A Two-Factor, Preference-Free Model for Interest Rate Sensitive Claims

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INTRODUCTION

Many studies have shown that one-factor models that use the instantaneous short rate as the factor and therefore result in perfect correlation among bonds are not suitable in modeling the current term structure.[1] The emphasis of past research has been on developing a term structure model that can well describe the curvature of the observed term structure. Several different approaches have been taken to resolve this problem. One approach is to take the current term structure as given and incorporate term structure fluctuations in various ways. Ho and Lee (1986) model the uncertainty by putting perturbation functions on the forward prices. Black, Derman, and Toy (1990) model the uncertainty through a volatility curve.[2] The Ho–Lee model is easier to calculate

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[1]The well-known one-factor models include Vasicek (1977); Cox, Ingersoll, and Ross (1985); Dothan (1978); and Longstaff (1989). The empirical studies that show one-factor models can not fit the yield curve are Chen and Scott (1991) and Pearson and Sun (1989) for maximum likelihood; Heston (1989) and Gibbons and Ramaswamy (1993) for generalized method of moments: Letterman and Scheinkman (1991) for factor analysis.

[2]The volatility curve is also called the term structure of the volatility. This assumption is identical to random volatility.

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but generates negative interest rates. The Black–Derman–Toy model does not generate negative interest rates but needs to solve a series of simultaneous equations. The continuous time model by Heath, Jarrow, and Morton (1992) also takes the initial term structure as given. Their model is in spirit similar to the Ho–Lee model. They let the forward rate follow a stochastic process and the term structure is a function of the forward rate. Their model cannot generate a closed form solution of the term structure unless the normality assumption is adopted and that generates negative interest rates. All of these studies take the current term structure as a given to perfectly fit the current term structure.

Another approach is to improve the one-factor model so that all shapes of the yield curve can be generated. Hull and White (1990) let the parameters in the stochastic process of the instantaneous rate be deterministic functions of time. This generalization, maintained closed form, will give the term structure model more flexibility to generate yield curves. However, their model provides no closed form solutions and has to rely on numerical methods. Constantinides (1992), by modeling the marginal rate of substitution, provides a one-factor model where bonds are not perfectly correlated; hence, all shapes of the yield curve can be generated. His model is able to prevent negative interest rates and produce simple solutions to the term structure and its contingent claims at the same time. The drawback of the model is that the option formula depends on an unobservable factor. Therefore, it is difficult to be tested empirically.

Yet another approach is to add more factors to the term structure model. Richard (1978); Brennan and Schwartz (1979); Cox, Ingersoll, and Ross (1985); and Longstaff and Schwartz (1992) have developed two-factor term structure models where either an arbitrage-free or utility-based methodology is used.[3] Richard uses the real rate of interest and inflation as the two factors and argues that the sum of the two factors will yield the instantaneous rate. Since bond prices are functions of the instantaneous interest rate, the two sources of risk will produce bond prices with enough flexibility to fit any shape of the yield curve. Brennan and Schwartz, on the other hand, believe that the most important factors that characterize a term structure are its two ends, short and long rates. Bond prices along the yield curve, therefore, are explained by short and long rates. By modeling these two rates with two

[3]It is understood now that these two methods are consistent if markets are complete. If markets are not complete, then it is impossible to implement the no-arbitrage argument without a utility-based asset pricing theory.

different stochastic processes, they build a term structure model with two factors. A common characteristic of these two models is that factors are chosen arbitrarily. There is little theoretical support to such choices. Cox, Ingersoll, and Ross (CIR hereafter), on the other hand, construct a model under a general equilibrium framework in which they derive factors and their dynamics from production technology and an additive utility function of a representative agent. They find that in equilibrium, the instantaneous rate can be expressed by separate factors. There are two ways of constructing two factors in their model. The first method is to directly decompose the instantaneous rate into two factors, each of which follows some stochastic process. This decomposition produces a simple solution of the term structure. As pointed out by Chen and Scott (1993), this method is tractable empirically and maximum likelihood estimation is possible. Another method is to observe that the CIR model can be interpreted as a stochastic volatility model because the volatility of the instantaneous interest rate is a function of the two factors. Longstaff and Schwartz (1992) adopt this approach and derive their two-factor term structure model.

This article is similar to the first method of CIR where one decomposes of the instantaneous rate into two correlated factors. Each of the factors is assumed to follow an Ornstein–Uhlenbeck (O–U hereafter) process, as does Vasicek (1977). Langetieg (1980) derived a similar Gaussian model with multiple factors, but he does not provide any explicit solutions to the term structure and its claims.[4]

This study derives simple, closed-form (in the Black–Scholes sense), preference-free formulas for interest rate contingent claims. It also shows alternative derivations of contingent claim prices. Although this article focuses on a two-factor model, it can accommodate any number of factors. Allowing more factors in the term structure makes the model more flexible and, therefore, both the initial interest rate term structure and the volatility term structure can be reasonably fitted.

The following section presents the general, two-factor model followed by various pricing formulas for interest rate claims such as options, futures, forwards, and futures options. The third section illustrates how this two-factor model can be shown as a preference-free model and the fourth contains a discussion of the implications of the model. An appendix includes some derivations.

[4]Langetieg does provide a closed form solution to discount bond options with Merton's (1973) result.

THE TWO-FACTOR MODEL

Following the suggestion by CIR, the instantaneous rate, r, is decomposed into two correlated factors, y_1 and y_2 (i.e., $r = y_1 + y_2$) each of which follows an Ornstein–Uhlenbeck (O–U) process as follows:

$$\begin{cases} dy_1 = \alpha_1(\mu_1 - y_1)\,dt + \sigma_1\,dW_1 \\ dy_2 = \alpha_2(\mu_2 - y_2)\,dt + \sigma_2\,dW_2 \end{cases} \tag{1}$$

where α, μ, and σ are constants, $W(t)$ is a standard Wiener process with mean 0 and variance dt, and $E[dW_1 dW_2] = \rho\,dt$. According to CIR, the decomposition can take place for n factors. The next section shows that the number of factors in the decomposition will not alter the final result but only introduces more parameters and provides more flexibility. The O–U process was first used by Vasicek (1977) in modeling the term structure. The transition density of the process is well studied and used [see Arnold (1974), Section 8.2]. It is normal with the conditional mean and variance–covariance as follows:

$$E[\,y_i(s)\,|\,y_i(t)] = \mu_i + (\,y_i(t) - \mu_i)e^{-\alpha_i(s-t)}$$

$$\operatorname{cov}[\,y_i(s), y_j(s')\,|\,y_i(t), y_j(t)] = \frac{\rho\sigma_i\sigma_j}{\alpha_i + \alpha_j}\,e^{-\alpha_i(s-t)}e^{-\alpha_j(s'-t)}(e^{(\alpha_i+\alpha_j)(s'-t)} - 1)$$

where $i, j = 1, 2$ and $s' > s$. The following simplified notation is used throughout the analyses in this study:

$$m_i = E[\,y_i(s)\,|\,y_i(t)] = \mu_i + (\,y_i(t) - \mu_i)e^{-\alpha_i(s-t)}$$

$$v_i^2 = \operatorname{var}[\,y_i(s)\,|\,y_i(t)] = \frac{\sigma_i^2}{2\alpha_i}(1 - e^{-2\alpha_i(s-t)})$$

$$c_i = \operatorname{cov}[\,y_i(s), y_i(s')\,|\,y_i(t)] = \frac{\sigma_i^2}{2\alpha_i}\,e^{-\alpha_i(s+s'-2t)}(e^{2\alpha_i(s'-t)} - 1)$$

$$c_{ij} = \operatorname{cov}[\,y_i(s), y_j(s)\,|\,y_i(t), y_j(t)] = \frac{\rho\sigma_i\sigma_j}{\alpha_i + \alpha_j}(1 - e^{-(\alpha_i+\alpha_j)(s-t)})$$

The following notation for various prices is used throughout the article:

$P(\{y_1, y_2\}, t; T)$ = Pure discount bond price at time t expiring at time T,

$C(\{y_1, y_2\}, t; T_c, T)$ = Call premium on $P(\{y_1, y_2\}, T_c; T)$,

$\Phi(\{y_1, y_2\}, t; T_f, T)$ = Futures price of $P(\{y_1, y_2\}, T_f; T)$,

$\Psi(\{y_1, y_2\}, t; T_f, T)$ = Forward price of $P(\{y_1, y_2\}, T_f; T)$, and

$C(\{y_1, y_2\}, t; T_c, T_f, T)$ = Futures call premium on $\Phi(\{y_1, y_2\}, T_c; T_f, T)$.

Let a pure discount bond price at time t, a function of the two factors, be the current value of \$1 paid at time T. Then, by Ito's rule, the dynamics for the bond prices can be written as follows:

$$dP = \mu_p\, dt + \sigma_1 P_1\, dW_1 + \sigma_2 P_2\, dW_2$$

where

$$\mu_p = P_t + \alpha_1(\mu_1 - y_1)P_1 + \alpha_2(\mu_2 - y_2)P_2 + \frac{\sigma_1^2}{2} P_{11} + \frac{\sigma_2^2}{2} P_{22} + \rho\sigma_1\sigma_2 P_{12}$$

Subscripts of P represent partial derivatives with respect to the state variables with the same subscripts. Following either the standard arbitrage argument in Vasicek (1977) or the general equilibrium argument in CIR (1985), one can derive the following partial differential equation (PDE) for the bond price:

$$P_t + [\alpha_1(\mu_1 - y_1) - q_1\sigma_1]P_1 + [\alpha_2(\mu_2 - y_2) - q_2\sigma_2]P_2 + \frac{\sigma_1^2}{2} P_{11} + \frac{\sigma_2^2}{2} P_{22} + \rho\sigma_1\sigma_2 P_{12} = (y_1 + y_2)P \quad (2)$$

This equation requires a two-factor asset pricing model in continuous time of the form:[5]

$$\mu_p = \left(r + q_1\sigma_1 \frac{P_1}{P} + q_2\sigma_2 \frac{P_2}{P}\right)P$$

where q_1 and q_2 are market prices of risk associated with two factors, y_1 and y_2, respectively. Equation (2) is an initial-boundary value problem in differential equations. Standard separation of variables technique is used to solve the above PDE with the boundary condition, $P(\{y_1, y_2\}, T; T) = 1$.[6] The solution is the following function of the two factors and time:

$$P(\{y_1, y_2\}, t; T) = e^{-y_1 F_1(t,T) - y_2 F_2(t,T) - G(t,T)} \quad (3)$$

[5]This equation is used extensively in the option literature when the underlying state variables are not publicly traded [for example, see Richard (1978, p. 47–84)].
[6]See the appendix for a sketch of how to solve eq. (1).

where

$$F_i(t,T) = \frac{1 - e^{-\alpha_i(T-t)}}{\alpha_i}$$

$$G(t,T) = \sum_{i=1}^{2}\left(D_i[T - t - F_i(t,T)] + \frac{\sigma_i^2 F_i^2(t,T)}{4\alpha_i} - \frac{\rho\sigma_1\sigma_2}{\alpha_1\alpha_2} \cdot \left(T - t - F_i(t,T) + \frac{1 - e^{-(\alpha_1+\alpha_2)(T-t)}}{\alpha_1 + \alpha_2}\right)\right)$$

$$D_i = \mu_i - \frac{\sigma_i q_i}{\alpha_i} - \frac{\sigma_i^2}{2\alpha_i^2}$$

This solution was first provided by Chaplin (1987b). It is apparent that if the correlation between two state variables is zero, then the solution will reduce to a product of two Vasicek formulas (1977), each of which depends on one of the state variables. Also, eq. (3) will return to the one-factor model of Vasicek when one of the factors disappears.

The option on the above pure discount bond will have to satisfy the same PDE as eq. (2) because both of them are contingent claims on the same factors. If the bond price dynamics can have a closed form transition density, then the PDE that the option has to satisfy can be written in terms of the bond price. This is precisely the key reason why preference-free formulas exist for bond options, futures, forwards, futures options, and forward options in the next section. If the processes cannot result in a closed form distribution for the bond price, solutions are more complex.[7] The PDE is the same as eq. (2) with a switch of notation:

$$C_t + [\alpha_1(\mu_1 - y_1) - q_1\sigma_1]C_1 + [\alpha_2(\mu_2 - y_2) - q_2\sigma_2]C_2 + \frac{\sigma_1^2}{2}C_{11} + \frac{\sigma_2^2}{2}C_{22} + \rho\sigma_1\sigma_2 C_{12} = (y_1 + y_2)C \quad (4)$$

The same PDE is reached because all arbitrage arguments that apply to bonds can apply to their options as well. What makes the solution different is the boundary condition. The boundary condition for options is $\max\{P(\{y_1, y_2\}, t; T_c, T) - K, 0\}$ for calls and $\max\{K - P(\{y_1, y_2\}, t; T_c, T), 0\}$ for puts. Here, the formula is presented in a

[7] See Chen and Scott (1992) for the solutions under the square root processes. They are already complex when the correlation is zero.

consistent notation for call as follows:[8]

$$C(\{y_1, y_2\}, t; T_c, T) = P(\{y_1, y_2\}, t; T)N(d_1) - P(\{y_1, y_2\}, t; T_c)KN(d_2) \tag{5}$$

where

$$\begin{aligned} d_1 = \frac{1}{\sigma_p}\Bigg(&\sum_{i=1}^{2} y_i F_i(T_c, T)e^{-\alpha(T_c - t)} - F_i(T_c, T) \\ &\cdot [W_i(T_c, T) - W_i(t, T)e^{-\alpha(T_c - t)}] \\ &- G(T_c, T) - \ln K\Bigg) \end{aligned}$$

$$\begin{aligned} d_2 = \frac{1}{\sigma_p}\Bigg(&\sum_{i=1}^{2} y_i F_i(T_c, T)e^{-\alpha(T_c - t)} - F_i(T_c, T) \\ &\cdot [W_i(t, t) - W_i(t, T_c)e^{-\alpha(T_c - t)}] \\ &- G(T_c, T) - \ln K\Bigg) \end{aligned}$$

$$W_i(t, s) = D_i - \frac{\sigma_i^2 F_i(t, s)}{2\alpha_i} - \frac{\rho\sigma_1\sigma_2}{\alpha_i(\alpha_1 + \alpha_2)} - \frac{\rho\sigma_1\sigma_2}{\alpha_1 + \alpha_2} F_i(t, s)$$

$$\sigma_p = \sqrt{\sum_{i=1}^{2} v_i^2 F_i^2(T_c, T) + 2c_{12}F_1(T_c, T)F_2(T_c, T)}$$

It is clear that the solution is a Black–Scholes type. The first term of the solution is the current asset price multiplied by a normal probability. The second term is the discounted strike multiplied by another normal probability. Note that the discount factor that discounts the strike price is a discount bond price which reflects the uncertainty of the two factors. Comparing this option model with the option model developed by Chen and Scott (1992), it can be seen that the univariate integration of a normal density is very straightforward and accurate closed form approximations can be found.[9] This is because the linear combination of two normal variates is still normally distributed. The same statement can not be made for two non-central chi-squared variates. In the next section, this property is explicitly used to derive a preference-free

[8]The verifications of the two formulas can be found in Chaplin (1987b).

[9]See, for example, Hull (1993) and Stoll and Whaley (1993).

representation of eq. (5). The difficulty of deriving a density for the linear combination of two non-central chi-squared variates is essentially the reason why it is difficult to arrive at a preference-free form in CIR (1985) for the one-factor option model and Chen and Scott (1992) for the two-factor model.[10]

As pointed out in a number of articles [e.g., Carr (1988)], the futures price will satisfy a different PDE due to the zero-endowment nature of futures contracts. The no-arbitrage argument implies the following PDE for the futures price:

$$\Phi_t + [\alpha_1(\mu_1 - y_1) - q_1\sigma_1]\Phi_1 + [\alpha_2(\mu_2 - y_2) - q_2\sigma_2]\Phi_2 + \frac{\sigma_1^2}{2}\Phi_{11} + \frac{\sigma_2^2}{2}\Phi_{22} + \rho\sigma_1\sigma_2\Phi_{12} = 0 \quad (6)$$

It should be noted that eq. (6) differs from eq. (2) or (4) on only the right-hand side. In eq. (6), the right-hand side is zero to reflect the return of a zero-dollar investment. The solution can be obtained by solving the PDE with the boundary condition, $\Phi(\{y_1, y_2\}, T_f; T_f, T) = P(\{y_1, y_2\}, T_f; T)$,. The technique of solving the above PDE is the same as before (see an appendix) and the solution is:

$$\Phi(\{y_1, y_2\}, t; T_f, T)e^{-y_1X_1(t)-y_2X_2(t)-Y(t)} \quad (7)$$

where

$$X_i(t) = F_i(t, T) - F_i(t, T_f)$$

$$Y(t) = \sum_{i=1}^{2}\left(D_i[T - T_f - X_i(t)] - \frac{\sigma_i^2}{2\alpha_i^2} \cdot [X_i(t) - \frac{\alpha_i}{2}X_i^2(t) - F_i(T_f, T)] + \frac{\rho\sigma_1\sigma_2}{\alpha_1\alpha_2}F_i(T_f, T)\right) - \frac{\rho\sigma_1\sigma_2}{\alpha_1\alpha_2}\left(T - T_f - \frac{\alpha_1\alpha_2}{\alpha_1 + \alpha_2}[X_1(t)X_2(t) - F_1(T_f, T)F_2(T_f, T)] + \frac{1 - e^{-(\alpha_1+\alpha_2)(T-T_f)}}{\alpha_1 + \alpha_2}\right)$$

[10]Note that the square-root process does not generate a non-central chi-squared distribution. Rather, it gives a *scaled* non-central chi-squared distribution. Therefore, the sum (although independent) of the two factors is actually a linear combination of non-central chi-squared distributions, which does not give a non-central chi-squared distribution.

Similar to the bond pricing formula, when there is no correlation, this solution will collapse back to a product of two one-factor equations as in Chen (1992).

Finally, the option on futures will have to satisfy the same PDE as described in eq. (4). The solution is different due to the different boundary condition for futures options. The call option will have the terminal payoff function, $C(\{y_1, y_2\}, T_c; T_c, T_f, T) = \max\{\Phi(\{y_1, y_2\}, T_c; T_f, T) - K, 0\}$. Therefore, the solution is:

$$
\begin{aligned}
C(\{y_1, y_2\}, t; T_c, T_f, T) = H(\{y_1, y_2\}, t)N(d_3) \\
- P(\{y_1, y_2\}, t; T_c)KN(d_4)
\end{aligned}
\tag{8}
$$

where

$$
\begin{aligned}
H(\{y_1, y_2\}, t) &= e^{-y_1 A_1(t) - y_2 A_2(t) - B(t)} \\
A_i(t) &= F_i(t, T_c) + X_i(t) \\
B(t) &= \sum_{i=1}^{2}\Bigg(D_i[T - T_f + T_c - t - A_i(t)] - \frac{\sigma_i^2}{2\alpha^2} \\
&\quad \cdot \left[X_i(T_c) - \frac{\alpha_i}{2} A_i^2(t) - F_i(T_f, T)\right] \\
&\quad + \frac{\rho\sigma_1\sigma_2}{\alpha_1\alpha_2}[F_i(t, T_c) + F_i(T_f, T) \\
&\quad + X(T_c) - X_i(t)]\Bigg) - \frac{\rho\sigma_1\sigma_2}{\alpha_1\alpha_2} \\
&\quad \cdot \Bigg(T - T_f + T_c - t + (1 - \alpha_1 X_1(T_c)) \\
&\quad \cdot (1 - \alpha_2 X_2(T_c)) \frac{1 - e^{-(\alpha_1+\alpha_2)(T_c - t)}}{\alpha_1 + \alpha_2} \frac{\alpha_1\alpha_2}{\alpha_1 + \alpha_2} \\
&\quad \cdot [X_1(T_c)X_2(T_c) + F_1(T_f, T)F_2(T_f, T)] \\
&\quad + \frac{1 - e^{-(\alpha_1+\alpha_2)(T_f - t)}}{\alpha_1 + \alpha_2}\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
d_1 = \frac{1}{\sigma_h}\Bigg(\sum_{i=1}^{2} & -\ln K + Y(T_c) - y_i X_i(t) \\
& - (\alpha_i \mu_i - q_i \sigma_i)(X_i(T_c) - X(t)) \\
& + \frac{\sigma_i^2 X_i(T_c) F_i(t, T_c)^2}{2} + X_i(T_c) v_i^2 \\
& - \frac{\rho \sigma_1 \sigma_2}{\alpha_1 \alpha_2}\Bigg[\sum_{i=1}^{2} - \alpha_i X_i(T_c) \\
& \cdot \left(F_i(t, T_c) - \frac{1 - e^{-(\alpha_1 + \alpha_2)(T_c - t)}}{\alpha_1 + \alpha_2}\right) \\
& - \frac{\alpha_1 \alpha_2 (X_1(T_c) X_2(T_c) - X_1(t) X_2(t))}{\alpha_1 + \alpha_2}\Bigg]\Bigg) \\
d_2 = d_1 - \sigma_h & \\
\sigma_h = \sqrt{\sum_{i=1}^{2}} & \overline{v_i^2 X_i^2(T_c) + 2c_{12} X_1(T_c) X_2(T_c)}
\end{aligned}
$$

Note that in the above equation, the function H is not a futures price, nor a discounted futures price. It will be shown in the next section that H is actually a discounted forward price of an asset that pays the futures price at the maturity date of the option contract. Technically, H satisfies the PDE of eq. (2) but uses the futures price as the boundary condition. The rest of the formula shows a Black–Scholes type solution with more complex arguments in normal cumulative densities. In the next section, it is shown that with an alternative method, these pricing formulas can be shown to be precisely identical to those under the one-factor framework. It allows more parameters to better fit the term structure. It also solves the major problem of perfect correlation among bonds in the one-factor model.

THE PREFERENCE-FREE FORM

In this section, an alternative method is used to re-derive pricing formulas in the previous section. Each price is computed by an expectation. This method was first discovered by Cox and Ross (1976) as a risk-neutral pricing method. They observe that when the preference parameter drops out of the valuation process, all assets should earn what a risk-neutral individual would expect, the risk-free rate. Then they solve the transition density by solving the forward equation (a PDE) of the stochastic process and then compute the expectation. It should be noted that solving for the density and then computing the expectation to obtain the option formula is identical to solving the PDE for the option directly. No mileage is gained in the Cox–Ross method. However, an advantage can be obtained if a few similar but different contingent claims that share the same transition density but satisfy different PDEs or boundary conditions are solved by the Cox–Ross method. If one tries to solve the PDEs for all the claims, then one has to repeatedly go through all the steps mentioned in the appendix, which is time consuming. On the other hand, if one computes expectations, since they all use the same density, the solutions can be obtained more easily because the PDE needs to be solved only once for the density. Interest rate claims are perfect examples because various claims are all coming from the same stochastic processes (therefore the same density). Moreover, the Cox–Ross method allows a preference-free representation of all the pricing formulas in the previous section.

First of all, it should be recognized that the PDE for bonds and options (and futures options) is the Kolmogorov backward equation (KBE) for the Kac functional. Therefore, the solutions to these differential equations are nothing but expectations of the discounted final payoff [see Karlin and Taylor (1981, pp. 222–224)], taken over the density of a process that has the coefficient of the first derivative term as the drift and the coefficient of the second derivative term as the diffusion coefficient. Comparing the two drifts and two diffusion coefficients results in the conclusion that the distribution type remains the same and the drift is adjusted by the term, $q\sigma$. This adjustment is known as risk-neutralization. In a more modern martingale theory by Harrison and Kreps (1979), this adjustment transforms the original probability measure to an equivalent measure in which all normalized assets are martingales. Therefore, the current value is a conditional expectation of the discounted payoff as a consequence of the martingale theory.

The PDE for the futures price is the KBE and the solution is the expected final payoff without discounting [see Karlin and Taylor (1981, p. 214–216)]. The reason for dropping the discount factor is that, on the right-hand side of the PDE, there is no $(y_1 + y_2)\Phi$, which comes from the fact that trading futures requires no initial investments.

According to Karlin and Taylor, the discount bond price can be computed as the following expectation:[11]

$$P(r,t;T) = \hat{E}\left[\exp\left(-\int_t^T r(s)\,ds\right)\right]$$
$$= \hat{E}\left[\exp\left(-\int_t^T [y_1(s) + y_2(s)]\,ds\right)\right] \tag{9}$$

where $\hat{E}[\cdot]$ is the expectation taken under the following "adjusted" processes:

$$\begin{cases} dy_1 = \alpha_1\left(\mu_1 - y_1 - \dfrac{\sigma_1 q_1}{\alpha_1}\right)dt + \sigma_1\,d\hat{W}_1 \\ dy_2 = \alpha_2\left(\mu_2 - y_2 - \dfrac{\sigma_2 q_2}{\alpha_2}\right)dt + \sigma_2\,d\hat{W}_2 \end{cases}$$

with the same correlation. To compute this expectation, define R as the accumulation of the instantaneous return in the period (t, T):

$$R = \int_t^T r(s)\,ds = \int_t^T y_1(s)\,ds + \int_t^T y_2(s)\,ds$$

It follows that R is normally distributed with mean and variance computed as follows:

$$\hat{E}[R] = \int_t^T \hat{E}[r(s)]\,ds = \int_t^T m_1 + m_2\,ds$$
$$= \sum_{i=1}^{2} y_i F_i(t,T) + \left(\mu_i - \frac{q_i\sigma_i}{\alpha_i}\right)(T - t - F_i(t,T)) \tag{10}$$

and

$$\hat{V}[R] = \hat{\text{cov}}[R,R] = \int_t^T \int_t^u 2\,\hat{\text{cov}}[r(u), r(s)]\,ds\,du \quad (\forall\ u > s)$$
$$= \int_t^T \int_t^u \sum_{i=1}^{2} \sum_{j=1}^{2} 2\,\hat{\text{cov}}[y_i(u), y_j(s)]\,ds\,du$$
$$= \sum_{i=1}^{2} \sum_{j=1}^{2} 2 \int_t^T \int_t^u$$

[11]All expectations from now on are conditional expectations condition on current values (at t). The condition part is dropped for notational simplicity.

$$\cdot \frac{\rho\sigma_i\sigma_j e^{-\alpha_i(s-t)} e^{-\alpha_j(u-t)}(e^{(\alpha_i+\alpha_j)(u-t)} - 1)}{\alpha_i + \alpha_j}\, ds\, du$$

$$= \sum_{i=1}^{2}\sum_{j=1}^{2} \frac{2\rho\sigma_i\sigma_j}{\alpha_i(\alpha_i + \alpha_j)}$$

$$\cdot \left[T - t + \frac{e^{-(\alpha_i+\alpha_j)(T-t)}}{(\alpha_i + \alpha_j)} - \frac{e^{\alpha_i(T-t)}}{\alpha_i} - \frac{e^{-\alpha_j(T-t)}}{\alpha_j} + \frac{(\alpha_i + \alpha_j)^2 - \alpha_i\alpha_j}{\alpha_i\alpha_j(\alpha_i + \alpha_j)} \right] \tag{11}$$

The bond price is nothing more than a moment generating function of the random variable, R, with a coefficient -1. Then it follows that:

$$P(\{y_1, y_2\}, t; T) = \exp\left(-\hat{E}[R] + \frac{\hat{V}[R]}{2} \right)$$
$$= \text{eq. (3)}$$

Note that the bond price follows a log-normal process as a consequence of the normally distributed R. Thus, any claim written on the bond will have the Black–Scholes type of result that assumes a log-normally distributed stock price.

It is clear that if the underlying asset is traded and its distribution can be identified, one can write its contingent claims in terms of the asset and a pure discount bond price. This result can be clearly seen from the expectation operation. Explicit formulas for various bond claims are provided in the following sub-sections.

Options on Pure Discount Bonds

Since eq. (4) is the KBE for the Kac functional, it is clear that the solution to this differential equation will be the expectation of the discounted payoff function at maturity. The drift and diffusion coefficients should be consistent with those in the KBE. Since the two processes follow a joint O–U process, the conditional density function is normal with the means, variances, and covariances described in eq. (1).[12] Therefore, one can compute the call price by finding the

[12] See Sections 8.2 and 8.3 in Arnold (1974).

following expectation:

$$
\begin{aligned}
&C(\{y_1, y_2\}, t; T_c, T) \\
&\quad = \hat{E}\left[\exp\left(-\int_t^{T_c} r(s)\,ds\right)\max\{P(\{y_1, y_2\}, T_c; T) - K, 0\}\right] \\
&\quad = \hat{E}\left[\exp\left(-\int_t^{T_c} r(s)\,ds\right)(P(\{y_1, y_2\}, T_c; T) - K)1_{\{P>K\}}\right] \\
&\quad = \hat{E}\left[\exp\left(-\int_t^{T_c} r(s)\,ds\right)P(\{y_1, y_2\}, T_c; T)1_{\{P>K\}}\right] \\
&\qquad - K\hat{E}\left[\exp\left(-\int_t^{T_c} r(s)\,ds\right)1_{\{P>K\}}\right] \\
&\quad = \hat{E}\left[\exp\left(-\int_t^{T_c} r(s)\,ds\right)P(\{y_1, y_2\}, T_c; T)\right]\hat{E}[\eta_1 1_{\{P>K\}}] \\
&\qquad - K\hat{E}\left[\exp\left(-\int_t^{T_c} r(s)\,ds\right)\right]\hat{E}[\eta_2 1_{\{P>K\}}] \\
&\quad = \hat{E}\left[\exp\left(-\int_t^{T_c} r(s)\,ds\right)P(\{y_1, y_2\}, T_c; T)\right]\overline{E}[1_{\{P>K\}}] \\
&\qquad - K\hat{E}\left[\exp\left(-\int_t^{T_c} r(s)\,ds\right)\right]\tilde{E}[1_{\{P>K\}}] \qquad (12)
\end{aligned}
$$

where

$$
\eta_1 = \frac{\exp\left(-\int_t^{T_c} r(s)\,ds\right)P(\{y_1, y_2\}, T_c, T)}{\hat{E}\left[\exp\left(-\int_t^{T_c} r(s)\,ds\right)P(\{y_1, y_2\}, T_c, T)\right]}
$$

$$
\eta_2 = \frac{\exp\left(-\int_t^{T_c} r(s)\,ds\right)}{\hat{E}\left[\exp\left(-\int_t^{T_c} r(s)\,ds\right)\right]}
$$

are Randon–Nykodim derivatives where $\hat{E}[\eta_1] = \hat{E}[\eta_2] = 1$. The expectation operation under the two probability measures, $\tilde{E}[\cdot]$ and $\overline{E}[\cdot]$

are performed according to the following processes, respectively:

$$\begin{cases} dy_1 = \left(\alpha_1(\mu_1 - y_1) - \sigma_1 q_1 - \dfrac{\sigma_1 P_1}{P}\right) dt + \sigma_1\, d\tilde{W}_1 \\ dy_2 = \left(\alpha_2(\mu_2 - y_2) - \sigma_2 q_2 - \dfrac{\sigma_2 P_2}{P}\right) dt + \sigma_2\, d\tilde{W}_2 \end{cases}$$

and

$$\begin{cases} dy_1 = \left(\alpha_1(\mu_1 - y_1) - \sigma_1 q_1 - \dfrac{\sigma_1 P_1^*}{P}\right) dt + \sigma_1\, d\overline{W}_1 \\ dy_2 = \left(\alpha_2(\mu_2 - y_2) - \sigma_2 q_2 - \dfrac{\sigma_2 P_2^*}{P}\right) dt + \sigma_2\, d\overline{W}_2 \end{cases}$$

where P and P^* are the shorthand notation of $P(r, t; T_c)$ and $P(r, t; T)$, respectively. The proof is provided in the appendix. Note that all these changes of measure are done in the drift part of the processes only. They will not affect either the variances or the covariances. These changes of measure facilitate the computation of the pricing formula. It follows that any claim with stochastic interest rates can be priced this way. To reach the final solution, it should be understood that, by the law of iterative expectations, the two bond prices can be derived as follows:

$$\begin{aligned} &\hat{E}\left[\exp\left(-\int_t^{T_c} r(s)\,ds\right) P(\{y_1, y_2\}, T_c; T)\right] \\ &\qquad = \hat{E}\left[\exp\left(-\int_t^{T_c} r(s)\,ds\right)\exp\left(-\int_{T_c}^{T} r(s)\,ds\right)\right] \\ &\qquad = \hat{E}\left[\exp\left(-\int_t^{T} r(s)\,ds\right)\right] \\ &\qquad = P(\{y_1, y_2\}, t; T) \end{aligned}$$

and

$$\hat{E}\left[\exp\left(-\int_t^{T_c} r(s)\,ds\right)\right] = P(\{y_1, y_2\}, t; T_c)$$

A similar separation property was proposed by Jamshidian (1987, 1989). Jamshidian observes that under the "$\sim$" probability measure, the expectation will give the forward price of the asset. Therefore, the "$\sim$" process can be called a forward-adjusted process. This process differs from the risk-neutralized process by an additional term. In Jamshidian's language, it can be seen that the first term of the equation can also be expressed as:

$$\hat{E}\left[\exp\left(-\int_t^{T_c} r(s)\,ds\right)\right]\tilde{E}[P(\{y_1,y_2\},T_c;T)]$$

$$= P(\{y_1,y_2\},t;T_c)\frac{P(\{y_1,y_2\},t;T)}{P(\{y_1,y_2\},t;T_c)}$$

$$= P(\{y_1,y_2\},t;T)$$

so that the two approaches are consistent. However, since the second expectation in the equation requires one more change of measure, the approach in this study is more direct. Due to the fact that the expectation of an indicator function results in a probability, the following formula for the call can be obtained by working out explicitly the two probabilities. Given the normal density of the O–U process, the final solution can be written as follows:

$$C(\{y_1,y_2\},t;T_c,T) = P(\{y_1,y_2\},t;T)N(d_p) - P(\{y_1,y_2\},t,T_c)KN(d_p-\sigma_p) \qquad (13)$$

where

$$d_p = \frac{1}{\sigma_p}\left(\ln\frac{P(\{y_1,y_2\},T_c,T)}{P(\{y_1,y_2\},T_c,T)K}+\frac{\sigma_p^2}{2}\right)$$

$$\sigma_p^2 = \mathrm{var}[\ln P(\{y_1,y_2\},T_c,T)] = \text{as defined}$$

Note that this formula is precisely Jamshidian's one-factor model (1989). The only difference between this two-factor model and Jamshidian's one-factor model is that the bond price in Jamshidian needs to be the two-factor bond price of eq. (3). The similarity between this solution and the Black–Scholes stock option model has been described by Jamshidian (1989). The first bond price serves as the asset price and the second one serves as the discount factor. It is natural for the two formulas to look alike, given the log-normal distribution of the underlying asset in both cases.

Futures on Pure Discount Bonds

The futures price can be derived in the same way, but the expectation is taken with no discounting. This PDE is the Kolmogorov backward equation described in Karlin and Taylor and the solution is:

$$\begin{aligned}\Phi(\{y_1, y_2\}, t; T_f, T) &= \hat{E}[P(\{y_1, y_2\}, T_f; T)]\\ &= \hat{E}[e^{-y_1F_1(T_f,T)-y_2F_2(T_f,T)-G(T_f,T)}]\end{aligned}$$

The distribution of $Z = y_1F_1(T_f, T) + y_2F_2(T_f, T)$ is normal with mean and variance as follows:

$$\begin{aligned}\hat{E}[Z] &= m_1F_1(T_f, T) + m_2F_2(T_f, T)\\ \hat{V}[Z] &= v_1^2F_1(T_f, T)^2 + v_2^2F_2(T_f, T)^2\\ &\quad + 2c_{12}F_1(T_f, T)F_2(T_f, T)\end{aligned}$$

Thus, the futures pricing model is a moment generating function of the random variable Z with a coefficient of -1, which leads to the following function:

$$\begin{aligned}\Phi(\{y_1, y_2\}, t; T_f, T) &= e^{-\hat{E}[Z]+\hat{V}[Z]/2-G(T_f,T)}\\ &= \frac{P(\{y_1, y_2\}, t; T)}{P(\{y_1, y_2\}, t; T_f)}\, e^{-J} \qquad (14)\end{aligned}$$

where

$$\begin{aligned}J &= Y^*(t) - [G^*(t, T) - G^*(t, T_f)]\\ Y^*(t) &= Y(t) - \sum_{i=1}^{2} D_i[T - T_f - X_i(t)]\\ G^*(t, T) &= G(t, T) - \sum_{i=1}^{2} D_i[T - t - F_i(t, T)]\\ G^*(t, T_f) &= G(t, T_f) - \sum_{i=1}^{2} D_i[T_f - t - F_i(t, T_f)]\end{aligned}$$

Note that none of $G^*(t, T)$, $G^*(t, T_f)$, or $Y^*(t)$ carries the preference parameter, q_1 or q_2. Therefore, eq. (14) is preference-free.

Options on Pure Discount Bond Futures

Like bond options, bond futures options should satisfy the Kolmogorov backward equation for the Kac functional and the solution is therefore the conditional expectation of the discounted terminal payoff:

$$
\begin{aligned}
&C(\{y_1, y_2\}, t; T_c, T_f, T) \\
&\quad = \hat{E}\left[\exp\left(-\int_t^{T_c} r(s)\,ds\right)\max\{\Phi(\{y_1, y_2\}, T_c; T_f, T) - K, 0\}\right] \\
&\quad = \hat{E}\left[\exp\left(-\int_t^{T_c} r(s)\,ds\right)\Phi(\{y_1, y_2\}, T_c; T_f, T)\right]\overline{E}[1_{\{\Phi>K\}}] \\
&\qquad - \hat{E}\left[\exp\left(-\int_t^{T_c} r(s)\,ds\right)\right]K\tilde{E}[1_{\{\Phi>K\}}] \\
&\quad = H(\{y_1, y_2\}, t)N(d_h) - P(\{y_1, y_2\}, t, T_c)KN(d_h - \sigma_h) \qquad (15)
\end{aligned}
$$

where

$$
\begin{aligned}
d_h &= \frac{1}{\sigma_h}\left(\ln \frac{H(\{y_1, y_2\}, T_c)}{P(\{y_1, y_2\}, t, T_c)K} + \frac{\sigma_h^2}{2}\right) \\
\sigma_h^2 &= \mathrm{var}[\ln H(\{y_1, y_2\}, T_c)] = \text{as defined}
\end{aligned}
$$

Again, this solution is the same as Chen's one-factor model (1992) except for a two-factor H serving as the primary asset. Although this solution is the same as eq. (8), it is more intuitive and comparable to the Black model (1976) for futures options. The function, H, is an expected discounted futures price, corresponding to the Black model. In the Black model, since the interest rates are assumed constant, discounting is trivial in contrast to the H function here where stochastic interest rates are considered. With Jamshidian's separation theorem, the function H can be decomposed into:

$$
\begin{aligned}
&\hat{E}\left[\exp\left(-\int_t^{T_c} r(s)\,ds\right)\Phi(\{y_1, y_2\}, T_c; T_f, T)\right] \\
&\qquad = \hat{E}\left[\exp\left(-\int_t^{T_c} r(s)\,ds\right)\right]\tilde{E}[\Phi(\{y_1, y_2\}, T_c; T_f, T)] \\
&\qquad = P(\{y_1, y_2\}, t; T_c)\tilde{E}[\Phi(\{y_1, y_2\}, T_c; T_f, T)]
\end{aligned}
$$

The second term is a forward price of a contract where a futures contract is delivered at maturity, T_f. Note that H is preference-free:

$$
H(\{y_1, y_2\}, t) = P(\{y_1, y_2\}, t, T_c)\Phi(\{y_1, y_2\}, t, T_f, T)e^{-Z} \qquad (16)
$$

where

$$Z = B^*(t) - G^*(t, T_c) - Y^*(t)$$

$$B^*(t) = B(t) - \sum_{i=1}^{2} D_i[T - T_f + T_c - t - A_i(t)]$$

It is clear that the preference parameters, q_1 and q_2 drop out of the equation. They are embedded explicitly in the first term, the bond price, and implicitly the second term, the futures price, which contains two bond prices. See eq. (14).

Forward and Forward Options on Pure Discount Bonds

Forward and futures contracts themselves have no value. Thus, unlike pricing options, pricing forward or futures contracts is not finding the values of the contracts. Instead, it is to find the forward or the future price that the long agrees to pay. Since the forward contract will not be settled until maturity, it is equivalent to a long call and a short put with the forward price as their strike prices. Since the forward contract has no value, the call premium will have to equal the put premium. As a consequence, the forward price is the strike price that will make two premiums the same. Formally,

$$C(\{y_1, y_2\}, t; T_f, T) - P(\{y_1, y_2\}, t; T_f, T) = 0$$

By the put–call parity, this is equivalent to:

$$P(\{y_1, y_2\}, t; T) - P(\{y_1, y_2\}, t; T_f)K = 0$$

$$\frac{P(\{y_1, y_2\}, t; T)}{P(\{y_1, y_2\}, t; T_f)} = K = \Psi(\{y_1, y_2\}, t; T_f, T) \quad (17)$$

Options on forward are similar to those on futures. The solution can be derived by substituting the forward price into eq. (15).

$$C_\psi(\{y_1, y_2\}, t; T_c, T_f, T) = L(\{y_1, y_2\}, t)N(d_l) - P(\{y_1, y_2\}, t, T_c)KN(d_l - \sigma_l) \quad (18)$$

where

$$d_l = \frac{1}{\sigma_l}\left(\ln \frac{L(\{y_1, y_2\}, T_c)}{P(\{y_1, y_2\}, t, T_c)K} + \frac{\sigma_l^2}{2}\right)$$

$$\sigma_l^2 = \text{var}[\ln L(\{y_1, y_2\}, T_c)] = \sigma_h^2$$

The L function is also a preference-free function:

$$\begin{aligned} L(\{y_1, y_2\}, t) &= \hat{E}\left[\exp\left(-\int_t^{T_c} r(s)\,ds\right)\Psi(\{y_1, y_2\}, T_c; T_f, T)\right] \\ &= \hat{E}\left[\exp\left(-\int_t^{T_c} r(s)\,ds\right)\right]\tilde{E}[\Psi(\{y_1, y_2\}, T_c; T_f, T)] \\ &= P(\{y_1, y_2\}, t; T_c)\Psi(\{y_1, y_2\}, t; T_f, T) \\ &= P(\{y_1, y_2\}, t; T_c)\,\frac{P(\{y_1, y_2\}, t; T)}{P(\{y_1, y_2\}, t; T_f)} \end{aligned}$$

Using this result, eq. (17) will have the following result:

$$\begin{aligned} C_\psi(\{y_1, y_2\}, t; T_c, T_f, T) = {} & \frac{P(\{y_1, y_2\}, t, T_c)}{P(\{y_1, y_2\}, t, T_f)} \\ & \cdot [P(\{y_1, y_2\}, t; T)N(d_l) \\ & \quad - P(\{y_1, y_2\}, t, T_f)KN(d_l - \sigma_l)] \end{aligned}$$

The bracketed term resembles the bond option formula with the option maturing at T_f instead of T_c. This bracketed term is similar to Turnbull and Milne (1991, Theorem 2). However, the option on forward formula shown here has an adjustment term similar to a forward price.

Finally, put formulas can be seen with the put–call parity:

$$\zeta = C + P(\{y_1, y_2\}, t; T_c)K - A$$

where A equals $P(\{y_1, y_2\}, t; T)$ for bond option, $H(t)$ for bond futures option, and $L(t)$ for bond forward option.

DISCUSSION AND CONCLUSIONS

It is clear that whether or not a solution can be a preference-free form depends on whether or not it can be expressed in terms of traded assets. In theory, if the underlying asset is traded, then there exists a riskless trading portfolio of this underlying asset and its contingent claim (instantaneously). Therefore, the claim price can be expressed as a function of this underlying asset. Interest rate claims can be viewed as claims on a non-traded asset, the instantaneous rate, or a traded asset, the pure discount bond. Therefore, interest rate claims can be preference-free or non-preference-free, depending upon which asset is chosen as the underlying state variable. In this article, it is shown that since bond prices are themselves contingent claims, the preference-free format of other claims (especially options) may or may not exist. If the distribution of the bond price is a known distribution, then the solutions based upon bond prices are possible. Since bonds are traded assets, the preference-free format can be obtained. On the other hand, if the bond price distribution is not known (such as CIR), solutions can only be expressed in terms of the instantaneous rate. The preference-free form will not be reachable. This study shows also that in other preference-free option models (e.g., Black–Scholes) risk preferences do not actually disappear but are embedded in traded assets.

In this study, the two-factor model with a joint normality assumption is turned into a "one-factor" model. This enriched "one-factor" model preserves the simplicity of the one-factor models derived previously. But it takes into account the correlation of the two factors which allows one to describe the term structure more accurately. The decomposition of the one-factor into two, correlated factors successfully eliminates the major problem that one-factor models suffer from—perfect correlation among different-maturity bonds. Furthermore, this two-factor model is as easy to use as the one-factor model because of its preference-free form.

The preference-free form of the option model provides an interesting result. If the bond price is taken from the market place as given, then it makes no difference whether the bond is priced by the one-factor or the two-factor model. The key ingredient lies in the bond volatility. In other words, pricing an option accurately depends upon whether the bond volatility can be correctly specified.[13] The two-factor model certainly gives the volatility a much broader base. To illustrate, write eq. (3) as:

$$P(\{y_1, \overline{y}_2\}, t; T) = \overline{P}P(y_1, t; T)$$

In this equation the second factor is fixed and only the first factor is used to fit the volatility. This is similar to a one-factor model. The only difference is that the bond price is lowered by $\overline{P}$. But this changes only the mean. α and σ which determine the volatility, are not altered. The second factor is then used to minimize the probability of negative interest rates. The probability of obtaining negative interest rates for any fixed time is:

$$p(r < 0) = N\left(-\frac{\overline{m}_1 + m_2}{\sqrt{\overline{v}_1^2 + v_2^2 + 2c_{12}}}\right)$$

[13]See Dybvig (1989) for excellent discussions on the theoretical development of this issue.

In this equation, the first factor is fixed because it is used to fit the volatility already. To see that the probability is less than that in the one-factor model, one can choose μ so that $m_2 = 0$. Then, as long as the correlation between the two factors is negative and

$$|2c_{12}| > v_2^2$$

the probability of negative interest rates is smaller. These two equations are used recursively to jointly determine the best parameters. This shows that the two-factor model can do at least as well as the one-factor model and simultaneously provides lower probability of negative interest rates. Since the generalization to multiple factors is an easy task, the term structure is surely better fit and the negative interest rates problem is alleviated. Negative interest rates are the result of inappropriately specified volatility. More factors give more flexibility to the term structure which allows a better fit to the volatility and thereby alleviates the problem of inappropriately specified volatility.

If y_1 and y_2 follow square root processes, a preference-free form is difficult to define. This is because the density of the bond price is unknown, although it exists. It is well known that any linear combination of non-central chi-squares does not lead to any known density. One can certainly work out the density of the linear combination or the characteristic function; but none of them can be expressed in terms of any known density function.

Finally, it follows from the analysis in this study that one can do a n-factor decomposition of the instantaneous rate without affecting the options formula. The underlying assets (bond, forward, and futures) will have to be only slightly modified.[14]

APPENDIX

Method of Separation of Variables for Eq. (2)

Solving any partial differential equation requires guessing a form of the solution. If a specific form can solve the partial differential equation, then this must be *the* solution according to the existence and uniqueness theorem in differential equations.

A solution is guessed to take the following form:

$$P(\{y_1, y_2\}, t) = e^{-y_1 F_1(t) - y_2 F_2(t) - G(t)}$$

[14]Sharp (1987, Chapter 5) shows n-factor bond and bond option models.

Then, all the partial derivatives become:

$$\frac{\partial P}{\partial y_1} = P \cdot (-F_1)$$
$$\frac{\partial P}{\partial y_2} = P \cdot (-F_2)$$
$$\frac{\partial P}{\partial t} = P \cdot \left(-y_1 \frac{\partial F_1}{\partial t} - y_2 \frac{\partial F_2}{\partial t} - \frac{\partial G}{\partial t}\right)$$
$$\frac{\partial^2 P}{\partial y_1^2} = P \cdot F_1^2$$
$$\frac{\partial^2 P}{\partial y_2^2} = P \cdot F_2^2$$
$$\frac{\partial^2 P}{\partial y_1 \partial y_2} = P \cdot F_1 \cdot F_2$$

Plugging these partials back into the differential equation, one can obtain three separate ordinary differential equations to solve:

$$\alpha_1 F_1 - \frac{dF_1}{dt} - 1 = 0$$
$$\alpha_2 F_2 - \frac{dF_2}{dt} - 1 = 0$$
$$\frac{dG}{dt} + (\alpha_1\mu_1 - q_1\sigma_1)F_1 + (\alpha_2\mu_2 - q_2\sigma_2)F_2 - \frac{\sigma_1^2}{2}F_1^2 - \frac{\sigma_2^2}{2}F_2^2 - \rho\sigma_1\sigma_2 F_1 F_2 = 0$$

All of them are first order ordinary differential equations in t which are easy to solve. Solving the first two ordinary differential equations for F_1 and F_2 and using them to solve for G in the third ordinary differential equation, one obtains a homogenous solution. The original boundary condition, $P(\{y_1, y_2\}, T) = 1$, can be translated into conditions for F's and G as follows

$$F_1(T) = F_2(T) = G(T) = 0$$

This helps to obtain the particular solution, eq. (3).

The solution for futures can be approached in exactly the same way. However, the solution for options (or futures options) requires extra effort. d_1 and d_2 (or d_3 and d_4) themselves will satisfy certain ordinary differential equations. Detailed derivations can be obtained from the author on request. The reader can refer to Chen (1992, appendix) for a one-factor version of the differential equations.

Change of Probability Measure

This appendix presents the changes of measure in eq. (12). By Ito's rule, the second change of measure can be operated as follows:

$$
\begin{aligned}
0 &= \ln P(r, T_c; T_c) \\
&= \ln P(r, t; T_c) + \int_t^{T_c} \frac{1}{P}\Big(P_u\, d_u + P_1\, dy_1 + P_2\, dy_2 \\
&\qquad + P_{12}\, dy_1\, dy_2 + \frac{1}{2} \\
&\qquad \cdot [P_{11}(dy_1)^2 + P_{22}(dy_2)^2]\Big) \\
&= \ln P(r, t; T_c) + \int_t^{T_c} \frac{1}{P}\bigg(P_u + \alpha_1\Big(\mu_1 - y_1 - \frac{\sigma_1 q_1}{\alpha_1}\Big) \\
&\qquad \cdot P_1 + \alpha_2\Big(\mu_2 - y_2 - \frac{\sigma_2 q_2}{\alpha_2}\Big) \\
&\qquad \cdot P_2 + \rho\sigma_1\sigma_2 P_{12} + \frac{\sigma_1^2}{2} P_{11} \\
&\qquad + \frac{\sigma_2^2}{2} P_{22}\bigg) du \\
&\quad + \int_t^{T_c} \sigma_1 \frac{P_1}{P} \int_t^{T_c} \sigma_2 \frac{P_2}{P} \int_t^{T_c} \frac{1}{2}\Big(\sigma_1 \frac{P_1}{P}\Big)^2 - \int_t^{T_c} \frac{1}{2}\Big(\sigma_2 \frac{P_2}{P}\Big)^2 \\
&= \ln P(r, t; T_c) + \int_t^{T_c} (y_1(u) + y_2(u))\, du \\
&\quad + \int_t^{T_c} \sigma_1 \frac{P_1}{P} \int_t^{T_c} \sigma_2 \frac{P_2}{P} \int_t^{T_c} \frac{1}{2}\Big(\sigma_1 \frac{P_1}{P}\Big)^2 - \int_t^{T_c} \frac{1}{2}\Big(\sigma_2 \frac{P_2}{P}\Big)^2
\end{aligned}
$$

This proves the second change of measure that:

$$
\begin{aligned}
\eta_2 &= \frac{e^{-\int_t^{T_c} r\, du}}{P(\{y_1, y_2\}, t; T_c)} \\
&= \exp\bigg[\int_t^{T_c} \sigma_1 \frac{P_1}{P} + \int_t^{T_c} \sigma_2 \frac{P_2}{P} - \int_t^{T_c} \frac{1}{2}\Big(\sigma_1 \frac{P_1}{P}\Big)^2 \\
&\qquad - \int_t^{T_c} \frac{1}{2}\Big(\sigma_2 \frac{P_2}{P}\Big)^2\bigg] \\
&= \frac{d\tilde{\wp}}{d\hat{\wp}}
\end{aligned}
$$

The first change of probability measure goes through the same process:

$$
\begin{aligned}
\ln P(r, T_c; T) = \ln P(r, t; T) & \\
&+ \int_t^{T_c} (y_1(u) + y_2(u))\, du \\
&+ \int_t^{T_c} \sigma_1 \frac{P_1^*}{P^*} + \int_t^{T_c} \sigma_2 \frac{P_2^*}{P^*} \\
&- \int_t^{T_c} \frac{1}{2}\left(\sigma_1 \frac{P_1^*}{P^*}\right)^2 - \int_t^{T_c} \frac{1}{2}\left(\sigma_2 \frac{P_2^*}{P^*}\right)^2
\end{aligned}
$$

Therefore,

$$
\frac{d\overline{\wp}}{d\hat{\wp}} = \eta_1
$$

The Black–Scholes Result Applied to Eq. (12)

Although the first change of measure in eq. (12) can be done directly, the change of measure used in deriving the Black–Scholes result is more convenient. Since the change of measure in the Black–Scholes formula can apply to all log-normal processes, it can be applied here. Let $S(t)$ follow a risk-neutralized log-normal process that satisfies the stochastic differential equation:

$$
dS = rS\, dt + \sigma S\, d\hat{W}
$$

where r is the risk-free rate and σ is the volatility parameter. Then the call option formula can be written as the following conditional expectation:

$$
\begin{aligned}
C &= e^{-r(T-t)}\hat{E}[\max\{S_T - K, 0\}] \\
&= e^{-r(T-t)}\hat{E}[(S_T - K)1_{\{S>K\}}] \\
&= e^{-r(T-t)}\hat{E}[S_T 1_{\{S>K\}}] - e^{-r(T-t)}K\hat{E}[1_{\{S>K\}}] \\
&= e^{-r(T-t)}\hat{E}[S_T]\hat{E}\left[\frac{S_T}{\hat{E}[S_T]} 1_{\{S>K\}}\right] \\
&\quad - e^{-r(T-t)}KN(d_2) \\
&= S_t\hat{E}[\eta 1_{\{S>K\}}] - e^{-r(T-t)}KN(d_2) \\
&= S_t\overline{E}[1_{\{S>K\}}] - e^{-r(T-t)}KN(d_2) \\
&= S_t N(d_1) - e^{-r(T-t)}KN(d_2)
\end{aligned}
$$

where $N(d_2)$ is a normal probability defined upon the process $dS = rS\,dt + \sigma S\,d\hat{W}$ and therefore $d_2 = [\ln(S/K) + (r - \sigma^2/2)(T - t)]/\sigma\sqrt{T-t}$. Now, write:

$$\ln S_T = \ln S_t + \int_t^T \left(r - \frac{\sigma^2}{2}\right) du + \int_t^T \sigma\,d\hat{W}$$

$$\ln S_T = \ln S_t + \int_t^T r\,du = \int_t^T \sigma\,d\hat{W} - \int_t^T \frac{\sigma^2}{2}\,du$$

$$\frac{S_T}{S_t e^{r(T-t)}} = \frac{S_T}{\hat{E}[S_T]} = \eta$$

This means $N(d_1)$ is defined upon $dS = (r + \sigma)S\,dt + \sigma S\,d\overline{W}$ and, therefore, verifies that $d_1 = d_2 + \sigma\sqrt{T-t}$. This gives eq. (12) an alternative derivation:

$$\begin{aligned}
&\hat{E}\left[\exp\left(-\int_t^{T_c} r(s)\,ds\right)P(\{y_1, y_2\}, T_c; T)1_{\{P>K\}}\right] \\
&\quad = \hat{E}\left[\exp\left(-\int_t^{T_c} r(s)\,ds\right)\right]\tilde{E}[(P(\{y_1, y_2\}, T_c; T)1_{\{P>K\}}] \\
&\quad = P(\{y_1, y_2\}, t; T_c)\tilde{E}[P(\{y_1, y_2\}, T_c; T)] \\
&\qquad \cdot \tilde{E}\left[\frac{P(\{y_1, y_2\}, T_c; T)}{\tilde{E}[P(\{y_1, y_2\}, T_c; T)]}1_{\{P>K\}}\right] \\
&\quad = P(\{y_1, y_2\}, t; T_c) \cdot \frac{P(\{y_1, y_2\}, t; T)}{P(\{y_1, y_2\}, t; T_c)} \\
&\qquad \cdot \tilde{E}\left[\frac{P(\{y_1, y_2\}, T_c; T)}{\tilde{E}[P(\{y_1, y_2\}, T_c; T)]}1_{\{P>K\}}\right] \\
&\quad = P(\{y_1, y_2\}, t; T)\tilde{E}\left[\frac{P(\{y_1, y_2\}, T_c; T)}{\tilde{E}[P(\{y_1, y_2\}, T_c; T)]}1_{\{P>K\}}\right] \\
&\quad = P(\{y_1, y_2\}, t; T)\tilde{E}[\eta 1_{\{P>K\}}]
\end{aligned}$$

Note that P follows a log-normal process. Hence, the previous result can be applied directly and will result in $P\,N(d_1)$ and $\sigma p = \sqrt{\mathrm{var}(\ln P)}$. This approach is convenient because any variable that follows a log-normal process will have this result. As a consequence, both futures options and forward options can be derived using the above procedure because both H and L follow log-normal processes.

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