DISTORTION-FREE FUTURES PRICE SERIES

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INTRODUCTION

The 1993 volume of this journal included ten articles employing lengthy time series derived from futures prices. Five of the articles followed prices of the near contract up to the expiration date and then rolled over to the next nearest contract's prices. Other writers rolled over to the next contract either at the beginning of the delivery month or on a date up to one month before the expiration date. These ten articles also considered several different price variables. One author employed the five-day average price for the contract with the most open interest. Others used priced changes, price levels, price relatives, or logarithms of price relatives. A few writers indicated an economic reason for selecting a rollover date, but none discussed sensitivity of the results to the choice of variables or contract linking method.

The lack of consensus on the proper method of constructing a futures price time series coupled with so little discussion of the propriety of the method actually employed suggests that it does not matter much which variable or linking approach is used. However, it often *does* make a difference. Ma, Mercer, and Walker (1992) showed that typical statistical tests of futures series can be very sensitive to the choice of rollover date and linking method. The computations of Ma, Mercer, and Walker have been replicated but alternative price variables, e.g.,

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yields or price levels were substituted. The results show that *both* the linking method and the form of the price variable have a significant impact on most statistical tests. Some of the range of statistical outcomes created by alternative specifications often can be reduced because one particular form of the price variable will be indicated by the nature of the problem. Selection of the linking method remains, however; and no rigorous theoretical justification has been given yet to suggest that one linking method is better than another.

The most appropriate way to connect expiring futures or option contract prices depends on the use to which the data will be put. The first portion of this article demonstrates that the linking process must be of a single, specific functional form if the series are to be treated as time series of numbers appropriate for use in a regression or other statistical tests. The linking formula is known. However, the choice of a key parameter—the last day observed prices may be used in determining the series' values—is not established by theory. The second part of the article indicates the condition necessary to create a futures series that is both distortion-free and unique.

CONCEPTUAL PROBLEMS WITH CONTRACT LINKING

An expiring contract's prices usually are linked to the prices of later contracts by subtracting a constant or scaling down or up the new or old prices. The resulting series is smooth in prices, but the numbers have no economic meaning. An extreme example is the negative numbers that could result form the use of the common linking rule, "subtract from subsequent prices the difference between the price of the expiring contract and the next near contract." The resultant negative numbers cannot represent the value of a financial asset.

The series of numbers produced by linking contracts generally cannot be interpreted as prices—that is, as ratios of a monetary unit to a commodity unit—even if the numbers are not negative. The linking process does cause the monetary amounts to be transformed into a continuous sequence. If the commodity units are not transformed by the linking procedure also, the transformed "prices" no longer will be exchange rates between corresponding money and commodity units. But suppose the commodity units *were* adjusted to preserve the meaning of the prices. The new series is continuous as numbers, but the attributes of the financial instrument being priced will vary as the contract approaches expiration and will experience periodic discrete jumps as the contract is rolled over. One price in the series cannot be compared with another because the qualities of the contract have not been standardized even though both numbers are proper prices. Also, prices in such a linked futures series cannot be related to, say, prices of the spot commodity because the spot commodity has uniform attributes while the linked futures commodity does not.

A FUTURES PRICE INDEX FOR THE COMMODITY

The linking process concentrates on joining observed prices sequentially: the correct linked price today is the value which follows yesterday's price in some appropriate manner. An alternative approach, one which avoids the complexities just outlined, focuses on the meaning of each number in the series rather than on the relation between sequential numbers.

The futures price time series can be viewed as a sequence of scalars, each identified as the level of this commodity's futures prices on a particular date. There are several prices observed at a given moment, one for each contract with a unique expiration date. The problem, then, is to define a rule which generates a representative value from the observed prices. Candidates for the representative value of the contract include: the price of the most liquid contract, the mean price of all traded contracts, the price of the contract nearest expiration, etc. Each possibility represents a rule which maps all prices into a single number or index. This suggests that the challenge of forming a lengthy futures price time series is equivalent to the problem of finding a price index for observed futures prices.

Price index number formulas have been developed from three distinct starting points: representative agent theory, statistical models, or tests. Most discussions of the Consumer Price Index are examples of the representative agent approach to price index formation. The agent of the Consumer Price Index is the rational consumer. Afrait (1977, pp. 122–127) and others have shown that revealed preferences imply that the Consumer Price Index is the upper bound measure of the impact of price changes if a representative consumer is assumed to exist. The Consumer Price Index, or any Laspeyres index based on a representative buyer, assumes that the agent purchases goods in a competitive market. The index measures the effect of exogenous price changes on the consumer. However, these assumed conditions are not met in futures markets. The representative agent may just as easily sell as buy. Also, many agency models of futures markets identify several classes of actors: producers, commercial users scalpers, etc. There would be a different price index for each type of agent, and each index would represent the real cost to that particular agent. Futures time series are generated to represent the time path of the value of the contract, not to measure real effects of price changes. Therefore, the representative agent model is not appropriate for this type of index. (However, the index actually selected may have an economic interpretation even though the index did not begin from an economic premise).

Theil's (1954, pp. 28–34) linear aggregates represent the statistical approach to price index formulation. Microvariables are mapped onto macrovariables with some statistical error. The use of such a statistical index in most futures problems would mean that any test of the model would be a joint test of: the statistical assumptions of the index; efficiency of the market; and the specific hypotheses of the model. These added complexities are not necessary. The statistical index approach will not be considered further here because it is possible to form a well-defined index of futures contract prices without making any additional assumptions about the distribution of price errors.

The third basis for an index formula, and the method employed here, is related to Irving Fishe's (1922) tests. Fisher viewed the index series as a tool to be employed in another problem. The index, to be appropriate in its use, must possess certain properties. Each desired characteristic of the index is defined by a test. For example, suppose an index of price changes needs to be constructed to be used in an application that requires that there be a natural zero point. The phrase, "there has been no change in prices" (as measured by the index), must have unambiguous meaning. This property can be imparted to the index by an appropriate mathematical condition, such as "when all observed price changes are zero, the index must be zero."

The properties of lengthy futures price time series are divided into two categories. The first set of properties asserts that index magnitudes must act as numbers usable in statistical hypothesis testing. Multiplication, division, addition, and subtraction of index values must have meaning. That one index value is greater than another must make sense, and there must be a determinate value for the index given a set of observed prices. The second group of price index properties relates to the role of time. The statement, "The price of the January contract increased by two dollars this week," defines a connection between the passage of time and the change in a single contract's price. It is necessary that a corresponding link exist between the price index and time if any time series analysis is to be performed on the set of index numbers.

All of the characteristics of a futures price index necessary to create a meaningful lengthy time series may be defined by seven properties. The remaining question is whether *any* index function can possess all of these properties at the same time. (If not, there does not exist a single number which represents prices in this application.) A family of index functions does exist which has all the traits necessary to define a meaningful, lengthy futures price time series.

INDEX VALUES MUST POSSESS THE TRAITS OF NUMBERS

Four properties of the price index must be preserved if the index values are to act as numbers.

Relevant Information Property

A futures contract time series has the Relevant Information Property if the only price data that determine the value of the index of today's prices are the actual prices observed this day. Other variables may enter into the computation of the index, but those are parameters, not data. The distinction between parameters and data is that parameters can be known without observing the data. Variables that might be parameters in a futures price index include: the date; number of days to expiration of the near contract; whether delivery notice can be given on this contract; etc.

The Relevant Information Property can be assured if the following condition is met:

Let P_t be the value of the price index at time t. The index function, Φ, depends on the vector of observed prices, $p_t = (p_{t,1}, \ldots p_{t,n})'$, and, possibly, t or other parameters.

$$P_t = \Phi(p_t; t, \ldots.) = \Phi_t(p_t) \tag{1}$$

The Information Property acts as a philosophical contention that prices in this time period are not a function of prices which materialize in the next period. A symmetric argument asserts that if today's prices cannot enter into yesterday's price level index, then yesterday's prices do not belong in today's index.

The condition given by eq. (1) does not require that all prices be measured at the same instant or even be measured over the same day. Closing prices on the last trading day of the week may be the data for weekly prices. Also, "the average of closing prices from Monday through Friday" or "Wednesday's opening price" or "the average of all observed prices during the week" produce time series that would be consistent with the Information Property because observed prices fit into one and only one time period. The rule, "Wednesday's price is the average of Monday's through Friday's closing prices, and Thursday's price is the average of Tuesday through Monday's, etc.," is a centered moving average which places each price observation in five different time periods for the index. Any rule such as this, which acts like a moving window, cannot generate a functionally determinate, distortion-free time series. The Relevant Information condition eliminates functional indeterminacy by placing the value of the index, P_t, and the data, p_t, in exactly the same, unique time compartment.

Condition (1) also is related to the assertion that preprocessing of data should be performed in a way that preserves as much information as possible. The potential for information loss through preprocessing may be seen by converting the series of observed contract prices, $p_{t,i}$, into an infinite Fourier series:

$$p_{t,i} = p_i(t) = \sum_{j=0}^{\infty} [a_j \cos j\omega t + b_j \sin j\omega t]$$

The price time series is exactly represented by a sum of cycles of various periods. The cycle with the lowest frequency is associated with the major trend line, and high frequency cycles are necessary to represent sharp changes in direction of the $p_{t,i}$ time series.

Every index which does not preserve the Relevant Information Property can be shown to be a function of some sort of moving average. The moving average dampens or completely eliminates all cycles of length less than the period of the average, which means that index formulations which fail the Information condition tend to remove the high-frequency components of the time series. This damping does help smooth out noise and may be appropriate in some cases, but discarded with the noise is information about the tops, bottoms, and sharp breaks in the time series. These points may represent exactly those events that are most important. The Relevant Information Property implies that if a time series is to be smoothed, the conditioning should be applied to the final, lengthy time series directly. In this way, any distortions can be observed; and alternative fitting or smoothing functions may be tested. On the other hand, violating the Relevant Information condition means that indirect, nonobservable and irreversible smoothing must be occurring within the function which forms the index.

Price Scale Property

Lengthy futures price time series are created for use in predicting, testing, or similar processes involving calculations. The elements of the time series can be multiplied and divided in a meaningful way because the values of the price index are real numbers. Similarly, the observed prices are real numbers and can be multiplied and divided. The index function is real-valued; and observed prices, the data of the index, also are real. However, there is no assurance that, in general, an index function will preserve the scaling of observed prices even though the domain and range of the function are real.

The Scale Property maintains that multiplications must have the same meaning for observed contract prices and the index. This is equivalent to saying that any information imbedded in a price index time series formed from prices measured in "dollars per ounce" should be expressed without distortion in an index formed from prices measured in units of "cents per ounce." The Scale Property insures that meaningful differences in magnitude are captured by the index while meaningless changes in scale have no real impact.

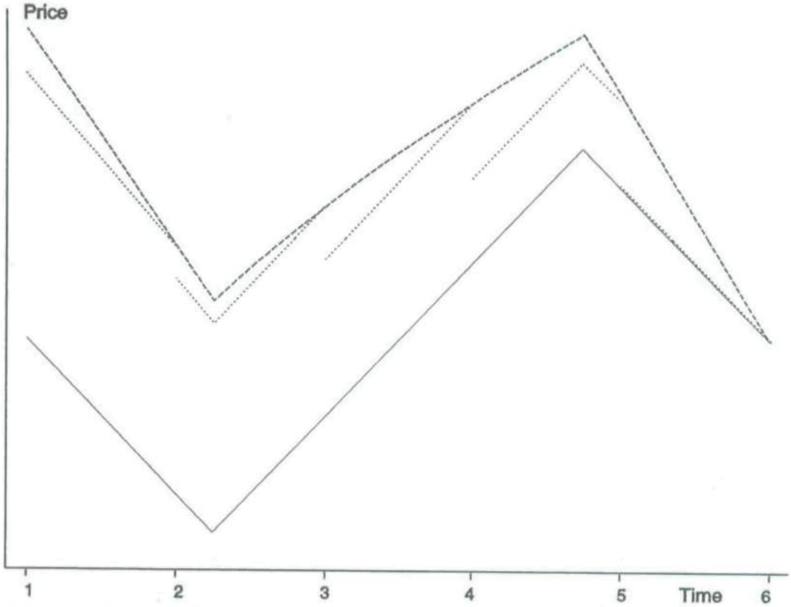
An index which possesses the Price Scale Property must meet this condition:

If $\tilde{p}_t = \lambda p_t$ for some positive number, λ, then

$$\Phi_t(\tilde{p}_t) = \lambda \Phi_t(p_t) \tag{2}$$

The linking rule, "on the rollover date add to all preceding prices the difference between the price of the expiring contract and the price of the next near contract," will create a time series that does not preserve price scale. The distortion introduced by this rule may be seen in Figure 1. The solid line shows the true, stylized price cycle as captured by a distortion-free price index. The dashed line is a linked series produced by the additive linking rule given above. The linking process has produced a series that is continuous across the rollover dates, which are indicated by numbers on the time axis. The dashed line does capture the general shape of the true price cycle but distorts both the price level and the rate of change in prices.

Point 6 on the time axis of this figure represents the observed price of the nearest contract at expiration. There has been no rollover of this contract, so nothing has been added to the observed price at this end point or at any moment over the entire period from time 5 to time 6. Even though no linking adjustment has been made in this interval, the true price and the linked series are not equal because the



The true time series (solid line) versus a series created by additive linking (dashed line).

FIGURE 1

Distortions caused by additive linking. The true time series (solid line) traces a stylized price cycle. The additively linked series (dashed line) is formed by adding to preceding prices the difference between contract prices at rollover. The additively linked series differs from the true cycle because: (1) carrying costs imbedded in linked prices decrease as time passes; and (2) the linking procedure shifts earlier prices from the observed values. The amount of price translation caused by linking is shown by the dotted segments, which are the linked series with carrying costs removed.

linked prices are responding to two distinct forces: the underlying price is passing through the stylized cycle and the contract is approaching expiration. (This must be a normal carry market because the futures price approaches the spot price from above. The market places a ceiling on the observed futures prices through arbitrage between the futures contract and the cost of carrying the actual commodity to expiration. Carrying costs fall as the contract approaches expiration, which causes the linked price series to converge to the true price level.)

The dotted line in Figure 1 shows the linked contract price with carrying costs removed. The dotted line reflects only underlying price changes and any linking adjustment. No linking occurs between times 5 and 6; therefore, the true price series and the dotted line are coincident. A rollover occurs at time period 5 so all prices before this moment must be shifted by the additive linking rule being employed. The dotted line, with the expiration effect eliminated, shows that all prices between periods 4 and 5 are shifted up by the difference between the near and the next near contact prices at time 5. The dotted line is parallel to the true price line because additive linking changes level, but not slope. The linked series differs from the true series by greater and greater amounts as each rollover occurs. Time 1 prices shift up four times by the linking process and differ markedly from the true price, which is the same in periods 1 and 6.

The additive linking rule—an example of an index which does not possess the Scale Property—distorts the rate of change of prices because the rule does not hold fixed the convenience yields or carrying costs that accrue until roll over. Price *levels* also are distorted by additive linking, but the additive shifting process does not distort the slope of the time series any further. It is the shifting of price levels that causes additive linking to distort proportional changes in all prices and to fail condition (2).

Price Level Property

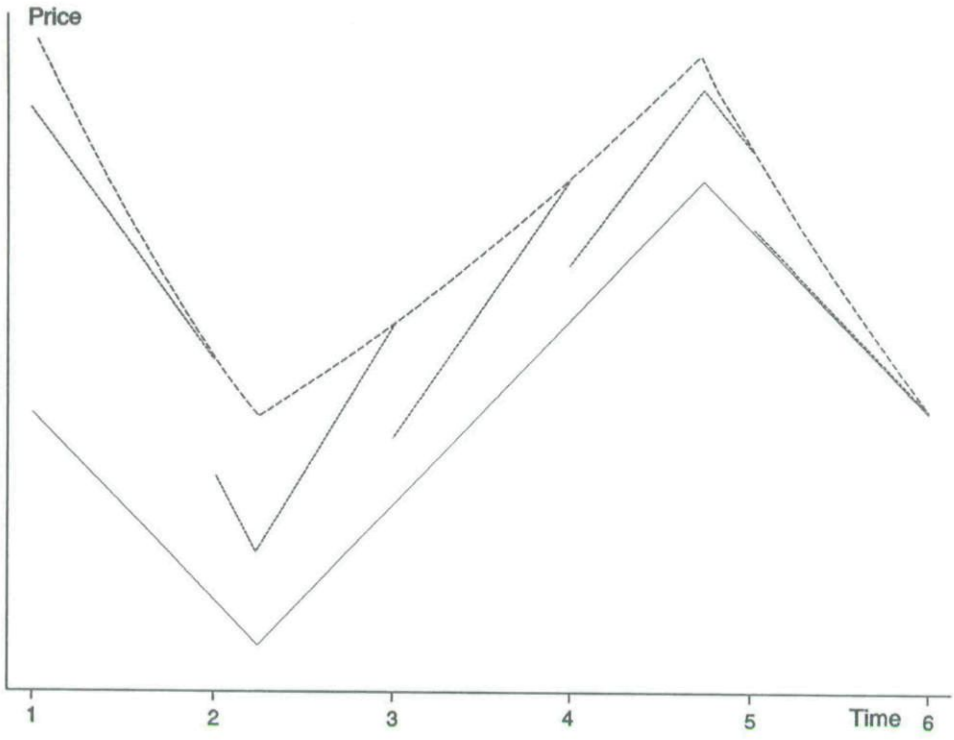
Two distinct computational operations act on real numbers—multiplication and addition. The Price Scale Property asserts that multiplication must have the same meaning for observed prices and the price index. The Level Property preserves addition and substraction in the same fashion.

If $\tilde{p}_t = (p_{t,1} + \kappa, p_{t,2} + \kappa, \ldots, p_{t,n} + \kappa)$ for all $p_{t,i}$ and some number, κ, then

$$\Phi_t(\tilde{p}_t) = \Phi_t(p_t) + \kappa \tag{3}$$

A stylized linear price cycle is used again in Figure 2 to demonstrate the distortion caused by an index which does not appropriately retain the connection between the levels of observed prices and the resulting index. The dashed line in this figure is the time series produced by the linking rule, "multiply all previous prices by a number equal to the ratio of the next near contract price divided by the observed price of the near contract." This linking method fails condition (3).

Between times 5 and 6, the additively (Fig. 1) and the muliplicatively (Fig. 2) linked time series appear to be the same and to be different from the true, underlying price series. This happens because both linking methods do not hold convenience yields net of costs constant. The linked series in Figure 2 approaches the true price line continuously as the contracts expire. At time 5 a rollover occurs. All preceding prices are multiplied by a number, which in this normal carry market is somewhat greater than one. The rescaling that occurs at rollover changes both



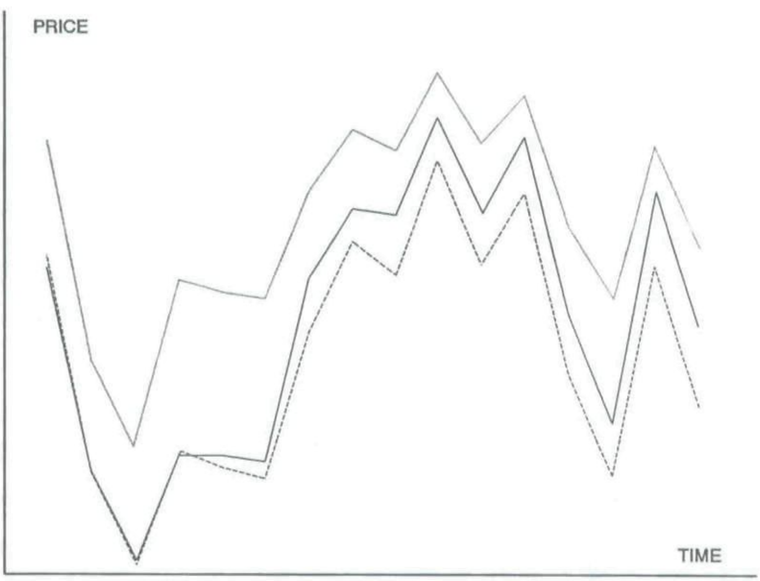
The true time series (solid line) versus a series formed by multiplicative linking (dashed line).

FIGURE 2
Distortions from multiplicative linking. The true time series (solid line) traces a stylized price cycle. A multiplicatively linked series (dashed line) is formed by multiplying preceding prices by the ratio of the contract prices at rollover. The multiplicatively linked series differs from the true cycle because: (1) carrying costs imbedded in linked prices decrease as time passes; and (2) the linking procedure rescales prices causing changing prices to appear to move more rapidly. The slope distortion caused by linking is shown by the dotted segments, which are the linked series with carrying costs removed.

price levels and the rat of change of prices. When prices are positive, as in Figure 2, this multiplicative scaling causes rising prices to increase faster and falling prices to have a greater negative slope.

The multiplicatively linked time series contains a systematic negative rate of change distortion (relative to true prices) caused by the effect of the time to expiration. A complicated shape and level distortion is caused by the repeated rescaling at each rollover occasion. The exact scaling distortion depends on price magnitudes, price rates of change, and the convenience yields net of costs. The multiplicatively linked time series is not distortion-free and does not retain the Price Level Property because if all observed prices were increased by one dollar, the index's price change would not be exactly one dollar after rollover-related rescaling occurs.

An indication of the biases induced by the linking process for real data is shown in Figure 3. Gold futures contract prices are additively linked by subtracting from all succeeding observations the difference between the near and next near contracts on the rollover day, which is the first day of the delivery month.[1] A multiplicatively linked series is formed by dividing all subsequent prices on the rollover date. The distortion-free index provides a reference. All three series shown have the same general shape. The index which retains all desired properties has a time rate of change of $1.28 per week. A smaller, $0.77 per week change is found for the additively linked series because that index does not remove the systematic negative distortion caused by the contracts'



Conforming (solid line), Additive (dashed line) and Multiplicatively (dotted line) Linked Gold Futures Price Series

FIGURE 3

Three gold futures price series: distortion-free (solid line), additively linked (dashed line), and multiplicatively linked (dotted line). All three have the same general shape but produce differing values for common statistical measures. The time rate of change is 1.27, 0.77, and 0.49 $/week, respectively. Serial correlation coefficients are 0.545, 0.459, and 0.439, respectively. Standard deviations are 5.96, 5.22, and 3.94.

[1] The data are Chicago Mercantile Gold beginning in January, 1984. The points shown in Figure 3 are weekly closing prices from October 29, 1993 through February 4, 1994. The distortion-free index is shifted by $132 to facilitate comparisons.

approach toward expiration. The multiplicatively linked series has a \$0.49 weekly rate of change, which is less than the \$1.28 true rate, partly because of the expiration effect. In this case, the multiplicatively linked series shows a lower time rate of change than the additive series because the overall trend is up and this multiplicative linking rule shrinks those prices at later times compared to observations early in the series. (The last price in this series is rescaled by linking to about 70% of its observed value.)

Linked series also produce other common statistical measures that are biased when compared to the values obtained from the distortion-free series. The standard deviation of the reference series is 5.96 compared to 3.94 for the scaled down multiplicatively linked series. The additive series, with a standard deviation of 5.22, appears to be less volatile than the true series. Actually, the distortion-free and the additive series are approximately parallel and have a small positive trend. The expiration distortion in the additively linked series tends to reduce the value of the additive series relative to the conforming sequence on the right-hand side of the figure. This biases the additive series toward the mean, which causes the volatility measure to be lower than the true value. Serial correlation for the true series is 0.545 compared to 0.459 for the additive series and 0.439 for the multiplicative series. The negatively biased serial correlations of the linked series are caused by the expiration effect embedded in these price sequences, which have an upward trend. The additional slope distortion characteristic of multiplicative linking explains the greater bias in the multiplicative series. If the two linked series are joined by adjusting *preceding* observations rather than *succeeding* prices (the later method is used in Fig. 3) and if only prices after the December, 1993 rollover are considered, the true serial correlation coefficient is 0.035. The additive and multiplicative serial correlation coefficients are 0.124. Both linked series have the same correlation coefficients because no rollover has occurred. True prices are very weakly correlated, but linked prices show a stronger serial correlation because the expiration effect induces persistent into the series.

Weak Monotonicity Property

If observed prices change very little, the Monotonicity Property asserts that the index also should not change much. In addition, this trait requires that when any single contract price increases, the price index either increases or is unchanged.[2]

If $\tilde{p}_{\bar{t}} \geq p_{\bar{t}} \geq 0$, for some $\bar{t}$, then

$$+\infty > \Phi_{\bar{t}}(\tilde{p}_{\bar{t}}) \geq \Phi_{\bar{t}}(p_{\bar{t}}) \geq 0 \tag{4}$$

The two previous conditions require the index to preserve the scale and level of prices. The Monotonicity Property asks that any change in prices not cause a counterintuitive change in the index. All of the more than 250 price index functions investigated by Irving Fisher (1922) pass the Monotonicity test. Also, every linking method known to this author is smooth in every individual contract price at each point in time. All of these price index or linking schemes have some *other* defining characteristic which causes the index to possess the Monotonicity Property; therefore, this condition—though essential for mathematical reasons to insure that unique indices can be identified—is not a practical restriction in itself.

The price index defined by the rule, "select the closing price of the most active contract," might appear to be quite discontinuous if two contracts have a similar level of activity and the price to be used jumps back and forth between two contracts. However, this index is discontinuous in *time* and is continuous in *prices*. The Monotonicity Property considers a single, fixed time period and asks how the index would change if a price had been higher. For the "most active contract" time series, either the observed price is not that of the most active contract and any change in the price would have no effect on the index, or the contract is the most active and an increase in that price (compared to what the price might have been at the same time) causes the an identical change in the index because the price is exactly the index at that moment.

THE UNIQUE PRICE DISTORTION-FREE INDEX

Only one family of functional forms for the price index will possess the Information, Scale, Level, and Monotonicity Properties. The index must be a weighted average of observed prices. Some price weights may be

[2]The Monotonicity Property is somewhat weaker than the assumption that the index function is continuous in each price; yet, either assumption, coupled with conditions (1)–(3), produces the same index function. Several combinations of assumptions other than (1)–(4) also would produce the same index functional form. The essential features defining the index are the preservation of scaling and translation of prices plus continuity (or monotonicity, or at least a weak form of boundedness and monotonicity) of the Φ-function.

identically zero, implying that those prices do not enter into the determination of the index. Negative weights would not be allowed because the price index would decrease whenever the associated observed price increases. This would cause the index to fail the Monotonicity condition. At least one of the weights must be positive, and the sum of the weights must equal one. If the weights do not sum to one, an increase of all prices by one dollar would not be reflected in a one-dollar increase of the index; and the index would not retain the Level Property. The weights may be constants or they might change over time or may adjust in response to any other parameter.

A straightforward application of the methods of Aczel (1966, pp. 236–237) proves that a convex combination of observed prices is the *only* possible functional form of the index.[3]

$$\Phi_t(p_t) = c_t p_t$$

where $c_{t,i} = c_i(t, \ldots) \geq 0$ and $\sum_{i=1}^{n} c_{t,i} = 1$ and if $p_{t,i} \equiv 0$ then $c_{t,i} = 0$.

The index function, $\Phi_t(p_t) = c_t p_t$, allows a wide class of possible formulations and includes many of the procedures commonly used to link contracts. All of these methods would be feasible: "select the price of the contract nearest expiration"; "select the price of the most active contract"; or "report the average of the prices of the two contracts which are closest to expiration but have not reached the first notice day." Some other natural linking procedures do not preserve all the necessary traits. For example, the additive linking rule, "subtract from all later prices the difference between the expiration price of the old contract and the price of the next oldest on that date," was tested by Ma, Mercer, and Walker (1992) because the procedure is quit typical. This rule distorts the price scale and violates condition (2).[4]

THE ADDITIONAL ROLE OF TIME IN THE PRICE SERIES

Some price indexes are of the form, $\Phi_t(p_t) = c_t p_t$, and yet generate a time series with little useful content. The rule, "each day select the

[3]A complete proof is contained in Geiss (1994).

[4]This linking rule distorts price levels but not first differences. Some elimination of index distortion can be achieved by properly specifying the problem. If the model in which the time series will be used is concerned with price changes rather than price levels, then the index should be constructed to preserve the qualities of the price differences. Observed contract prices would be diferenced, and an index of differences employed. That index preserves the scale and level of differences. Similarly, if yields are important then the index should be made from observed yields. This yields index, which preserves scale and level of yields, would distort first differences and vise versa.

highest price from any of the commodity's contracts," does meet tests (1)–(4). Another feasible but improbable index would be generated by the procedure, "randomly select a single observed price each day." These examples suggest that it is not enough just to retain the numerical qualities of prices. The proper role of time in the index must be considered also. Time enters the first four properties in a static way. Each property defines the traits an index must have at each moment in time. However, these four conditions fail to insure that the effect of the passage of time on observed prices is captured in an undistorted manner by the index values. The following three essential index attributes assign an appropriate role to the time parameter. The result will be an index which is seamless and has a natural economic interpretation.

Time Continuity Property

Price continuity (from the Monotonicity Property) asserts that if time were fixed, minor changes in observed prices should not induce major jumps in the index. Time continuity is a comparable notion. If prices do not change, then arbitrarily small changes in time would be associated with, at most, minuscule responses in the index. The formal assumption, made in this spirit, is:

$$c_{t,i} = c_i(t, \ldots) \text{ is continuous in time} \tag{5}$$

Time continuity is a characteristic of the price index, while condition (5) requires that the relative weight assigned to each contract evolves smoothly. These conceptually distinct notions are equivalent when the other properties of a distortion-free index are present. Jointly, these properties require that the index be of linear form, $\Phi_t(p_t) = \sum_{i=1}^{n} c_{t,1} p_{t,i}$. This equation implies that continuity of $c_{t,1}$ is sufficient for time continuity of $\Phi_t(p_t)$, and vice versa.

The Time Continuity Property does not imply that the observed prices are produced by a continuous process; nor does this assume that the entire index series is continuous. Also, condition (5) does not require that identical sets of prices observed at different times generate the same price index. In fact, if the price index is to have economic meaning, the index should increase as time passes when observed contract prices are constant, because, as the contract approaches expiration, the avoided carrying costs are less and the implied underlying contract value will be greater.[5]

[5]A normal carry market is assumed.

All time series based on rules such as "use the price of the nearest contract (or nearest contract not in the delivery month, etc.)" fail the Time Continuity test because these rules cause discontinuous switching from near months to later delivery months. On the rollover date, the smallest possible change in time will cause a jump in the index as the near contract is dropped.[6] Additive and multiplicative linking rules force the time series to appear to be continuous in prices on the rollover day; but these linking methods also fail to preserve the Time Continuity Property (and others) because any rolling over from one single price to another must cause a time discontinuity in the values of the price weights, $c_{t,i}$.

A future time series based on the rule "use the price of the most active contract" also does not possess the Time Continuity characteristic because contracts enter and leave the computation of the index in a discontinuous fashion. In addition, the weights are a function of volume (or open interest) and not of time, which allows time to change a small amount, and, if a new most active contract appears, causes the index to shift by a relatively great amount. Finally, a rule which commonly does have the Time Continuity Property (and all four price properties) is "let each contract's price weight be equal to the open interest of that contract divided by the total open interest." If the contract's open interest changes in a reasonably smooth manner over time, the price weights will be indirect, continuous functions of time.

Seamlessness Property

Suppose some sort of market adjustment mechanism is producing a series of observed contract prices. As long as the price-generating process continues to operate in the same manner, the representation of the process by the price index time series should be unchanged. For instance, suppose a correlation exists among contract prices one day apart and that the price index captures this correlation. Then, the time series created from index values should reflect the correlation without regard to the date the observations began—even if that date happens to be at the expiration of a contract.

A sufficient and somewhat weaker "seamlessness" condition is:

[6] If the near and next near contracts always have the same prices, there will be no discontinuity in the index; but if all contract prices are the same there is no "index number problem."

If, for any two price vectors, p_t and $\tilde{p}_{t+\lambda}$,

$$\Phi(p_{t+\lambda}) - \Phi(p_t) = \kappa$$

then, whenever $p_{t+\lambda+\lambda} = p_{t+\lambda} + (p_{t+\lambda} - p_t)$, it must also be true that

$$\Phi(p_{t+2\lambda}) - \Phi(p_{t+\lambda}) = \kappa \tag{6}$$

The multiplicatively linked time series defined by the rule, "on the rollover date scale all preceding prices so that the rescaled expiring contract's price equals the price of the next near contract," produces a time series that is smooth but is not seamless because rescaling modifies preexisting relationships. To illustrate, suppose all contract prices are increasing by one dollar each day. If today is the last observation of the series, the price index, by this rule, will be equal to the price of the near contract. The index will be increasing at one dollar per day also. Let the next near contract have a premium of 5% over the near contract's price. On the rollover date, all preceding prices are increased by a factor of 1.05. The price index will continue to reflect a dollar per day rate of change for all days after this rollover but will show a \$1.05 per day for earlier dates. There is no change in whatever force is causing all market prices to increase in a dollar-per-day systematic fashion over the entire period; however, the multiplicative linking method induces what appears to be a shift in the rate of change of prices. The Seamlessness Property prevents this sort of distortion.

History Relived Property

The previous two conditions place structure on the evolution of the index with the passage of time. However, if history happens to repeat itself, the index should capture this also. If one year's observed contract prices are replicated exactly the next year, the data would be identical except for a systematic one-year shift in the observation date and the expiration date of the relevant contracts. The History Relieved Property asserts that the price index series for both years should be the same. Differences which are no more than shifts in date labels will not change the price index.

This property is captured by the condition:

Let τ be the number of days between the expiration of contract i and contract $i + 1$. If $p_{t,i} + p_{t+\tau,i+1}$ for all i, then:

$$\Phi_t(p_t) = \Phi_{t+\tau}(p_{t+\tau}) \tag{7}$$

The Seamlessness, Continuity, and History Relived properties create an internal time consistency that results in a price index series which is just as smooth as any linked series. The Continuity Property imposes the requirement that every single piece of data must: have no role in the index; have a uniform role; or have a steadily changing role. This implies that at each moment the index must be smoothly changing with the passage of time. The Seamlessness trait requires that any "time → observed price → price index" connection which holds over one interval must be captured the same way over any other comparable finite increment of time. The joint effect of these two conditions is to extend the time continuity which exists at a point in time to entire intervals, which are bounded by the date some contract first begins to trade and the date another contract ceases trading. The History Relieved characteristic forces these intervals to join together in a systematic manner. Linking is achieved through time consistency rather than by direct assumption.

THE SEAMLESS, DISTORTION-FREE INDEX

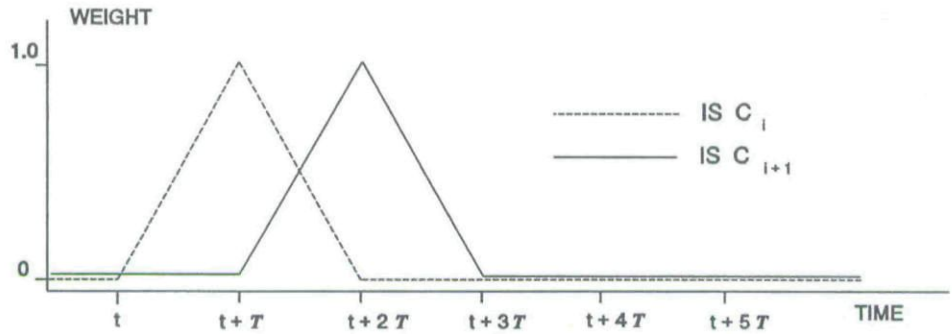
Only one class of price index functions will provide a seamless series which preserves the numerical qualities of the observed prices. The most general index function which meets conditions (1)–(7) is (Geiss, 1994):

$$\Phi_t(p_t) = c_t p_t$$

$$c_i(t) = \alpha_{i,t} + \beta_{i,t} t$$

In addition, these index properties imply that the weight functions, $c(t)$, are composed of connected, straight lines. The weight must be zero if the contract has not started trading. As the contract's price enters the index, the weight begins to increase at a steady rate. There may be several discrete changes in the slope of the weight function, but eventually the weight must decrease to zero before the rollover date.

The simplest weight function is the tent shape shown in Figure 4. Two prices determine the index at any point in time. At time t, the price of contract i enters the index with a weight of zero. The observed price of the ith contract increases and reaches a maximum of 1.0 at time $t + \tau$. After $t + \tau$, weight c_i begins to fall and weight c_{i+1} increases at the same rate, which guarantees that the weights always sum to one. Contract i is dropped from the index at time $t + 2\tau$. The price of contract $i + 1$ is the index at this moment. An instant later, contract $i + 2$ will begin to enter the index.



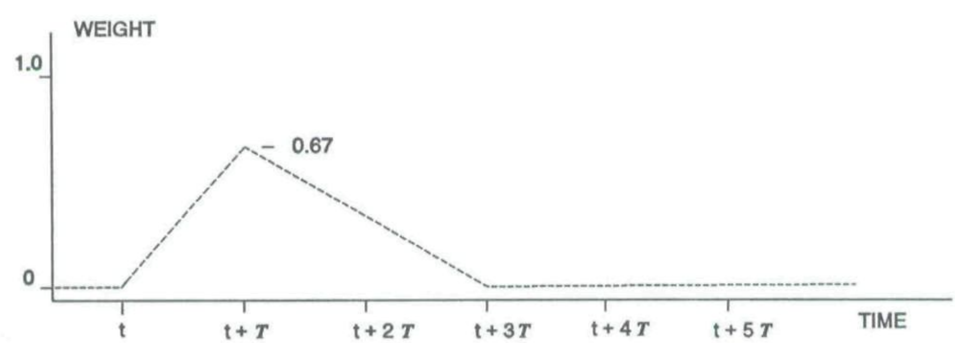
The Simplest Weight Function with Two Prices

FIGURE 4

A profile of the simplest weight functions. Contract i (the dashed line) enters the index at time t. The importance of contract i increases to a maximum of 100% at time $t + \tau$ and then begins to decrease. Contract $i + 1$ (solid line) increases in weight, replacing contract i.

Many other shapes of the weight function are possible. Figures 5–7 indicate some weight profiles that produce distortion-free indexes. Profiles may be any connected shape composed of linear segments that: is never negative; begins at zero; changes slope at times which are multiples of the period between the new contracts; and has a shape and spacing which insure that the sum of the weights always is one.

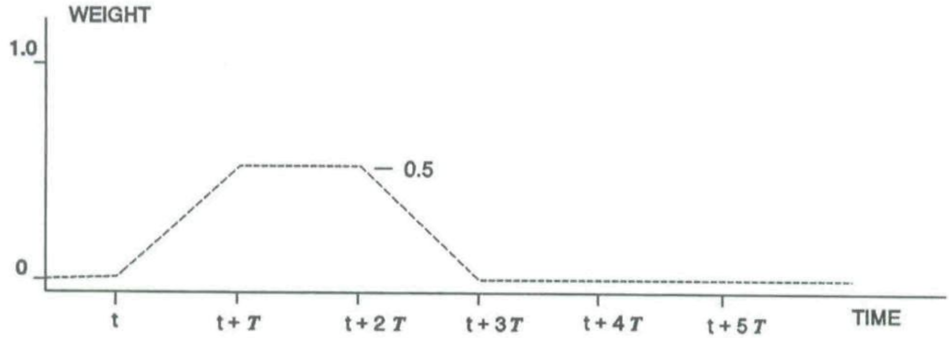
The remainder of this article considers only the two-price, tent-shaped weight profile of Figure 4. This is the simplest shape, and there is no theoretical reason to select a more complex profile. Also, a series of tests suggests that actual indexes are not sensitive to changes in the



One Possible Weight Function with Three Prices

FIGURE 5

Weight profile with three contracts in index. The maximum weight attached to a single price is 2/3. Another price has 1/3 of the weight and a new third price is just entering the index. The newest price rapidly becomes more important as the weights of the older two contracts both fall at half that rate.



Another Three-Price Weight Function

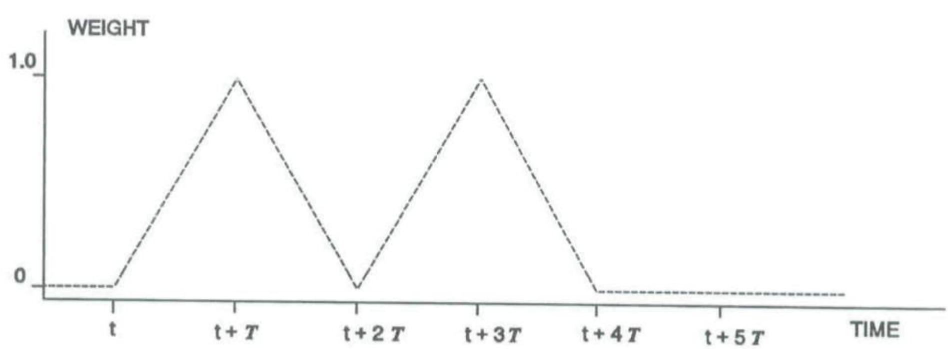
FIGURE 6
An alternative three-contract weight profile. The "tent" shape profile of Figure 5 causes each price to be 2/3 of the value of the index at the maximum weight. This profile assigns less peak weight to a single price but maintains that weight for the time between two rollover dates.

weight formula. Sixty price indexes are formed using a variety of weight functions and six different commodities. No identifiable difference found between indexes based on different weight profiles.

THE DISTORTION-FREE INDEX IS RELATED TO A PORTFOLIO OF CONTRACTS

The simple, two contract price index is a convex combination of the prices for the two contracts nearest expiration.

$$\Phi_t(p_t) = c_1 p_{t,i} + c_2 p_{t,i+1}$$



One of Many Possible Four-Price Weight Functions

FIGURE 7
One of many possible four-contract weight profiles. This legitimate profile causes a contract's price to increase in importance, decrease, and then increase and decrease again. No economic justification could exist for such a profile. When tested, these complex profiles produce indexes that are indistinguishable from indexes based on simple weight functions.

Let E_i be the date of expiration for contract i plus one day. If $E_i \leq (t + \Omega) \leq E_{i+1}$, then,

$$c_1 = [E_{i+1} - (t + \Omega)]/[E_{i+1} - E_i] \tag{8}$$

for some nonnegative integer, Ω, and

$$c_2 = 1 - c_1$$

This price index may be interpreted as the market value of a portfolio composed of the near and next near contracts. The proportion of the portfolio in the near contract is c_1. This fraction decreases linearly as time passes. The proportion allocated to the next near contract increases in a steady manner from zero to 100% and then back to zero. This index could be interpreted to be the value of a perpetual contract with a fixed time to maturity as employed by Herbst, Kare, and Caples (1989). The days to expiration of this created perpetual contract is the number of days between expiration dates minus one plus the value of the constant, Ω.

Ω represents the number of days between the last day the contract's prices are included in the index and the expiration date. This variable is called the cutoff parameter. If the economic problem suggests that the last two weeks of prices are not representative of the usual pricing mechanisms, Ω would be set to 14. The last day the oldest contract's price would be included in the index is 15 days before expiration. After that day, the included contracts would roll over to the second and third oldest contracts.

CHOICE OF THE CUTOFF PARAMETER

Economic Rationale

Equation (8) shows that the distortion-free price index can be specified exactly for this weight profile if the value for the cutoff parameter is known. Every positive value of the parameter, Ω, theoretically is consistent with the formation of a seamless price index, but ranges of value usually can be eliminated by the nature of the futures contract. Very large parameter values may not be possible, because, when one contract is removed from the index Ω days before expiration, a new contract must be available to enter the index. The contract newly added to the index must be trading $\Omega + \tau$ days before expiration. Futures contracts with that many months to expiration might not exist for that product.

Additional constraints on the selection of the cutoff parameter may come from the economic nature of the problem in which the index number is employed. Discontinuous changes in the contract's features are particularly important since such jumps are not consistent with the continuity induced by the time-related properties. As a specific example, suppose an index is formed from the two nearest contracts for palladium traded on the New York Mercantile Exchange. This commodity's near contracts are one month apart. The near contract has no limits on daily price changes and delivery notice may be made. The next year contract does have daily price limits and has no delivery notice.

The number of daily units of carrying charges and convenience yields is held constant by the nature of the palladium price index. The mean number of days in the perpetual contract depends on the number of days between contracts, in this case 30, and the cutoff parameter, Ω. Suppose this parameter is set to zero, which means that the near contract's price remains in the index until the last trading day. If there are ten days until the expiration of a contract, eq. (8) gives the index weights as $c_1 = 11/30$, and $c_2 = 29/30$. The near contract has ten days of net carrying costs to expiration; the next near price reflects 40 days of carrying costs to the expiration of that contract. The price index has $[11/30] \times 10 + [29/30] \times 40 = 29$ days of carrying costs. The next day, nine days to expiration, the index weights would adjust as required by eq. (8) and the number of days of costs in each observed price would fall by one day for each contract. The index would maintain the same number of days of carrying costs imbedded in the price ($[10/30] \times 9 + [20/30] \times 39 = 29$).

The price index for palladium can be standardized in terms of carrying costs because daily costs are reasonably constant between contracts and the index reflects a constant number of days of cost in the portfolio. Deliverability, because delivery notice either can or cannot be made, is discontinuous in time. The palladium contract price index ten days from expiration with the cutoff parameter set at zero is the value of a portfolio with 36.6% of the members deliverable and the rest not deliverable. With nine days to expiration, 33% of the portfolio is deliverable. The price index with this value of the cutoff parameter cannot hold constant "deliverability" or "existence of a daily price limit" or any other attribute that changes as a nonlinear function of time.

A meaningful palladium price index might be formed, however. To do so, the problem of discontinuity must be avoided by setting the cutoff parameter at a value which retains only contracts with attributes which are identical or that change by the same amount each day. One feasible index could be generated if the cutoff parameter is set to 30. Suppose the time is still ten days before expiration. The near contract would have been dropped from the index, $c_1 = 0$. The next near contract would have a weight of 11/30 and the third nearest contract a weight of 19/30. The number of days of carrying costs included in the index is: $0 \times 10 + [11/30] \times 40 + [19/30] \times 70 = 59$. The amount of deliverability in the index is: 0 × "deliverably" + [11/30] × "not deliverable" + [19/30] × "not deliverable" = "completely not deliverable." This palladium contract appears to have a meaningful futures price index whenever the cutoff parameter is at least 30 because all of the attributes of contracts with more than 30 days to expiration are either the same (e.g., presence of price limits) or are a homogeneous linear function of time (e.g., carrying costs).

Independence of the Choice of Cutoff Date

The nature of the connection between the cutoff date and the value of the index depends on how the market prices perpetual portfolios of different durations. If $\Phi_t^{29}(p_t)$ denotes the price index created by a portfolio with 29 days duration, then $\Phi_t^{29}(p_t) - \Phi_t^{28}(p_t)$ indicates the impact on the price index of a one-day change in the cutoff parameter. This difference, $\Phi_t^{29}(p_t) - \Phi_t^{28}(p_t)$, also is the market value of one extra day of net carrying costs.

When daily net carrying costs are constant, all distortion-free price indexes are related by:

If d is the duration of the portfolio,

$$\Phi_t^d(p_t) = \gamma + \delta d \tag{9}$$

The slope parameter is the market-revealed daily net carrying costs, and the intercept is the imputed market value of a portfolio with zero days duration—a portfolio with all net carrying costs removed.[7] The zero-duration portfolio, $\Phi_t^0(p_t)$, cannot be observed directly or even approached, because the minimum duration for a two-price portfolio is the number of days between the expiration dates minus one.

[7] Daily convenience yields may be a constant function of the number of units of the commodity in the contract while carrying costs could be related to the current value of the contract. Both of these types of factors can be captured by regressing price indices on duration. If a linear relation exists, Φ^0 has meaning.

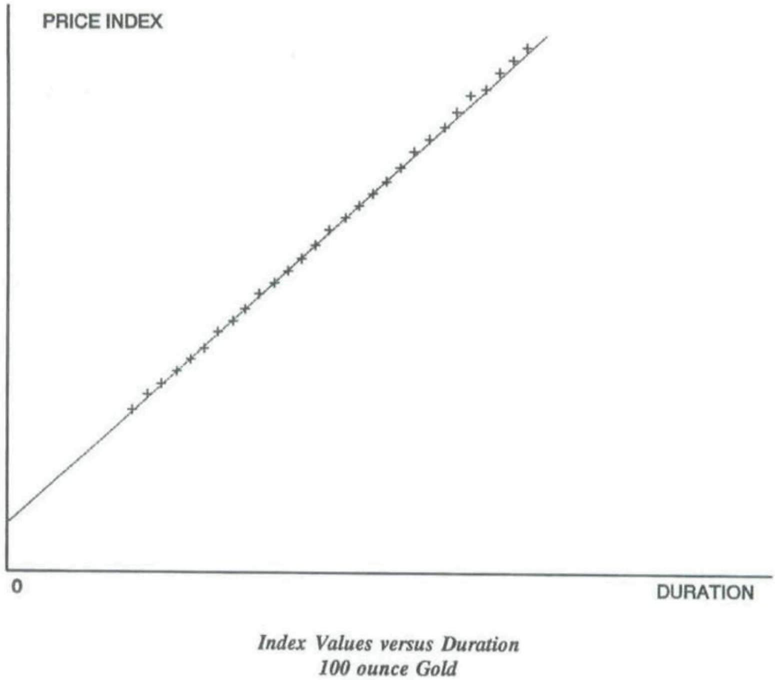


FIGURE 8
Index values versus duration of the index for 100 ounces of gold. The average value of the index is a linear function of duration. The natural normalization for this price index is the portfolio with zero days duration, the vertical intercept of the regression.

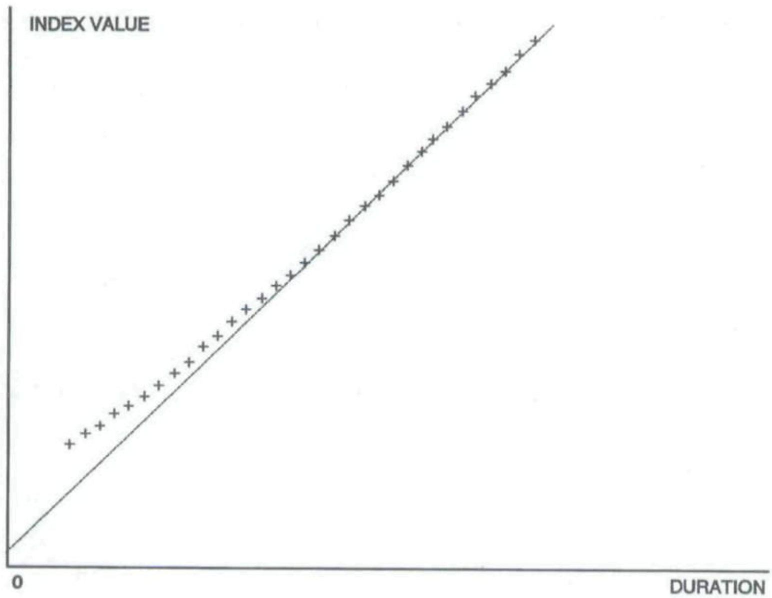
The zero-duration portfolio is meaningful and a natural normalization of the family of indexes as long as the market data for this commodity reveal that duration and price are linearly related. Figure 8 shows that this linear connection does hold for the gold contract.[8] The estimated relation is:[9]

$$\Phi_t^d(p_t) = 361.586 + 0.0302d$$

The three cents per day net carrying costs implied by this regression are consistent with the intercontract price spreads during this period.

The live cattle contract (Fig. 9) is an example of a commodity that does not have a natural, unique, distortion-free price index because

[8]The observation dates range from August 31, 1992 through January 5, 1994 for the 100-ounce gold contract on the Chicago Mercantile Exchange.
[9]t-values for the intercept and slope coefficients are 115,466 and 548. $R^2 = 0.9992$. The error terms are stable with the maximum individual error equal to 1.7 standard deviations for 100 indexes (duration for 90 to 198). When this equation is restated as a second order polynomial, the quadratic duration term has a coefficient not different from zero.



Index Values versus Duration
Live Cattle

FIGURE 9
Index values versus duration of for live cattle. Price and duration are not linearly related. No single price index is appropriate for all cutoff dates for this commodity.

duration and the price index are not linearly related.[10] One practical implication is that the user of a live cattle price index should select a cutoff parameter value that is in the linear segment of the graph. All indices in this range would be identical except for an additive constant. Statistical tests should be robust to the choice of index (but would reflect properties of cattle prices only for contracts not near expiration). The alternative approach suggested by Figure 9 is to perform all statistical tests several times each with a different cutoff parameter.

CONCLUSIONS

A lengthy futures prices time series is a set of numbers which represent prices on each date and are consistent in the sense that the price on one day has meaning with respect to the price on another day. Most prices series are generated by linking together sets of observed prices of

[10]The observation dates are the same. The contract is CME live cattle. A second order polynomial fits these data significantly better than does a linear form. The slope of the price index-duration relation for cutoff parameters from 0 to 20 is statistically different from the slope with the cutoff greater than 40.

the nearest or the most liquid contract. The resulting time series will never both be seamless and preserve the level and scale of prices no matter what method is used to roll over from one contract to the next. An alternative is to produce a price index with the desired properties. This study shows that very natural characteristics generate exactly one well-defined family of possible index functions.

The price index must be a weighted average of observed prices for contracts with varying expiration dates. The index must reduce the weight attached to near contracts as the expiration date approaches. Weight is shifted systematically to more distant contracts. At expiration (or the number of days before expiration selected for the cutoff date), the near contract price is given no weight and the value of a more distant contract automatically enters the calculations.

One could interpret this price index as a special type of linking function. Most linking procedures employ the prices of one contract exclusively and then shift to another contract on the rollover day. The distortion-free price index brings in each new contract's prices progressively over time and then tapers down the role of those prices as the contract approaches expiration. The link becomes a smooth, seamless splice.

The price index also can be viewed as the value of a portfolio formed from contracts with varying expiration dates. Composition of the portfolio is continuously adjusted so that the average number of days to expiration is a constant. The average life of this perpetual contract is held fixed, which eliminates problems caused by changing valuation due to the approach of expiration.

Only one important variable in the price index is not predetermined by theory. This is the number of days before expiration that a contract will no longer be given weight in the index. The correct cutoff day may be suggested by the problem; but if not, the data often will reveal that an index can be formed which is independent of the cutoff day.

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