On the Conventional Definition of Currency Hedge Ratio

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INTRODUCTION

The standard textbook example of futures hedging incorporates a cash asset and a futures contract. The hedge ratio is defined as the ratio of the futures position to the cash position. Moreover, the minimum variance hedge ratio prescribes the number of futures contracts required for a unit of cash position to minimize the risk (variance). To determine the usefulness of the futures contract as a hedging vehicle, one relies upon the concept of hedging effectiveness, which is defined as the proportional risk reduction generated by the risk-minimizing hedge strategy. Under certain conditions, a monotonic relationship can be established between hedge ratio and hedging effectiveness.

When extending the above definition of hedge ratio to the case of multiple cash assets, one needs to choose among many cash assets to calculate the hedge ratio. This is unavoidable, as there is only one futures position but many cash positions. For example, in currency hedging there is only one foreign currency futures but two cash assets (i.e., a domestic asset and a foreign asset). The convention in the institutional investment

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industry is to define the hedge ratio as the ratio of the futures position to the cash position of the foreign asset; the cash position of the domestic asset has no bearing whatsoever.

This article attempts to evaluate the appropriateness of the conventional definition of hedge ratio when there are multiple cash assets. The problem of such a definition is highlighted in Kritzman (1993). He shows that under the most likely circumstance the minimum variance hedge ratio decreases with foreign asset exposure. In other words, the more foreign assets (either in units or in allocations) the hedger has, the smaller is the resultant hedge ratio. Kritzman concluded that "it is more relevant to hedge currency risk for small allocations to foreign assets than for large allocations."

Note that Kritzman's conclusion is based upon the premise that the magnitude of the conventional hedge ratio is positively related to the usefulness of the futures contract. Although this positive relationship is found in the case of a single cash asset, no such relationship exists when there are at least two cash assets. Consequently, the comparison of hedge ratios tells nothing about the relative importance of currency hedging. In fact, the conventional hedge ratio (defined as the ratio of the futures position to the cash position of the foreign asset) lacks the usual meaning, in the sense that it does not prescribe the amount of futures contracts required to match a unit of foreign asset. This property needs to be treated carefully when interpreting any statements regarding the conventional hedge ratio.

MINIMUM-VARIANCE HEDGE RATIO

Consider a single-period world. A hedger takes positions in two cash assets at the beginning of the period and lifts them at the end of the period. Let W_1 and W_2 denote the two positions; and let P_1 and P_2 denote the cash prices at the end of the period for asset 1 and 2, respectively. There is a futures contract available, the trading of which is based on the second asset. Let V denote the futures position, and let f denote the price of the futures contract at the end of the period. The cash positions are assumed to be nontradeable. The hedger then chooses the futures position to minimize his overall risk, which is measured by the variance of $(P_1W_1 + P_2W_2 + fV)$. The resultant minimum-variance futures position is

$$V^* = [-\mathrm{Cov}(P_1, f)/\mathrm{Var}(f)]W_1 + [-\mathrm{Cov}(P_2, f)/\mathrm{Var}(f)]W_2 \quad (1)$$

Alternatively, V^* may be written as $V_1^* + V_2^*$ with

$$V_i^* = -[\mathrm{Cov}(P_i, f)/\mathrm{Var}(f)]W_i, \quad i = 1, 2 \tag{2}$$

where V_i^* is the minimum-variance futures position for a direct or cross hedge of the ith cash asset. When there is only one cash position, $-V_i^*/W_i$ is a well-defined hedge ratio.

In the context of foreign asset exposure and currency hedging, the first asset is referred to as the domestic asset, whereas the second asset is referred to as the foreign asset. The institutional investment industry defines this hedge ratio as $-V/W_2$. Thus, the minimum-variance hedge ratio is

$$\begin{aligned} h^* &= -V^*/W_2 \\ &= [\mathrm{Cov}(P_1, f)/\mathrm{Var}(f)](W_1/W_2) + \mathrm{Cov}(P_2, f)/\mathrm{Var}(f) \end{aligned} \tag{3}$$

It is easily seen from eq. (3) that h^* increases whenever W_2 decreases, provided $\mathrm{Cov}(P_1, f) > 0$. That is, if the price of the domestic asset is positively correlated with the price of the currency futures contract, the minimum-variance hedge ratio decreases with an increasing cash position in the foreign asset. Alternatively, let $W = W_1 + W_2$ be the total asset holding and let s denote the allocation to the domestic asset. Thus, $W_1 = sW$ and $W_2 = (1 - s)W$. This means that the minimum-variance hedge ratio decreases with increasing foreign asset exposure. Kritzman (1993) concluded, "it is more relevant to hedge currency risk for small allocations to foreign assets than for large allocations." His conclusion appears to be counterintuitive.

HEDGING EFFECTIVENESS

Nevertheless, the relevance of futures hedging should be evaluated on the basis of its effect rather than on the magnitude of the hedge ratio. A popular measure for the effect is the so-called hedging effectiveness [Ederington (1979)]. It represents the reduction in the variance of the portfolio (in percentage relative to the variance of the cash portfolio). For the case considered here, the variance of the cash portfolio is $\mathrm{Var}((sP_1 + (1 - s)P_2)W)$ and the variance of the total portfolio is $\mathrm{Var}((sP_1 + (1 - s)P_2)W + fV)$. From eq. (1), one can show that the minimum-variance futures position reduces the risk by

$$\begin{aligned} \Delta &= \mathrm{Var}((sP_1 + (1 - s)P_2)W) - \mathrm{Var}((sP_1 + (1 - s)P_2)W + fV^*) \\ &= W^2\{s\,\mathrm{Cov}(P_1, f) + (1 - s)\mathrm{Cov}(P_2, f)\}^2/\mathrm{Var}(f) \end{aligned} \tag{4}$$

Note that the trading of the futures contract is based on the second asset. It is, therefore, reasonable to assume that $\mathrm{Cov}(P_2, f) > 0$. As a result, Δ increases with s when $\mathrm{Cov}(P_1, f) > \mathrm{Cov}(P_2, f)$, and Δ decreases with s otherwise. That is, the risk reduction produced by the minimum-variance hedge strategy increases (decreases) with foreign asset exposure when $\mathrm{Cov}(P_1, f)$ is larger (smaller) than $\mathrm{Cov}(P_2, f)$.

To derive the hedging effectiveness (HE), the risk reduction is divided by the variance of the cash portfolio:

$$\begin{aligned}\mathrm{HE} =& \{s\,\mathrm{Cov}(P_1, f)\\ &+ (1-s)\mathrm{Cov}(P_2, f)\}^2/\{\mathrm{Var}(f)\mathrm{Var}(sP_1 + (1-s)P_2)\}\end{aligned} \tag{5}$$

The effect of different foreign asset exposures can be examined through the first derivative of HE with respect to s:

$$d(\mathrm{HE})/ds = 2A[(B_1 + B_2)s - B_1]\{\mathrm{Var}(f)\}^{-1} \tag{6}$$

where

$$A = (s\,\mathrm{Cov}(P_1, f) + (1-s)\mathrm{Cov}(P_2, f))/\{\mathrm{Var}(sP_1 + (1-s)P_2)\}^2 \tag{6a}$$

$$B_1 = \mathrm{Cov}(P_2, f)\mathrm{Cov}(P_1, P_2) - \mathrm{Var}(P_2)\mathrm{Cov}(P_1, f) \tag{6b}$$

$$B_2 = \mathrm{Cov}(P_1, f)\mathrm{Cov}(P_1, P_2) - \mathrm{Var}(P_1)\mathrm{Cov}(P_2, f) \tag{6c}$$

The signs of eq. (6) determine how the hedging effectiveness varies with different allocations to the foreign asset. The following properties are useful:

Proposition 1. *Let $\mathrm{Corr}(P_i, f)$ denote the correlation coefficient between P_i and f. That is,*

$$\mathrm{Corr}(P_i, f) = \mathrm{Cov}(P_i, f)/\{\mathrm{Var}(P_i)\mathrm{Var}(f)\}^{1/2}, \quad i = 1, 2 \tag{7}$$

Suppose all three prices, P_1, P_2, and f, are positively, imperfectly correlated. Then, (i) $B_1 < 0$ if $\mathrm{Corr}(P_1, f) \geq \mathrm{Corr}(P_2, f)$; (ii) $B_2 < 0$ if $\mathrm{Corr}(P_2, f) \geq \mathrm{Corr}(P_1, f)$; and (iii) B_1 and B_2 cannot both be positive.

In the presence of a single asset, say asset i, the minimum-variance hedge strategy generates the hedging effectiveness of $\mathrm{Corr}^2(P_i, f)$. Proposition 1 states that B_2 is negative when the currency futures is a better hedging instrument for the foreign asset than for the domestic asset. Similarly, B_1 is negative when the currency futures serves better than the foreign asset. Because both the foreign asset and the currency futures are denominated in the same currency unit, it is more likely that $\mathrm{Corr}(P_2, f) > \mathrm{Corr}(P_1, f)$, resulting in a negative B_2. On the other hand, $\mathrm{Corr}(P_1, f) > \mathrm{Cor}(P_2, f)$ is only a necessary condition for B_2

to be positive. A sufficient condition requires the difference to be large enough [see eq. (A2) in the Appendix]. This condition is in contrast to the general perception of the usefulness of the currency futures.

Proposition 2. *Suppose all prices,* P_1, P_2, *and* f *are positively imperfectly correlated. (i) When* $B_1 < 0$ and $B_2 < 0$ *and as foreign asset exposure increases, the hedging effectiveness first increases and then decreases. Maximal hedging effectiveness is achieved at the* $(s^*, 1 - s^*)$ *allocation where* $s^* = B_1/(B_1 + B_2)$; *(ii) when* $B_1 > 0$ *and* $B_2 < 0$, *the hedging effectiveness is an increasing function of foreign asset exposure. Maximal hedging effectiveness occurs at the complete allocation to the foreign asset; and (iii) when* $B_1 < 0$ *and* $B_2 > 0$, *the hedging effectiveness is a decreasing function of foreign asset exposure. Maximal hedging effectiveness occurs at the complete allocation to the domestic asset.*

Proposition 2 shows that, to argue for the relative importance of currency hedging for small allocations of foreign assets (based on hedging effectiveness) the futures contract must be a much better hedging instrument for the domestic asset than for the foreign asset. When the hedging effectiveness for the two assets are similar (i.e., the difference between $\mathrm{Corr}(P_1, f)$ and $\mathrm{Corr}(P_2, f)$ is small), both B_1 and B_2 are negative. Consequently, maximal hedging effectiveness occurs at an incomplete allocation.

PORTFOLIO EFFECTS

Recall that eq. (4) concludes that the maximal risk reduction occurs at one of the two complete allocations depending upon the comparison between $\mathrm{Cov}(P_1, f)$ and $\mathrm{Cov}(P_2, f)$. Nonetheless, maximal hedging effectiveness is expected to occur at an incomplete allocation. The divergence is due to the portfolio effects in the presence of multiple cash assets. For a general discussion of the portfolio effects, see Lypny (1988) and Lien (1990, 1993).

An important element in producing the portfolio effects is the correlation coefficient of the two asset prices. The correlation coefficient is irrelevant in the minimum-variance hedge decision; yet, it impacts the variance of the cash portfolio. As a result, it affects the magnitude of and the change in hedging effectiveness associated with a given foreign asset exposure. For example, when the two asset prices are uncorrelated, both B_1 and B_2 are negative. The maximal hedging effectiveness is achieved at an incomplete allocation.

FURTHER DISCUSSIONS

Given that the conventional hedge ratio and hedging effectiveness provide contradictory predictions, and given that hedging effectiveness is a well-accepted measure for the usefulness of a futures contract, it is reasonable to ask what information is contained in the conventional hedge ratio. In the case of a single cash asset, the minimum-variance hedge ratio dictates the amount of futures positions required to match a unit of cash position so that minimal risk is obtained. Moreover,

$$\mathrm{HE} = (h^*)^2[\mathrm{Var}(f)/\mathrm{Var}(P)] \tag{8}$$

where P is the asset price. In other words, if two cash assets have the same risk, comparisons of hedging effectiveness can be found from comparisons of minimum-variance hedge ratios. The cash asset with a larger hedge ratio benefits more from futures hedging.

When there are two cash assets, the conventional definition of the hedge ratio has a different meaning. Although it is defined as the ratio of the futures position to the cash position in the foreign asset, the futures position is composed of two components: one provides direct hedging for the foreign asset and the other cross-hedges the domestic asset. The calculation of the hedge ratio, therefore, is based upon the wrong presumption that all the futures positions are undertaken to hedge the foreign asset position.

Suppose a unit model is adopted. The hedge ratio represents the amount of futures contracts required to match with a cash portfolio consisting of a unit of the foreign asset and (W_1/W_2) units of the domestic asset. Hereby the calculation of the hedge ratio requires the knowledge of both cash positions, W_1 and W_2. When a percentage model is adopted, the hedge ratio describes the amount of futures contracts required to match with a cash portfolio consisting of a unit of the foreign asset and $s/(1 - s)$ units of the domestic asset. Knowing the cash position of the domestic asset sW is sufficient for the calculation of the hedge ratio. Based on those interpretations, it is clear why the hedge ratio decreases with increasing W_2 (or decreasing s). In both cases, the cash portfolio shrinks in size and fewer contracts are required for hedging purposes. Because W_1 remains unchanged in the unit model but declines with increasing W_2 in the percentage model, the hedge ratio is expected to decline more in the latter specification.

From eqs. (3) and (5), the relationship between hedging effectiveness and the hedge ratio can be written as follows:

$$\mathrm{HE} = (h^*)^2 \cdot [\mathrm{Var}(f)/\mathrm{Var}(sP_1 + (1 - s)P_2)] \tag{9}$$

While h^* decreases as s increases, the variance of the cash portfolio also changes as s changes. Consequently, the comparison of hedging effectiveness cannot be inferred from the comparison of hedge ratios. If one considers a cash portfolio (with a given allocation s) with a third cash asset, then larger hedge ratios imply greater hedging effectiveness provided the risk in the new cash asset and the risk in the given cash portfolio are the same. The provisional condition is rarely satisfied when comparing two cash portfolios with different allocations.

CONCLUSIONS

In sum, this note argues that the conventional definition of hedge ratio used by institutional investors in the presence of multiple cash assets is problematic. It does not have the usual meaning as in the case of a single cash asset. The positive relationship between hedge ratios and hedging effectiveness prevalent in the single cash asset case fails to exist in the presence of multiple cash assets. As a consequence, the comparison of hedge ratios among different allocations tells nothing about the relative importance of currency hedging. Ignoring these differences can lead to incorrect conclusions.

APPENDIX

To prove Proposition 1, one divides both sides of eq. (6b) by a factor of D_2:

$$D_2 = \mathrm{Var}(P_2)\{\mathrm{Var}(P_1)\mathrm{Var}(f)\}^{1/2}$$

The following relationship is obtained:

$$B_1/D_2 = \mathrm{Corr}(P_2, f)\mathrm{Corr}(P_1, P_2) - \mathrm{Corr}(P_1, f) \tag{A1}$$

Since D_2 is positive, the sign of B_1 is identical to the sign of the right-hand side of eq. (A1). Similarly, by dividing both sides of eq. (6c) by a factor of D_1:

$$D_1 = \mathrm{Var}(P_1)\{\mathrm{Var}(P_2)\mathrm{Var}(f)\}^{1/2}$$

one obtains

$$B_2/D_1 = \mathrm{Corr}(P_1, f)\mathrm{Corr}(P_1, P_2) - \mathrm{Corr}(P_2, f) \tag{A2}$$

Thus, the sign of B_2 is the same as the sign of the right-hand side of eq. (A2). Because $\mathrm{Corr}(P_1, P_2)$ lies between 0 and 1, Properties (i) and

(ii) are immediate from eqs. (A1) and (A2). Furthermore, when $B_1 > 0$, it must be the case that $\mathrm{Corr}(P_2, f) > \mathrm{Corr}(P_1, f)$ and, therefore, $B_2 < 0$. If $B_2 > 0$, then $\mathrm{Corr}(P_1, f) > \mathrm{Corr}(P_2, f)$, implying $B_1 < 0$. That is, B_1 and B_2 cannot both be positive.

Proof of Proposition 2 follows directly from eq. (6). Because A is positive, the sign of $d(\mathrm{HE})/ds$ is the same as the sign of $(B_1 + B_2)s - B_1$. When B_1 and B_2 are both negative, $(B_1 + B_2)s - B_1$ is positive if $s < B_1/(B_1 + B_2)$; and it is negative otherwise. The first derivative equals zero at $s^* = B_1/(B_1 + B_2)$, which attains the maximal hedging effectiveness. Because $(B_1 + B_2)s - B_1$ lies between $-B_1$ and B_2, the first derivative is always positive if $B_1 < 0$ and $B_2 > 0$. It is always negative if $B_1 > 0$ and $B_2 < 0$.

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