AN OPTIMAL PRICE INDEX FOR STOCK INDEX FUTURES CONTRACTS

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INTRODUCTION

Since by its nature every financial futures contract expires, time series work on futures markets will often be limited by the short period in which any particular contract exists. For stock index futures this is a period of about half a year.[1] Some types of analysis, however, would benefit from a far longer time series. To give just one example, the relative absence of friction-related market microstructure problems [Cohen et al. (1980)] and the symmetry of costs with respect to short and long positions [Karpoff (1987)] suggest that futures contract prices would be better than their underlying spot prices at discounting news, and so be more suitable for studying the news process and the daily price–volume relationship. Indeed this was the motivation for the indices proposed by Clark (1973) and subsequently used by Tauchen and Pitts (1983) in their seminal price–volume model.

There are a number of ways in which an index of futures contract prices may be calculated. This article defines an optimal price index and shows how optimal indices may be derived for different futures con-

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[1]Sometimes even this period will be inappropriately long, given the finding of Yadav and Pope (1990) that mispricing in stock index futures tends to increase with time to expiry.

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tracts. Interestingly, where there are more than two contracts available at any one time, there is more than one optimal index.

The article then compares the optimal index with a spliced index, created via the widely practiced method of taking the near contract, ignoring the others, and at expiration, splicing to the next near contract.

Another alternative, the Clark index, is examined and compared to the spliced and optimal indices. Practical differences among the three indices are illustrated using data from the FTSE 100 contract traded on London International Financial Futures Exchange (LIFFE). It is found that an optimal index is preferable to either the spliced index or Clark's index. The former is demonstrably suboptimal, while the latter is much harder to calculate and potentially unreliable.

AN OPTIMAL PRICE INDEX

The objective is to create an optimal price index from the prices of the available futures contracts. This begs the question "What do we mean by optimal?"

Definition (optimal futures price index). *An optimal futures price index preserves a constant proportionality between the change in the price index and the change in the underlying spot price, in the absence of market imperfections.*

In the absence of market imperfections, each futures contract trades at a "fair value" relative to the spot price. This fair value is maintained by arbitrage. In the case of stock index futures,

$$F_k = P(1 + (r - y)(k - t)) \tag{1}$$

where F_k is the price of the futures contract expiring at time k (where $t \leq k$), P the spot price, r the interest rate, and y the dividend yield. All periods are measured in years.[2]

The simplest index of futures contract prices would be the price of the near contract, F_k. This is the generally accepted method of creating a contiguous series.[3] Notionally, the price series of the near contracts are spliced together so that the daily return of the resulting price index is equal to the daily return of the near contract. The problem with the

[2]It is assumed in this article that the "cost of carry" $r - y$ is constant. For most of the time this is a good approximation. Occasionally, however, there will be some marked seasonality. For example, the interest rate will jump upwards at the end of each quarter if the yield curve is upward sloping. The empirical results of the section on Practical Differences demonstrate that the impact of this factor, and likewise the impact of transactions costs, is small.

[3]See Buckle, Clare, and Thomas (1994, footnote 2, p. 6).

spliced index is that the change in the index depends upon both the change in the spot price and the time to expiry of the near contract:

$$dF_k = (1 + (r - y)(k - t))dP - P(r - y)dt \tag{2}$$

This means that the price change ratio, dF_k/dP, is seasonal, regardless of whether dP is seasonal. It is this problem which motivates the definition of optimality given above. An optimal futures price index should have no seasonal variations other than those in the spot price.

The optimal futures price index, F^*, will be restricted to convex combinations of the prices of the currently available futures contracts:

$$F^* = \sum_{i=0}^{n-1} \lambda_i(t) F_{k+iv} \tag{3}$$

where there are n contracts available and the time between contract maturities is v. The weights in eq. (3) are written explicitly as functions of time, since without time variation in the weights there will always be seasonality similar to that of the spliced index. The weights satisfy the usual conditions

$$\sum_{i=0}^{n-1} \lambda_i(t) = 1, \quad \text{and } 0 \le \lambda_i(t) \le 1 \quad i = 0, \ldots, n - 1 \tag{4}$$

for all values of t. The spliced index is the simple case $\lambda_0(t) = 1$, and $\lambda_i(t) = 0$ for $i = 1, \ldots, n - 1$.

The following theorem gives the restrictions on the weights necessary to achieve an optimal stock index futures price index. The approach is easily generalized to other futures, with $r - y$ being replaced by the appropriate cost of carry.

Theorem (optimal weights). *For* F^* *to be an optimal futures price index the weights must satisfy eq. (4) and the two additional conditions*

$$\sum_{i=0}^{n-1} \lambda_i'(t) = 0\,, \tag{5}$$

$$v \sum_{i=1}^{n-1} i\lambda_i'(t) = 1\,, \tag{6}$$

where $\lambda_\mathrm{i}'(\mathrm{t})$ *is the first derivative of* $\lambda_\mathrm{i}(\mathrm{t})$.

Proof. The definition of optimality implies that

$$dF^*/dP = c \quad \text{for all } t \tag{7}$$

where c is some constant.[4] For eq. (7) to hold,

$$\frac{d}{dt}\left(\frac{dF^*}{dP}\right) = 0 \quad \text{for all } t \tag{8}$$

Substituting for F^* from eq. (3) and rearranging, one obtains:

$$\sum_{i=0}^{n-1} \lambda_i'(t)(1 + (r - y)(k + iv - t)) = r - y \tag{9}$$

However, the weights cannot be functions of $r - y$. This requires

$$\sum_{i=0}^{n-1} \lambda_i'(t) = 0 \tag{10}$$

which is condition eq. (5). Substituting eq. (5) into eq. (9) gives condition eq. (6), which completes the proof.

It can easily be confirmed that the spliced index is not optimal, since eq. (6) cannot be satisfied when $\lambda_i'(t) = 0$ for $i = 1, \ldots, n - 1$. Trivially, no index comprising just one futures price can be optimal.

The theorem can now be used to derive the optimal weights for different numbers of contracts. When $n = 2$, condition eq. (6) implies $\lambda_1'(t) = 1/v$; therefore, condition eq. (5) implies $\lambda_0'(t) = -1/v$. Solving these two equations with condition eq. (4) gives the solution:

$$\lambda_0(t) = \frac{k - t}{v} \qquad \lambda_1(t) = \frac{v - (k - t)}{v} \tag{11}$$

These weights can be substituted into F^* to identify the nature of the index, giving

$$F^* = P(1 + (r - y)v) \tag{12}$$

The optimal price index behaves like a notional forward contract which has v years until expiry. There is only one optimal price index using just two contracts.

[4]This condition may also be written

$$\frac{dF^*}{dt} = c\frac{dP}{dt} \quad \text{for all } t \tag{24}$$

In other words, the rate of change of the futures price index and the spot price should be in constant proportion no matter what the relationship between the current date and the expiry pattern of the futures contracts.

More Than Two Contracts

For stock index futures, there are often only two contracts available; and so optimal indices with $n > 2$ are not required. However, there are other futures markets where there are many more than two contracts available [e.g., the market for cotton futures examined by Clark (1973), where there were generally eight contracts available]. Therefore, the optimal indices for three contracts is derived to demonstrate the approach and illustrate the properties of the resulting indices.

Generally, solutions can be found by using the sufficient condition for eq. (6),

$$\lambda_i'(t) = 1/(iv(n-1)) \quad \text{for } i = 1, \ldots, n-1 \tag{13}$$

finding $\lambda_0'(t)$ by eq. (5), and then working back to the weights using eq. (4). For $n = 3$, there are three sets of weights found by this method:

$$\begin{aligned}
&\lambda_0(t) = \frac{v + 3(k-t)}{4v} \quad \lambda_1(t) = \frac{2v - 2(k-t)}{4v} \quad \lambda_2(t) = \frac{v - (k-t)}{4v} \\
&\lambda_0(t) = \frac{3(k-t)}{4v} \quad \lambda_1(t) = \frac{3v - 2(k-t)}{4v} \quad \lambda_2(t) = \frac{v - (k-t)}{4v} \\
&\lambda_0(t) = \frac{3(k-t)}{4v} \quad \lambda_1(t) = \frac{2v - 2(k-t)}{4v} \quad \lambda_2(t) = \frac{2v - (k-t)}{4v}
\end{aligned} \tag{14}$$

These correspond to notional forward contracts which have v, $(5/4)v$, and $(3/2)v$ years to expiry, respectively. Interestingly, the first set of weights can be generalized further, since none of the individual weights ever fall to zero. The generalization is

$$\lambda_0(t) = \frac{v + 3(k-t-a)}{4v}, \quad \lambda_1(t) = \frac{2v - 2(k-t-a)}{4v},$$
$$\lambda_2(t) = \frac{v - (k-t-a)}{4v} \tag{15}$$

where $0 \leq a \leq v/3$. The notional time to expiry of this contract is $v + a$. In other words, the investigator can choose any notional time to expiry between v and $(4/3)v$. This includes $(5/4)v$, the second of the indices in eq. (14), which would require $a = v/4$. Therefore two different sets of weights can give the same notional time to expiry. Investigators should be aware of this possibility when creating indices with three or more contracts.

Where there is more than one optimal index available, a general rule would be to choose that set of weights which gives a notional time to expiry closest to the mean time to expiry of all contracts in the market over the period in question.

CLARK'S METHOD

The deficiency of the splicing method, which takes just the near contract, has already been highlighted. Clark (1973) proposed an alternative solution which shares many of the properties of the optimal price indices described above. However, Clark's method is not optimal where there are only a small number of contracts available, which is the case with stock index futures. In Clark's defense, however, his method was applied to cotton futures, where the number of contracts available at any time was eight.

Clark's index is a convex combination of futures contract prices, as in eqs. (3) and (4). However, the weights are not derived from the theoretical properties of futures contracts. Rather, they are derived from the empirical distribution of the time until expiry across all contracts in the sample period. Denoting this distribution as $W[\cdot]$, Clark's index is

$$F^c = \frac{\Sigma_{i=0}^{n-1} W[k + iv] F_{k+iv}}{\Sigma_{i=0}^{n-1} W[k + iv]} \tag{16}$$

Since $dk/dt = -1$, the weights do change with time. However, they are clearly not optimal, satisfying neither eq. (5) nor eq. (6). Clark's approach attempts to create a notional contract with

$$\bar{w} = \frac{\Sigma_{i=1}^{\infty} iW[i]}{\Sigma_{i=1}^{\infty} W[i]} \tag{17}$$

years until expiry. But it has been shown in the previous section that only certain very specific times to expiry are possible with an optimal index: an empirical choice will generally not suffice.

Since Clark's index is not optimal, it will contain some seasonality in addition to that in the underlying spot price. The source of this seasonality is variation in the de facto time to expiry. This is $\bar{w}$, on average, but varies on a cycle of v years, because the contracts available at any given time provide just a small subset of the total range of possible times to expiry. In fact, the de facto time to expiry of the Clark index at time t is

$$w_t = \frac{\Sigma_{i=0}^{n-1} (k + iv) W[k + iv]}{\Sigma_{i=0}^{n-1} W[k + iv]} \tag{18}$$

and the Clark index might be written

$$F^c = P(1 + (r - y)w_t) \tag{19}$$

By comparing eqs. (17) and (18) it can be seen that w_t tends to the constant $\bar{w}$ when $n \rightarrow \infty$ and $v \rightarrow 1$ (which also implies $k \rightarrow 1$). This confirms that Clark's index is less suboptimal where there is a large number of different times to expiry available.

However, even where there is a large number of contracts available, Clark's index still has disadvantages over the optimal index. First, it is much harder to calculate, requiring an extra data set (the daily open interest on each contract) and the estimation of the distribution $W[\cdot]$. Second, it is sample-dependent since $W[\cdot]$ will depend upon the period of calculation. This sample-dependence would probably not be a problem in a mature market since $W[\cdot]$ would be stable. In a new or evolving market, the "seasoning" of the contract could make $W[\cdot]$ vary, in which case the Clark index calculated over a period might be substantially different from its values in a subperiod.

PRACTICAL DIFFERENCES

The notion of optimality depends upon there being no market imperfections. Since this is not the case in actual markets, it is important to examine whether there are empirical as well as theoretical differences among the three indices. There is also the issue of ease of use to be considered. This section examines these questions using the FTSE 100 contracts traded on LIFFE over the period, 1985–1994.

The optimal index is the two-contract specification eq. (11), since there are sometimes no more than two futures contracts available.[5] The Clark index requires the estimation of the time to expiry distribution, $W[\cdot]$, prior to calculating the index values. This is shown in Figure 1. The sharp falloff in $W[t]$ just before $t = 0.25$ indicates that most of the interest in FTSE 100 futures is in the near contract. The de facto times to expiry for the three indices are shown in Figure 2. The mean time to expiry for the Clark index, $\bar{w}$, is 0.169 years (42.7 days), but the availability of just two contracts causes the de facto time to expiry to vary from 0.253 years (63.8 days) to 0.123 years (31.0 days) over the course of one quarter.

[5]Indeed, sometimes at the start of a new quarter there is only one contract available. In this case the weights are set to $\lambda_0(t) = 1$ and $\lambda_1(t) = 0$; these are close to the theoretical values since $(k - t) \approx v$.

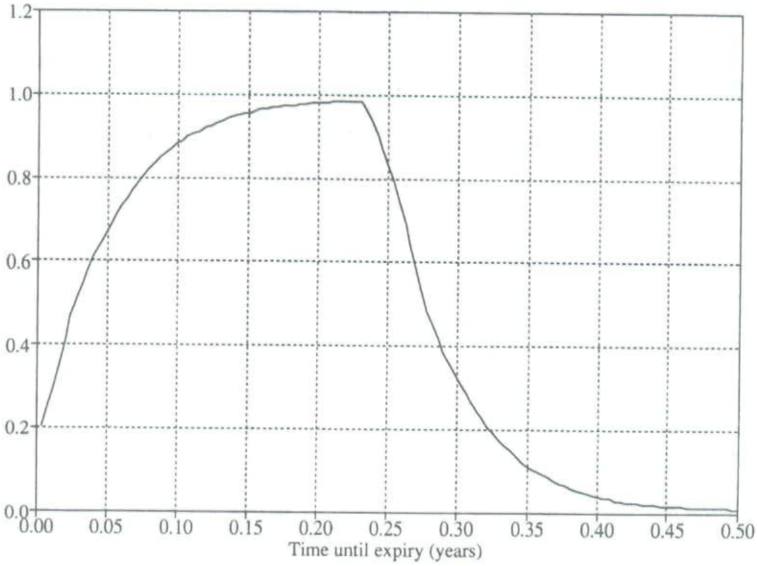

FIGURE 1
The distribution of contracts by time to expiry—FTSE 100 futures contracts, 1985–1994.

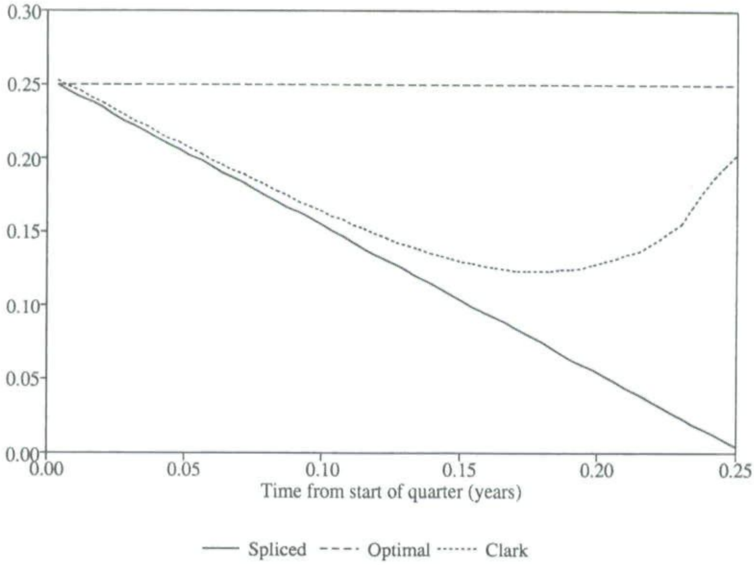

FIGURE 2
The Notional Time to Expiry over the Course of One Quarter.

From the properties of the three indices

$$\frac{dF_k}{dt} = -(r - y)P + (1 + (r - y)(k - t))\frac{dP}{dt} \tag{20}$$

$$\frac{dF^*}{dt} = (1 + (r - y)v)\frac{dP}{dt} \tag{21}$$

$$\frac{dF^c}{dt} = -(r - y)P\frac{dw_t}{dk} + (1 + (r - y)w_t)\frac{dP}{dt} \tag{22}$$

This suggests a bivariate regression of the form

$$\Delta F = \alpha + \beta\Delta P + u_t \tag{23}$$

where the daily change in each of the indices, denoted by Δ, is used to approximate the time derivatives.[6] Regression eq. (23) should only be correctly specified for the optimal index F^*, in which case, $\alpha = 0$ and $\beta > 1$. In the case of the spliced index, the significant time variation in β should cause the regression to be misspecified. It is hard to say, a priori, whether the variation in w_t over the course of a quarter is sufficient to have a significant impact on the specification of the Clark index regression.

On running the regressions, it was immediately apparent that all three models are characterized by strong negative first-order autocorrelation in the residuals. This is presumed to be a consequence of the arbitrage process. If a futures contract price is sufficiently in excess of its fair value, then arbitrage would cause $\Delta F_k < 0$ and $\Delta P > 0$ in the absence of news, and the disturbance in eq. (23) would be negative. If arbitrage next occurs in the other direction, the disturbance would be positive. Therefore, negative autocorrelation in the disturbances of eq. (23) suggests some form of "overshooting" by arbitrageurs.

The regressions are reestimated allowing for first-order autocorrelation in the disturbances. The results of these regressions are given in Table I. These show that the spliced index performs very differently from the optimal index and the Clark index. The optimal regression has precisely the properties predicted by the theory: a zero intercept and a gradient slightly but significantly in excess of 1.0. These properties are shared by the Clark index where, additionally, the $\hat{\beta}$ is slightly less than that of the optimal regression in accordance with the shorter mean time

[6]The change in the spot price, ΔP, is calculated from close to close of trading in the futures contracts at LIFFE (4.10 PM) rather than close to close in the spot market, since the two are not synchronous.

TABLE I
Regression Results, 1985–1994

	Dependent Variable		
	ΔF_k	ΔF^*	ΔF^c
$\hat{\alpha}$	-0.307^a	-0.041	-0.039
	(0.112)	(0.131)	(0.131)
$\hat{\beta}$	0.762	1.072^b	1.069^b
	(0.006)	(0.008)	(0.008)
$\hat{\rho}$	-0.285^a	-0.379^a	-0.375^a
	(0.019)	(0.018)	(0.018)
R^2	0.839	0.867	0.866
DW	2.102	2.174	2.172

[a]Significantly less than 0.0 at 5%.
[b]Significantly greater than 1.0 at 5%.

to expiry.[7] This contrasts with the spliced index regression, where the gradient is not significantly greater than 1.0, contradicting the theory. This suggests that the spliced index regression is misspecified.

The high R^2 values for the optimal and Clark regressions indicate that nearly 90% of the variation in the futures price can be explained by the fair value relationship. This leaves just over 10% to be explained by the sporadic actions of arbitrageurs, and slack introduced by transactions costs and variation in the interest rate and the dividend yield.

CONCLUSION

The need for a contiguous price index for futures contracts is well established. This article considers three such indices. The spliced index is the de facto standard and consists simply of the returns of each near contract in turn, notionally joined together into a price series. The Clark index is a convex combination of futures contract prices, where the weights are derived from the empirical distribution of time to expiry. By the definition provided in this article, neither of these indices is optimal, since they both introduce seasonality into the futures price index in addition to that present in the spot price.

The third index is optimal by the same criterion. The optimal index preserves a constant proportionality between spot price change and futures price index change. It is shown mathematically that there are several optimal indices. The investigator should choose that index with

[7]The hypothesis that the three parameters are the same in the optimal and Clark regressions has a test statistic of 1.30, which is comfortably below the critical value (χ^2_3 at 5%) of 7.82.

the most appropriate notional time to expiry, since it is in this respect that the indices differ from one another.

The empirical evidence suggests that the spliced index should be avoided. The Clark index and the optimal index perform almost identically, but market imperfections may be sufficient to hide the Clark index's suboptimality. However, given the much greater cost of calculating the Clark index and the uncertainty engendered by its sample-dependence, the optimal index should be unambiguously preferred when creating a price index of stock index futures contracts. Similar optimal indices can be calculated for other futures markets.

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