black–Scholes option prices, jump-diffusion option prices, and absolute diffusion option prices as functions of an underlying return standard deviation. This example is based on a strike price of $K = \$100$, a stock price equal to the discounted strike price; that is, i.e., $S = K\ exp(-rT)$, with an option expiration of $T = 3$ months, and an interest rate of $r = 5\%$. Annual return standard deviations range from 1%–99%. Jump-diffusion prices assume that a diffusion process and a jump process each contribute equally to total variance for the jump-diffusion process, and that the average time between log-normally distributed jumps is 3 months. Figure 1 reveals that at-the-money option prices from these three option pricing models are virtually linear functions of an underlying return standard deviation. Furthermore, for any given return standard deviation all option prices in Figure 1 are within 2% of one another, indicating that at-the-money option prices from these three option pricing models are all similar to one another.

The jump-diffusion and absolute diffusion models are two examples supporting a general observation that most European stock option pricing models yield option prices that are close to Black–Scholes prices for at-the-money options. Furthermore, most European stock option pricing models yield at-the-money option prices that are virtually linear functions

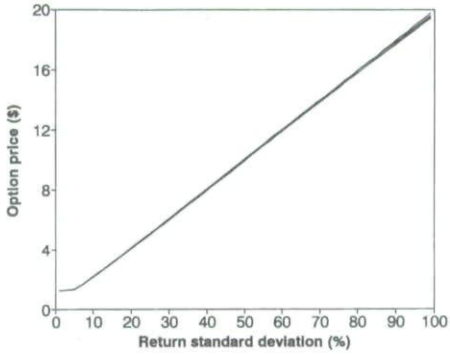

FIGURE 2

At-the-money American put option prices.

of an underlying return standard deviation. Consistent with this observation, Heynen et al. (1994), Johnson and Shanno (1987), Lamoureux and Lastrapes (1993), Rubinstein (1985), Scott (1987), Sheikh and Vora (1992), Stein and Stein (1991), and Wiggins (1987) show through extensive computer simulations that the Black–Scholes formula is a robust basis for obtaining implied standard deviations from at-the-money options. Jarrow and Wiggins (1989) reach a similar conclusion by analytical methods. The implication is that each of these models yields similar implied standard deviations from at-the-money options. In practice, the Black–Scholes model is generally preferred for reasons of simplicity and familiarity.

American Option Prices as Linear Functions of Return Standard Deviations

The Black–Scholes formula does not account for the early exercise value of American options. Because an American option price is not less than a European option price, Black–Scholes implied standard deviations obtained from American options are upwardly biased. Removing this bias requires an American option pricing model. As it turns out, the linear relationship between at-the-money option prices and return standard deviations is typically well preserved for American options. To illustrate this point, American put option values are calculated for several constant elasticity of variance (CEV) diffusion processes with the use of the modified binomial algorithm described in Gemmill (1993). Justification for this algorithm is provided by Nelson and Ramaswamy (1990), who show that a simple binomial representation of a CEV process converges in distribution to a true CEV diffusion process.

Figure 2 graphs American put option prices as functions of an underlying return standard deviation for CEV parameter values of $\psi = +2, +1, 0, -1$. The cases of $\psi = 1$ and $\psi = 0$ correspond to stock prices following geometric Brownian motion and arithmetic Brownian motion, respectively. These examples are based on a strike price of $K = \$100$, a stock price equal to the discounted strike price; that is, $S = K\,exp(-rT)$, an interest rate of $r = 5\%$, and an option expiration of $T = 3$ months.

Figure 2 reveals that at-the-money American put option prices are quite similar across CEV processes. For any given initial return standard deviation, that is, $\sigma(0)$, all option prices in Figure 2 are within 2% of one another. Also, for each CEV parameter value $\psi = +2, +1, 0, -1$, American put options display a virtually linear relationship between option prices and return standard deviations. This linear relationship is conjectured to be well preserved for most other American stock option pricing models.

Option Price Observation Errors

Boyle and Ananthanarayanan (1977), Butler and Schachter (1986), and Ingersoll (1976) point out that option prices embody observation errors that might cause biased implied standard-deviation estimates. However, the virtually linear relationship between at-the-money option prices and return standard deviations allays these concerns. For example, an option price based on a return standard deviation, σ, observed with error, ε, is expressed as $C = C(\sigma) + \varepsilon$. A basic reason why an error term, ε, might not be zero is that a correct option price could lie between a dealer's bid and asked prices, where the spread is a measure of friction in the trading process. Nevertheless, one might reasonably assume that rational options dealers set price quotes such that the expected error from the use of a simple average of bid and asked prices to represent a unique option price is not significantly different from zero. Even so, implied standard deviations are not, in general, unbiased estimates. This is shown by expanding an implied standard deviation (ISD) about a given return standard deviation (σ) in a Taylor series as

$$
\begin{aligned}
\mathrm{ISD} &= \sigma + \frac{\partial \sigma}{\partial C}\varepsilon + \frac{1}{2}\frac{\partial^2 \sigma}{\partial C^2}\varepsilon^2 + \cdots \\
\frac{\partial^2 \sigma}{\partial C^2} &= -\frac{\partial^2 C}{\partial \sigma^2}\left(\frac{\partial \sigma}{\partial C}\right)^3
\end{aligned}
\tag{9}
$$

Equation (9) reveals that even where expected errors are zero, squared errors might bias an implied standard deviation away from a correct volatility value. However, for at-the-money options the second derivative in (9) is close to zero and the expected bias is negligible for reasonable values of $Var(\varepsilon)$. Furthermore, Jarrow and Wiggins (1989) suggest that using implied volatilities from at-the-money options minimizes biases caused by transaction costs.

It is worth noting that the virtually linear relationship between at-the-money option prices and an underlying return standard deviation does not apply to all other option price parameters. For example, Cox and Rubinstein (1985) show that the functional relationship between option prices and an underlying stock price attains its greatest curvature when the stock price is in the neighborhood of the strike price. Consequently, mean-zero option price errors will produce a downward bias in implied stock prices obtained from at-the-money options. However, extensive evidence presented by Barone-Adesi and Whaley (1986), Kalaba, Langetieg, Weinstein, and Zubi (1987), and Kumar and Shastri (1990) suggests that implied option-price parameter biases are usually negligible.

In summary, the above section shows that when implied standard deviations are obtained from at-the-money options, any biases induced by stochastic volatilities and price observation errors are minimized. This conclusion supports the common research practice of using only options with strike prices close to a current cash price to obtain an implied standard deviation. For example, in addition to other studies cited elsewhere in this article, Chance (1986), Diz and Finucane (1993a), Fleming (1995), Jameson and Wilhelm (1992), Kleidon and Whaley (1992), Schwert (1990), and Xu and Taylor (1994) all use at-the-money options. In a stochastic volatility setting where return variances are uncorrelated with stock prices, at-the-money implied standard deviations can be interpreted as unbiased estimators of expected future return standard deviations. But where return variances are correlated with stock prices, this interpretation is not always correct. Nevertheless, even where return volatility is dependent on an underlying stock price, an at-the-money implied standard deviation still represents a virtually unbiased estimate of an initial return standard deviation for a variety of option pricing models. In the next section, minimum variance criteria are derived for implied standard deviation estimators, under the assumption that at-the-money implied standard deviations are unbiased.

EFFICIENT IMPLIED STANDARD DEVIATION (ISD) ESTIMATORS

Financial markets researchers most often obtain an implied standard deviation (ISD) by aggregating information across multiple option price observations. Commonly used estimator types include simultaneous equations estimators and weighted average estimators. The analysis attempts to discover which of these two estimator types is more efficient. It should be noted that an implied standard deviation from a single option price can be expanded in a Taylor series about a true return standard deviation as

$$\mathrm{ISD}_j = \sigma + \frac{\partial \sigma}{\partial C_j(\sigma)}(C_j - C_j(\sigma)) + \frac{1}{2}\frac{\partial^2 \sigma}{\partial C_j^2(\sigma)}(C_j - C_j(\sigma))^2 + \cdots \tag{10}$$

In eq. (10) and hereafter, observed option prices and theoretical option prices implied by the volatility value, σ, are represented by C_i and $C_j(\sigma)$, respectively. By definition of an implied standard deviation, an observed option price satisfies the identity, $C_j \equiv C_j(\mathrm{ISD}_j)$.

Based on Feinstein's (1989, 1992) argument and extensions discussed earlier, it is assumed that the second derivative, $\partial^2\sigma/\partial C^2$, deviates negligibly from zero for at-the-money options. This motivates ignoring second- and higher-order terms in eq. (10). It is shown that when only at-the-money options are used to estimate an implied standard deviation, simultaneous equations estimators and weighted average estimators have the same variance lower bound.

Simultaneous Equations Estimators

The variance of the generalized simultaneous equations estimator is derived as suggested by Beckers (1981). This estimator, denoted by σ^*, solves the following minimization problem:

$$\sigma^* = \underset{\sigma}{argmin} \sum_{j=1}^{N} W_j(C_j - C_j(\sigma))^2 \tag{11}$$

In eq. (11), N is the number of observed prices, C_j, in the sample. Theoretical option prices implied by the volatility value σ are stated as $C_j(\sigma)$. Weights that minimize the variance of the simultaneous equations estimator, $Var(\sigma^*)$, are selected. Before deriving variance-minimizing weights, the estimator variance is derived for an arbitrary set of weights.

For arbitrary weights, minimization of (11) with respect to σ implies the equality

$$\sum_{j=1}^{N} W_j(C_j - C_j(\sigma^*)) \frac{\partial C_j(\sigma^*)}{\partial \sigma} = 0 \tag{12}$$

Substituting a first-order Taylor expansion of option prices implied by the volatility value σ^*, that is, $C_j(\sigma^*)$, around prices implied by the true volatility value, σ, that is, $C_j(\sigma)$, into (12) yields

$$\sum_{j=1}^{N} W_j \left(C_j - C_j(\sigma) - \frac{\partial C_j(\sigma)}{\partial \sigma} (\sigma^* - \sigma) \right) \frac{\partial C_j(\sigma^*)}{\partial \sigma} = 0 \tag{13}$$

Substituting price error terms, $\varepsilon_j = C_j - C_j(\sigma)$, into (13) and rearranging yields the equality

$$\sum_{j=1}^{N} W_j \frac{\partial C_j(\sigma^*)}{\partial \sigma} \varepsilon_j = (\sigma^* - \sigma) \sum_{j=1}^{N} W_j \frac{\partial C_j(\sigma)}{\partial \sigma} \frac{\partial C_j(\sigma^*)}{\partial \sigma} \tag{14}$$

Squaring both sides of (14) and taking expected values yields this expression for the variance of a simultaneous equations ISD estimator:

$$Var(\sigma^*) = \frac{\displaystyle\sum_{j=1}^{N} \sum_{i=1}^{N} W_j W_i \frac{\partial C_j(\sigma^*)}{\partial \sigma} \frac{\partial C_i(\sigma^*)}{\partial \sigma} Cov(\varepsilon_j, \varepsilon_i)}{\left(\displaystyle\sum_{j=1}^{N} W_j \frac{\partial C_j(\sigma)}{\partial \sigma} \frac{\partial C_j(\sigma^*)}{\partial \sigma} \right)^2} \tag{15}$$

To facilitate algebraic development, the variance in (15) is restated in matrix notation. Based on a Cauchy–Schwartz inequality provided in Rao (1973), a variance lower bound is stated:

$$Var(\sigma^*) = \frac{\mathbf{W'P^*VP^*W}}{(\mathbf{W'PP^*1})} \geq \frac{1}{\mathbf{1'PV^{-1}P1}} \tag{16}$$

In eq. (16), $\mathbf{V}$ represents the covariance matrix of option price errors with elements denoted by $V_{ij} = Cov(\varepsilon_i, \varepsilon_j)$. $\mathbf{P}$ and $\mathbf{P}^*$ are matrices whose diagonal elements are option vegas; that is, $P_{jj} = \partial C_j(\sigma)/\partial \sigma$ and $P^*_{jj} = \partial C_j(\sigma^*)/\partial \sigma$. All off-diagonal elements of $\mathbf{P}$ and $\mathbf{P}^*$ are zero, and, therefore, $\mathbf{PP^*} = \mathbf{P^*P}$. $\mathbf{W}$ is a vector of weights.

The estimator variance in eq. (16) is equal to its lower bound for the weight vector, $\mathbf{W} = \mathbf{P^{*-1}V^{-1}P1}$, thereby defining the optimal variance-minimizing weights. This is demonstrated by referring to Rao (1973, p.

54), who shows that $\mathbf{X'AX}/(\mathbf{X'Y})^2 \geq 1/\mathbf{Y'A}^{-1}\mathbf{Y}$ holds with equality if $\mathbf{X} = \mathbf{A}^{-1}\mathbf{Y}$. If one lets $\mathbf{Y} = \mathbf{P^*P1}$, $\mathbf{X} = \mathbf{W}$, and $\mathbf{A} = \mathbf{P^*VP^*}$, one obtains the lower bound in (16).

Further insight is possible if sufficient linearity is allowed between theoretical option prices and standard deviation values such that the equality, $\mathbf{P^*} = \mathbf{P}$, is reasonably descriptive. This does not affect the variance lower bound, but allows stating optimal weights as $\mathbf{W} = \mathbf{P}^{-1}\mathbf{V}^{-1}\mathbf{P1}$. In this case, when option price errors, ε_j, are independent but heteroskedastic, the off-diagonal elements of $\mathbf{V}$ are zero, and it is easily verified that optimal weights are given by $\mathbf{W} = \mathbf{V}^{-1}\mathbf{1}$. In the important case of independent and identically distributed (i.i.d.) option price errors, equal weights are optimal; that is, $\mathbf{W} = \mathbf{1}$, and the estimator variance reduces to $Var(\sigma^*) = Var(\varepsilon)/\mathbf{1'P}^2\mathbf{1}$.

Harvey and Whaley (1991, 1992a) and Whaley (1982) assume i.i.d. option price errors, and, therefore, use a simultaneous equations estimator based on equal weights. Beckers (1981) uses vega weights; that is, $\mathbf{W} = \mathbf{P1}$, which implicitly assumes that $\mathbf{V} = \mathbf{P}^{-1}$ is the underlying covariance matrix. Day and Lewis (1988, 1992) weight each observation by the percentage of total daily trading volume represented by options in the same moneyness class. Volume weights effectively resemble vega weights, because trading volume is typically greatest for at-the-money options and diminishes as strike prices move away from cash prices.

Weighted Average Estimators

The variance of a weighted average ISD estimator is derived next. First suggested by Latané and Rendleman (1976), a weighted average estimator is based on implied standard deviations, ISD_j, separately obtained from individual options. Imposing the restriction that the weights, U_j, sum to one, a weighted average estimator is expressed as follows:

$$\bar{\sigma} = \sum_{j=1}^{N} U_j \mathrm{ISD}_j \tag{17}$$

Because $C_j \equiv C(\mathrm{ISD}_j)$ is an identity, a first-order Taylor expansion yields this expression for a weighted average ISD estimator:

$$\bar{\sigma} = \sigma + \sum_{j=1}^{N} U_j \frac{\partial \sigma}{\partial C_j(\sigma)} (C_j - C_j(\sigma)) \tag{18}$$

Substituting error terms, $\varepsilon_j = C_j - C_j(\sigma)$, into (18), subtracting σ from both sides, squaring, and taking expected values yields this expression for the variance of a weighted average estimator:

$$Var(\bar{\sigma}) = \sum_{j=1}^{N} \sum_{i=1}^{N} U_j U_i \frac{\partial \sigma}{\partial C_j(\sigma)} \frac{\partial \sigma}{\partial C_i(\sigma)} Cov(\varepsilon_j, \varepsilon_i) \tag{19}$$

Restating (19) in matrix notation and stating a variance lower bound yields

$$Var(\bar{\sigma}) = \frac{\mathbf{U}'\mathbf{P}^{-1}\mathbf{V}\mathbf{P}^{-1}\mathbf{U}}{(\mathbf{1}'\mathbf{U})^2} \geq \frac{1}{\mathbf{1}'\mathbf{P}\mathbf{V}^{-1}\mathbf{P}\mathbf{1}} \tag{20}$$

The lower bound in (20) is obtained from the inequality, $\mathbf{X}'\mathbf{A}\mathbf{X}/(\mathbf{X}'\mathbf{Y})^2 \geq 1/\mathbf{Y}'\mathbf{A}^{-1}\mathbf{Y}$, and letting $\mathbf{Y} = \mathbf{1}$, $\mathbf{X} = \mathbf{U}$, and $\mathbf{A} = \mathbf{P}^{-1}\mathbf{V}\mathbf{P}^{-1}$. The weighted average estimator variance in eq. (20) is equal to its lower bound for the weight vector, $\mathbf{U} = \mathbf{P}\mathbf{V}^{-1}\mathbf{P}\mathbf{1}/\mathbf{1}'\mathbf{P}\mathbf{V}^{-1}\mathbf{P}\mathbf{1}$, thereby defining its optimal variance-minimizing weights.

Comparison of Simultaneous Equations and Weighted Average Estimators

The variance lower bound for weighted average estimators in eq. (20) is equal to the variance lower bound for simultaneous equations estimators in eq. (16). This demonstrates that weighted average and simultaneous equations ISD estimators are potentially equally efficient. In general, the transformation of optimal weights for the simultaneous equations estimator to optimal weights for the weighted average estimator is stated as follows:

$$\mathbf{U} = \frac{\mathbf{P}\mathbf{P}^{*}\mathbf{W}}{\mathbf{1}'\mathbf{P}\mathbf{V}^{-1}\mathbf{P}\mathbf{1}} \tag{21}$$

Awareness of differing optimal weights for simultaneous equations estimators and weighted average estimators is not always reflected in the literature. For example, Latané and Rendleman (1976) and Merville and Pieptea (1989) use a vega-weighted average ISD estimator. A vega-weighted average ISD estimator is defined by $\mathbf{U} = \mathbf{P}\mathbf{1}/\mathbf{1}'\mathbf{P}\mathbf{1}$ and optimal weights for a weighted average ISD estimator are specified by $\mathbf{U} = \mathbf{P}\mathbf{V}^{-1}\mathbf{P}\mathbf{1}/\mathbf{1}'\mathbf{P}\mathbf{V}^{-1}\mathbf{P}\mathbf{1}$. By assuming vega weights are optimal, one can set the vector of vega weights equal to the vector of optimal weights to obtain an implicitly assumed covariance matrix, $\mathbf{V} = \mathbf{P}$. This covariance matrix has diagonal elements equal to option vegas, and zero off-diagonal elements. Because option vegas are greatest for at-the-money options, a vega-weighted average ISD estimator implicitly assumes that option price errors are *greatest* for at-the-money options. By contrast, Beckers (1981) uses a vega-weighted simultaneous equations estimator, which, as shown earlier, implicitly assumes the covariance matrix, $\mathbf{V} = \mathbf{P}^{-1}$. This covariance matrix has diagonal elements equal to inverses of option vegas, and zero off-diagonal elements. Here, a vega-weighted simultaneous equations ISD estimator implicitly assumes that option price errors are *smallest* for at-the-money options.

Similar analysis reveals that an equally weighted average ISD estimator; that is, $\mathbf{U} = \mathbf{1}/\mathbf{1}'\mathbf{1}$, implicitly assumes the covariance matrix, $\mathbf{V} = \mathbf{P}^2$, where option price errors are greatest for at-the-money options. However, an equally weighted simultaneous equations estimator; that is, $\mathbf{W} = \mathbf{1}$, is optimal for i.i.d. option price errors. For i.i.d. option price errors, the weight vector, $\mathbf{U} = \mathbf{P}^2\mathbf{1}/\mathbf{1}'\mathbf{P}^2\mathbf{1}$, is optimal for the weighted average ISD estimator.

Chiras and Manaster (1978) and Resnick, Sheikh, and Song (1993) use a weighted average ISD estimator based on option price-volatility elasticities. This estimator uses the weight vector, $\mathbf{U} = \mathbf{PO}^{-1}\mathbf{1}/\mathbf{1}'\mathbf{PO}^{-1}\mathbf{1}$, where $\mathbf{O}$ is a matrix with option price diagonal elements, and zero off-diagonal elements. Setting this weight vector equal to the optimal weight vector, $\mathbf{U} = \mathbf{PV}^{-1}\mathbf{P1}/\mathbf{1}'\mathbf{PV}^{-1}\mathbf{P1}$, yields the implicitly assumed covariance matrix, $\mathbf{V} = \mathbf{PO}$. Because the elements of $\mathbf{PO}$ monotonically decrease with an underlying strike price, elasticity weights implicitly assume that option price errors are smallest for out-of-the-money options and largest for in-the-money options.

In summary, optimal weights for the simultaneous equations ISD estimator are derived as

$$\mathbf{W} = \mathbf{P}^{-1}\mathbf{V}^{-1}\mathbf{P1} \tag{22}$$

and optimal weights for the weighted average ISD estimator are derived as

$$\mathbf{U} = \frac{\mathbf{PV}^{-1}\mathbf{P1}}{\mathbf{1}'\mathbf{PV}^{-1}\mathbf{P1}} \tag{23}$$

Weights for each estimator type are distinct. However, weights for one estimator type are a simple transformation of weights for the other type. Also, variances for both estimator types have the same attainable lower bound, and, therefore, are potentially equally efficient. Finally, optimal weights are uniquely determined by a covariance matrix of option price errors. Conversely, optimal weights implicitly assume a specific price error covariance matrix.

AN APPLICATION TO STOCK INDEX OPTIONS

As a specific application of the analysis in the previous section, an optimal implied standard deviation estimator is developed for stock index options where index option prices are not perfectly synchronous with an underlying index. The problem of nonsynchronous stock and option prices, first noted by Galai (1977) and recently examined in Boyle and Park (1994), is common in studies that rely on data from the financial press, for example, Choi and Wohar (1992), Day and Lewis (1988, 1992), and Park and Sears (1985). Data in the financial press exacerbate the nonsynchronous price problem because options trading in Chicago (CBOE) ends 15 minutes after the close of stock trading in New York (NYSE). Harvey and Whaley (1991) describe how asynchronous market closings cause negative correlation between call option and put option implied volatilities. Essentially, if good (bad) news reaches financial markets after a closing stock index level is recorded at 3:00 P.M., but before closing index options prices are recorded at 3:15 P.M., then implied call volatilities will be higher (lower) than implied put volatilities.

The problem of nonsynchronous index option prices and index levels is not limited to financial press data. Whaley (1993) points out that even with intraday data, reported stock index option prices lead concurrently reported stock index levels when index options trade more frequently than many stocks in the index. When significant news reaches financial markets, heavily traded index options immediately reflect the news, whereas stock indexes do not fully reflect the news until all stocks in the index have traded. Consequently, implied call volatilities are higher (lower) under good (bad) news than implied put volatilities.

As argued earlier, at-the-money options minimize bias. Therefore, in this example it is assumed that only four option contracts with the same expiration date are used to estimate an implied standard deviation. These are the two put/call pairs with discounted strike prices that bracket a dividend-adjusted cash price; that is, $K_L\, exp(-rT) < S\, exp(-\delta T) < K_H\, exp(-rT)$, where S is a currently observed index level and δ is the index dividend yield. For simplicity, a constant dividend yield and interest rate are assumed. In the special case where the dividend yield is equal to the interest rate, this at-the-money condition reduces to the simple form, $K_L < S < K_H$.

To capture the nonsynchronous price problem the following specification is assumed for call and put option price errors, denoted by ε_C and ε_P, respectively:

$$\varepsilon_C = \eta_C + \frac{\partial C}{\partial S}\, \Delta S(h_C), \qquad \varepsilon_P = \eta_P + \frac{\partial P}{\partial S}\, \Delta S(h_P) \tag{24}$$

In eq. (24), the first price error components, η_C and η_P, are noise terms assumed to be independent of each other and the underlying index. The second price error components are linear approximations of errors in option prices owing to unobserved changes in stock index levels over the intervals, h_C and h_P.

For the case of financial press data, unobserved index level changes between 3:00 and 3:15 P.M. are denoted by $\Delta S(h)$, where $Var(\Delta S(h)) = \sigma^2 S^2 h$ and $h = 15$ minutes. It is assumed that the put/call pair option prices are all current when recorded at 3:15. The 3:15 index option prices are based on an unobserved 3:15 index level, and, therefore, are in error with respect to the recorded 3:00 index level. Thus, abstracting from index staleness effects, the intervals, $h_C = h_P = h$, are all equal to 15 minutes for the four options considered.

Alternatively, for the case of intraday price data, the quantity, $\Delta S(h)$, is a measure of index staleness representing the difference between a reported index level based on last trade stock prices and an actual, but unrecorded, index level based on current stock prices. Assuming put/call pair option prices are observed simultaneously and are current when observed, the quantity, $\Delta S(h)$, is the same for all four options. Therefore, $h_C = h_P = h$ in this case also. However, h now corresponds to a time measure of index staleness, say, for example, the expected time interval over which $x\%$ of the stocks in the index trade at least once. To be consistent with earlier formulation, it is assumed that the measure for h is chosen to satisfy the equality, $Var(\Delta S(h)) = \sigma^2 S^2 h$. Within this framework, the case of 15-minute asynchronous stock and option market closings can be interpreted as simply a severe case of index staleness.

Assuming for simplicity that independent price error components have the same variance; that is, $Var(\eta_C) = Var(\eta_p) = Var(\eta)$, price error variances and covariances are then specified as

$$Var(\varepsilon_C) = Var(\eta) + \left(\frac{\partial C}{\partial S}\right)^2 \sigma^2 S^2 h$$

$$Var(\varepsilon_P) = Var(\eta) + \left(\frac{\partial P}{\partial S}\right)^2 \sigma^2 S^2 h$$

$$Cov(\varepsilon_C, \varepsilon_P) = \frac{\partial C}{\partial S}\frac{\partial P}{\partial S}\sigma^2 S^2 h \qquad (25)$$

Harvey and Whaley (1992a) suggest the magnitude of closing price asynchronicity errors. They report a return standard deviation of 0.18% for S&P 500 futures contracts over the last 15 minutes of daily trading. In the covariance term specified in eq. (25), this corresponds to $\sigma\sqrt{h} =$

0.0018. Assuming an index level of $S = 400$ and an at-the-money option where $\partial C/\partial S$ = one-half, the standard error component due to nonsynchronous prices is 36¢, which is significant. By comparison, George and Longstaff (1993) report average bid-ask spreads between 10.6¢ and 27.2¢ for at-the-money puts and calls on S&P 100 index options.

To facilitate algebraic development, European-style options are assumed to be in a setting where the Black–Scholes model applies. For American options, the analysis developed below is approximate, because the value of early exercise for at-the-money index options, while small, is not zero. For example, Harvey and Whaley (1992b) show that the early exercise premium for at-the-money American call options on the S&P 100 index is 2–3¢, on average. Substituting the terms, $\partial C/\partial S = N(d)$ and $\partial P/\partial S = N(d) - 1$, according to the Black–Scholes model into (25) yields this form for the price error covariance matrix, V:

$$\mathbf{V} = Var(\eta)\mathbf{I} + \sigma^2 S^2 h\mathbf{N} \tag{26}$$

In eq. (26), **I** is an identity matrix and the symmetric matrix, N, is defined as

$$\mathbf{N} = \begin{pmatrix} N^2(d_H) & N(d_H)(N(d_H) - 1) & N(d_H)N(d_L) & N(d_H)(N(d_L) - 1) \\ \cdots & (N(d_H) - 1)^2 & (N(d_H - 1)N(d_L) & (N(d_H) - 1)(N(d_L) - 1) \\ \cdots & \cdots & N^2(d_L) & N(d_L)(N(d_L) - 1) \\ \cdots & \cdots & \cdots & (N(d_L) - 1)^2 \end{pmatrix} \tag{27}$$

The matrix, N, multiplied by the scalar, $\sigma^2 S^2 h$, specifies the influence of nonsynchronous prices on the price error covariance matrix. The arguments, $d_H < d_L$, correspond to the higher and lower strike prices, $K_H > K_L$, respectively.

As stated in eqs. (22) and (23), optimal weights for a simultaneous equations estimator are $\mathbf{W} = \mathbf{P}^{-1}\mathbf{V}^{-1}\mathbf{P1}$, and optimal weights for a weighted average estimator are $\mathbf{U} = \mathbf{PV}^{-1}\mathbf{P1}/\mathbf{1}'\mathbf{PV}^{-1}\mathbf{P1}$, respectively. These optimal weights are a function of the price error covariance matrix, **V**. Selecting weights without reference to this covariance matrix can result in a loss of estimator efficiency. For the application examined here, however, when options that satisfy the at-the-money condition, $K_L\ exp(-rT) < S\ exp(-\delta T) < K_H\ exp(-rT)$, are used to estimate an implied standard deviation, the efficiency loss of an equally weighted estimator is small. To demonstrate this point in the current index option example, the efficiency of estimators based on equal weights is compared to an optimal estimator. As measures of efficiency, let $SE(\sigma^E)$ denote the standard error of a simultaneous equations estimator with equal weights, i.e., $W = \mathbf{1}$:

$$SE(\sigma^E) = \sqrt{\frac{\mathbf{1'PVP1}}{(\mathbf{1'P^2 1})^2}} \tag{28}$$

Also, let $SE(\bar{\sigma}^E)$ denote the standard error of an equally weighted average estimator; that is, $\mathbf{U} = \mathbf{1/1'1}$:

$$SE(\bar{\sigma}^E) = \sqrt{\frac{\mathbf{1'P^{-1}VP^{-1}1}}{(\mathbf{1'1})^2}} \tag{29}$$

From eqs. (16) and (20), the standard error lower bound (SELB) of an optimal simultaneous equations or a weighted average estimator is

$$SELB = \sqrt{\frac{1}{\mathbf{1'PV^{-1}P1}}} \tag{30}$$

Estimator standard errors specified in equations (28)–(30) are compared across a range of index levels around the strike prices, $K_H = 405$ and $K_L = 400$. For this example, the interest rate is set at $r = 5\%$ and an option maturity of $T = 1$ month. For simplicity, the dividend yield is assumed to be equal to the interest rate; that is, $\delta = r$, thereby reducing the at-the-money condition to $K_L < S < K_H$. To specify the independent price error variance, $Var(\eta)$, it is assumed that the correct option price occupies a random position uniformly distributed between bid and asked prices. Thus, when bid-ask price averages are available, the independent price error variance is $Var(\eta) = (\text{Ask} - \text{Bid})^2/12$. However, the financial press reports last sale prices. Assuming that a last sale price is equally likely to be either a bid price or an asked price, the independent price error variance is $Var(\eta) = (\text{Ask} - \text{Bid})^2/3$. Based on the George and Longstaff (1993) measurements, the bid-ask spread is assumed to be \$0.272, which implies that $Var(\eta) = 0.02466$ for last sale prices. To specify the impact of nonsynchronous stock index and index option prices on the price error covariance matrix, the Harvey and Whaley (1992a) measurement of S&P 500 futures volatility at the end of the trading day is used, which implies that $\sigma\sqrt{h} = 0.0018$. A true index return standard deviation of $\sigma = 0.20$ is assumed.

Estimator standard errors based on parameter values stated above are reported in Table I for index levels ranging from $S = 395$ to $S =$

TABLE I
Implied Standard Deviation (ISD) Estimator Standard Errors

Stock Index Level	*Optimal Estimator (Standard Error Lower Bound)*	*Equally Weighted Simultaneous Equations Estimator*	*Equally Weighted Average Estimator*	*Calls-Only Estimator*
	ISD estimate standard errors in basis points			
395	18.57	26.12	26.48	64.71
396	18.19	23.92	24.20	66.41
397	17.87	21.96	22.16	68.26
398	17.60	20.27	20.41	70.26
399	17.39	18.89	18.97	72.43
400	17.23	17.87	17.90	74.76
401	17.12	17.25	17.26	77.27
402	17.06	17.07	17.07	79.96
403	17.05	17.32	17.34	82.83
404	17.09	18.00	18.05	85.89
405	17.18	19.07	19.16	89.16
406	17.32	20.47	20.62	92.62
407	17.50	22.15	22.38	96.29
408	17.73	24.08	24.39	100.17
409	18.02	26.22	26.61	104.25
410	18.35	28.56	29.04	109.55

Comparison of standard errors for implied standard deviation (ISD) estimators based on two put/call option pairs with strike prices of 400 and 405, including an optimal estimator, an equally weighted simultaneous equations estimator, and an equally weighted average estimator. The calls-only estimator is based on two call options with strike prices of 400 and 405. The true market index return standard deviation is 20%, the interest rate is 5%, and the option maturity is 1 month. All ISD estimator standard errors are stated in basis points.

410. For each index level listed in Column 1, Column 2 lists the corresponding standard error lower bound of an optimal estimator; Column 3 lists the standard errors of an equally weighted simultaneous equations estimator; and Column 4 reports the standard errors of an equally weighted average estimator. To assess the merits of using put/call pairs, Column 5 lists the standard errors of an optimally weighted estimator based only on the two call options with strike prices, $K_H = 405$ and $K_L = 400$. Optimal weights for the calls-only estimator are determined by a calls-only price error covariance matrix that specifies the influence of nonsynchronous stock and option prices by this matrix:

$$\mathbf{N}_C = \begin{pmatrix} N^2(d_H) & N(d_H)N(d_L) \\ \cdots & N^2(d_L) \end{pmatrix} \tag{31}$$

The matrix, $\mathbf{N}_C$, in eq. (31) corresponds to the calls-only components of the matrix, N, defined in eq. (27).

As shown in Table I, optimal put/call estimator standard errors range from 17.05 basis points ($S = 403$) to 18.57 basis points ($S = 395$).

Equally weighted put/call estimator standard errors are somewhat larger, but for all index levels they are less than 30 basis points. Importantly, for index levels between the strike prices, $K_H = 405$ and $K_L = 400$, equally weighted put/call estimator standard errors are not larger than 20 basis points, suggesting that the efficiency loss of an equally weighted estimator is relatively small when used with at-the-money options. In contrast to the put/call estimators, the calls-only estimator standard errors are larger by several magnitudes, and monotonically increase from 64.71 basis points ($S = 395$) to 108.55 basis points ($S = 410$). Calls-only estimator standard errors monotonically increase because each element of the matrix, N_C, monotonically increases as the index level rises.

Stein (1989) first suggested the use of put/call pairs to mitigate the effects of nonsynchronously recorded stock index and index option prices. Essentially, nonsynchronous price errors for put and call options are near-perfectly negatively correlated, and their opposing effects almost perfectly cancel each other. To demonstrate this point in the context of the current index option example, suppose that an implied standard deviation is estimated with the use of only two call options with all parameter inputs the same as stated above. In this case, when $S = 403$, the standard error of a calls-only estimator is 82.83 basis points, which represents a considerable efficiency loss compared to estimators based on put/call pairs. Alternatively, suppose that all stock and option prices are perfectly synchronous, that is, $h = 0$, with all other parameters the same as stated above. In this case, when $S = 403$, the standard error of an optimal estimator is 17.03 basis points, which is only slightly smaller than the corresponding value of 17.05 basis points reported in Table I for an optimal put/call pair estimator. This suggests that the decision to use put/call pairs is important. Among studies using put/call pairs to estimate market index implied volatilities, Choi and Wohar (1992) use three put/call pairs with strike prices closest to a current cash price, and Diz and Finucane (1993b) and Stein (1989) use a single at-the-money put/call pair. Fleming, Ostdiek, and Whaley (1993) use the CBOE volatility index (VIX), which represents the implied volatility of a hypothetical at-the-money S&P 100 index option with 30 calendar days to expiration. VIX values are calculated from simultaneously recorded S&P 100 index levels and bid-ask option price averages for four put/call pairs (Whaley, 1993).

In summary, market index implied standard deviations that are estimated with the use of put/call pairs that satisfy the at-the-money condition, $K_L\, exp(-rT) < S\, exp(-\delta T) < K_H\, exp(-rT)$, yield relatively efficient implied standard deviation estimates from either an equally weighted simultaneous equations estimator or an equally weighted average estimator. Using put/call pairs mitigates the effects of nonsynchronous stock index and index option prices associated with daily closing prices. Nonsynchronous price effects are smaller when intraday data are available. However, Harvey and Whaley (1991) report noticeably improved estimator efficiency by aggregating intraday put and call prices. Important applications of put/call pairs to calculate market index implied volatilities include the CBOE volatility index (VIX) mentioned earlier, and implied volatility time-series data supplied by the Chicago Mercantile Exchange (CME).

Unfortunately, even with intraday data, mitigating the effects of nonsynchronous prices by using put/call pairs is more difficult with individual stock options. For example, the Berkeley option tapes report a last trade stock price with each update of an option price quote. To capture the benefits of put/call pairs, one must assume that option price quote updates occur somewhat more frequently than the underlying stock trades. Only under this assumption will the intervals, h_C and h_P, regularly overlap and lead to a systematic negative correlation between the price staleness measures, $\Delta S(h_C)$ and $\Delta S(h_P)$. When this assumption is not satisfied, the intervals, h_C and h_P, do not regularly overlap, and, therefore, the corresponding price staleness measures, $\Delta S(h_C)$ and $\Delta S(h_P)$, are typically uncorrelated and the off-diagonal elements of the covariance matrix, V, are equal to zero. However, the general analysis in the previous section still applies, because where the off-diagonal elements of V are zero, the optimal weights for a simultaneous equations estimator are given by $\mathbf{W} = \mathbf{V}^{-1}\mathbf{1}$.

SUMMARY AND CONCLUSION

An econometric analysis of efficient methods to estimate option-implied volatilities is presented. In this analysis, the virtually linear relationship between at-the-money option prices and return standard deviations is exploited to derive variance lower bounds for two popular implied standard deviation (ISD) estimator types: simultaneous equations ISD estimators, and weighted average ISD estimators. It is shown that simultaneous equations ISD estimators and weighted average ISD estimators have the same attainable variance lower bound, and, therefore, are equally efficient when used with appropriate weights.

Feinstein (1989, 1992) argues that the Black–Scholes (1973) option pricing model recovers virtually unbiased estimates of return standard deviations when return volatility is stochastic. Feinstein's argument is extended to alternative European and American option pricing models by showing that implied volatility estimates are nearly indistinguishable across many option pricing models for at-the-money options. It is shown also that implied standard deviations are not always unbiased estimates of future expected return standard deviations, but still represent virtually unbiased estimates of initial return standard deviations for a variety of option pricing models that allow dependencies between stock prices and volatilities.

Finally, an example derivation is provided of an efficient weighting scheme for a market index implied volatility estimator when option and stock price data are nonsynchronous. This example illustrates that a simple, equal weighting scheme is relatively efficient when put/call pairs of at-the-money stock index options are used to obtain implied standard deviation estimates.

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