Bivariate GARCH Estimation of the Optimal Hedge Ratios for Stock Index Futures: A Note

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Since the introduction of stock index futures markets in the early 1980s, several studies have investigated the optimal hedge for stock market portfolios using stock index futures, while restricting the hedge ratio to be constant over time.[1] However, if the joint distribution of stock index and futures prices is changing through time, estimating a constant hedge ratio may not be appropriate. Estimation of optimal or minimum risk hedges with futures contracts should use the time-dependent conditional variance models such as the Generalized Autoregressive Conditional Heteroscedastic (GARCH) framework. Recent studies find that the time-dependent conditional variance model improves the hedging performance in various futures contracts.[2] However, no study has shown

[1] Figlewski (1984), Lee, Bubnys, and Lin (1987), and Ghosh (1993).
[2] Cecchetti, Cumby, and Figlewski (1988) apply ARCH in estimating an optimal futures hedge with Treasury bonds. Baillie and Myers (1991) and Myers (1991) examine commodity futures and report improvements in hedging performance over the constant hedge approach by following a dynamic strategy based on the GARCH framework. Kroner and Sultan (1991, 1993) find similar results with currency futures.

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the economic viability of the GARCH hedging method with stock index futures in the presence of transactions costs. This study examines two of the most heavily traded stock index futures contracts in the U.S. and Canada: the S&P 500 Index Futures and Toronto 35 Index Futures.

ESTIMATION OF OPTIMAL HEDGE RATIOS

Hedging models can be constructed using a two-period investment decision based on utility maximization. Assuming that the only hedging instrument available to the investor is the futures contract, a hedge portfolio consisting of spot and futures is constructed. Let f_{t+1} and s_{t+1} be the changes in the futures price, F, and the spot price, S, between time t and $t+1$, respectively, and define b_t as the holdings of futures at time t. Then

$$x_{t+1} = s_{t+1} - b_t f_{t+1} \tag{1}$$

is the payoff at time $t+1$ to purchasing one unit of the spot and going short in b_t units of the futures at time t. The optimal hedge ratio, b_t^*, maximizes the investor's expected utility, or minimizes the risk of the hedge portfolio, under the assumption that futures prices are a martingale.

The GARCH specification requires modeling the first two conditional moments of the bivariate distributions of s_t and f_t. To account for the time-varying variances and covariances, the second moment can be parameterized with a bivariate constant correlation GARCH(1,1) model, as originally proposed in Bollerslev (1986) and applied in Baillie and Bollerslev (1990) and Kroner and Sultan (1991, 1993). This model parameterizes the conditional variances of two variables as ARMA models in squared residuals, while assuming constant correlations between the two. The bivariate distributions of spot and futures are assumed to be as follows.

$$\begin{aligned} s_t &= \alpha_0 + \alpha_1(S_{t-1} - \gamma F_{t-1}) + \epsilon_{st} \\ f_t &= \beta_0 + \beta_1(S_{t-1} - \gamma F_{t-1}) + \epsilon_{ft}, \end{aligned} \quad \left.\begin{bmatrix} \epsilon_{st} \\ \epsilon_{ft} \end{bmatrix}\right| \Psi_{t-1} \sim N(0, H_t) \tag{2}$$

$$H_t = \begin{bmatrix} h_{ss,t} & h_{sf,t} \\ h_{sf,t} & h_{ff,t} \end{bmatrix} = \begin{bmatrix} h_{s,t} & 0 \\ 0 & h_{f,t} \end{bmatrix} \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \begin{bmatrix} h_{s,t} & 0 \\ 0 & h_{f,t} \end{bmatrix} \tag{3}$$

$$\begin{aligned} h_{st}^2 &= c_s + a_s \epsilon_{s,t-1}^2 + b_s h_{s,t-1}^2 \\ h_{ft}^2 &= c_f + a_f \epsilon_{f,t-1}^2 + b_f h_{f,t-1}^2 \end{aligned} \tag{4}$$

Equation (2) represents the dynamics of spot and futures price changes. The term $(S_{t-1} - \gamma F_{t-1})$ is the error correction term, which is discussed below. The residuals, ϵ, follow a bivariate normal distribution with mean zero and conditional variance H, where Ψ_{t-1} is the information set at time $t - 1$. Equation (3) shows the structure of the bivariate conditional variance with constant correlation, ρ, between the two variances and eq. (4) is the bivariate GARCH(1,1) model.

Given the bivariate model of spot and futures prices changes, the time-varying hedge ratio can be expressed with the variance estimates from (3) and (4).

$$\hat{b}_t^* = \hat{h}_{sf,t}/\hat{h}_{ff,t}. \tag{5}$$

To estimate b_t^*, the parameters in eqs. (2), (3), and (4) are estimated first. The conditional variances and covariances can be forecasted at each time and provide the optimal hedge ratio as given above. In addition to the bivariate GARCH estimation, the conventional OLS hedge ratios can be obtained by setting $a_s = b_s = a_f = b_f = \alpha_1 = \beta_1 = 0$. Also a variation of the conventional OLS hedge that accounts for cointegration between the spot and futures prices can be obtained by setting $a_s = b_s = a_f = b_f = 0$.

COMPARISON OF DIFFERENT HEDGING TECHNIQUES

The spot market data consist of daily closing prices for the S&P 500 index and the Toronto 35 index. The futures data correspond to daily settlement prices for the three futures contracts. The first step is to construct series of weekly prices, including dividends, from every Wednesday.[3] To construct a dividend inclusive series for the U.S. indices, the daily dividend yields for the New York Stock Exchange are gathered from the Center for Research in Security Prices tapes to compute the nominal amount of daily dividends. For the Toronto 35 index, the dividend inclusive index (Total Return Index Value) is available from the TSE and is used without any adjustment. The price data begin on June 8, 1988, and end on December 18, 1991. Of the several futures contracts that are outstanding at any given time, the price of the nearest

[3] A Wednesday-to-Wednesday sequence is chosen because Wednesday data have the largest number of observations and do not coincide with regular announcements of public information. In addition, daily price limits are not imposed on the closing prices in any of the Wednesdays during the testing period.

contract is used. To avoid thin markets and expiration effects, however, we roll over to the next nearest contract one week prior to expiration of the current contract. This provides 185 weekly prices for the S&P 500 and 184 prices for Toronto 35.[4]

Several tests on the distributional properties of the data are performed and the results indicate the presence of one unit root and thus nonstationarity of each of the price series. Further tests reject two unit roots. Stationarity of first differences in S_t and F_t is confirmed. In addition, if the two series, the spot and futures prices, are cointegrated, then there exists a constant, γ, such that $S_t - \gamma F_t$ is stationary.[5] The constant, γ, is called the cointegrating parameter that links the spot and futures prices. If the spot and futures prices show a long-run or "equilibrium" relationship, such a cointegration measure needs to be added to eq. (2) to account for the long-run behavior of spot and futures price changes. The test results indicate that for the stock indices considered, the spot and futures prices are cointegrated at $\gamma = 1$. Because this estimate is consistent, $\gamma = 1$ is imposed in eq. (2).

The hedging effectiveness of four types of hedging models are compared. The first one is the naive hedging model where the hedge ratio is 1 over time. The second model is the OLS hedge, and the third one is the OLS with cointegration (OLS–CI) between spot and futures. The last one is the bivariate GARCH model. Maximum likelihood estimation is used to estimate the parameters in eqs. (2), (3), and (4) for the GARCH model and the parameters in eq. (2) for the OLS and the OLS–CI models.[6] The hedging effectiveness is measured by comparing the dynamic hedging performance of different hedging methods for the out-of-sample periods. To compare the out-of-sample hedging performance of the four strategies, the first 93 observations (one

[4] The logarithms of the prices, multiplied by 100, are used in this study because both spot and futures returns are conventionally assumed to follow log normal distributions.

[5] Recent studies [Wahab and Lashgari (1993) and Ghosh (1993)] document that the stock index and futures prices are cointegrated and show that an error correction representation for each series is appropriate. They stop short of estimating the hedge ratio with the time-varying conditional volatility. For the error correction term in (2), dividends are subtracted from the spot price, S_t, because the dividend exclusive spot price, $S_t - DIV_t$, represents the cost of carry model of futures prices more accurately than the dividend-inclusive price. For the price changes s_t, however, dividends are included because the dividend-inclusive return represents the actual return of stock index portfolios. Detailed results of cointegration and unit root tests are available from the authors upon request.

[6] The constant hedge ratio models are nested within the GARCH model. Likelihood ratio tests are used to compare the three hedging models. The GARCH model describes the distribution of spot and futures price changes better than both of the constant hedge ratio models. The detailed results of the estimation are available from the authors upon request.

half of the sample) are used to estimate the parameters. For the GARCH model, the estimated parameters are used to forecast the conditional variance and covariance of the following week. The forecasted hedge ratio is the one-period forecast of the covariance divided by the one-period forecast of the variance. With the OLS and the OLS–CI models, the variance and covariance of the first 93 observations are used to produce the optimal hedge ratios for the following week. Such forecasts are conducted for each week with the estimates from the proceding 93 observations up to the final week, totaling 92 (91) forecasts for S&P 500 (for TSE 35).

To compare the performances of each type of hedge, portfolios implied by the computed hedge ratios are constructed for each week. The variance of the returns to the constructed portfolios are calculated over the sample. Thus, $\sigma^2(s_t - b_t^* f_t)$ is evaluated where b_t^* is the computed hedge ratio for each hedge method. The portfolio variances are reported in Table IA. All four types of hedging reduce the variance of the spot portfolio significantly. The variance reduction is much greater for the S&P 500 Index futures. Table IA also reports the percentage variance reductions of the GARCH hedge over the other positions. The percentage variance reduction of the GARCH over the three alternative hedges shows noticeable improvement of hedging effectiveness through GARCH over the conventional methods for both S&P 500 and TSE 35 futures contracts.

The GARCH hedge is even more economically useful in improving the utility function of investors with a mean-variance expected utility

TABLE IA

Out-of-Sample Comparisons of Hedge Effectiveness

	Portfolio Variances		*% Variance Reduction of GARCH Hedge over*	
Hedging Methods	*S&P 500*	*TSE 35*	*S&P 500*	*TSE 35*
Unhedged position	5.0322	4.3500	97.916	77.471
Naive hedge ($b = 1$)	0.1089	1.1610	3.762	15.591
OLS conventional hedge	0.1150	1.0810	8.870	9.343
OLS–CI hedge	0.1113	1.0797	5.840	9.234
GARCH hedge	0.1048	0.9800		

The estimation period is from June 8, 1988–March 14, 1990. The forecast period is from March 21, 1990–December 18, 1991. The sample variances of portfolio returns are computed as $\sigma^2(s_t - b_t^* f_t)$, assuming that an investor forecasts the next week's hedge ratio using the data from the preceding 94 weeks at that time. The variance reduction is calculated as $(\sigma^2_{\text{others}} - \sigma^2_{\text{GARCH}})/\sigma^2_{\text{others}}$. The OLS–CI hedge is the OLS hedge that accounts for cointegration between the spot and futures prices.

TABLE IB

Utility Comparisons

Hedging Methods	*S&P 500*	*TSE 35*
OLS conventional hedge	45.24(2)	-48.05(2)
OLS CI hedge	43.27(1)	47.669(2)
GARCH hedge	41.55(10)	40.210(43)

This table presents the utility of the portfolio constructed by assuming that an investor rebalances his portfolio only when the potential gains in utility from the reduced variance offset the transactions costs that must be incurred. The transactions cost per rebalancing is assumed to be 0.0125%, which is the average of the high and the low quotes for transactions cost. The number of portfolio rebalancings is given in parentheses.

function. Assume that an investor faces the mean-variance expected utility function $EU(x) = E(x) - \lambda\,\mathrm{Var}(x)$, where x is the return from the spot-futures hedge portfolio and λ is the degree of risk aversion. In these simulations, λ takes the value of 4, and the expected return to the hedged portfolio is zero.[7] Each week, beginning with the 94th week in the sample, the utility from hedging is $-y - 4\,\mathrm{Var}(x)$, where $-y$ is the reduced returns caused by the transaction costs and $\mathrm{Var}(x)$ is the conditional variance based on the preceding 94 observations. A typical round trip transactions cost is around \$15–23 with one stock index futures contract, implying $y \approx 0.01$–0.015% with the level of index around 300 (price of one contract $= 500 \times 300 =$ \$150,000). Assume that an investor compares the utility function with costly rebalancing against that without rebalancing and takes the higher one. Thus, with this dynamic hedging strategy, an investor rebalances only if the potential utility gains from rebalancing offset the losses due to transactions costs. The sum of weekly utility functions with the three hedging strategies from March 21, 1990–December 18, 1991 are compared in Table IB. With both futures contracts, the utility increase is evident with the GARCH method. A mean-variance expected utility-maximizing investor would prefer the GARCH hedge to the conventional strategies. Therefore, the GARCH model gives an improved hedging strategy even after accounting for transactions costs.

BIBLIOGRAPHY

Baillie, R. T., and Bollerslev, T. (1990): "A Multivariate Generalized ARCH Approach to Modelling Risk Premia in Forward Foreign Exchange Rates Markets," *Journal of International Money and Finance*, 9:309–324.

[7] This assumption on λ is in line with most empirical studies in the literature [Grossman and Shiller (1981) and others].

Baillie, R.T., and Myers, R.J. (1991): "Bivariate GARCH Estimation of the Optimal Commodity Futures Hedge," *Journal of Applied Econometrics*, 6:109–124.

Bollerslev, T. (1986): "Generalized Autoregressive Conditional Heteroscedasticity," *Journal of Econometrics*, 31:307–327.

Cecchetti, S.G., Cumby, R.E., and Figlewski, S. (1988): "Estimation of Optimal Futures Hedge," *Review of Economics and Statistics*, 70:623–630.

Dickey, D.A., and Fuller, W.A. (1979): "Distribution of the Estimators for Autoregressive Time Series with a Unit Root," *Journal of American Statistical Association*, 427–431.

Engle, R.F. (1982): "Autoregressive Conditional Heteroscedasticity with Estimates of the Variance of United Kingdom Inflation," *Econometrica*, 50:987–1007.

Engle, R.F., and Granger, C.W.J. (1987): "Cointegration and Error Correction: Representation, Estimation and Testing," *Econometrica*, 55:251–276.

Figlewski, S. (1984): "Hedging Performance and Basis Risk in Stock Index Futures," *Journal of Finance*, 39:657–669.

Ghosh, A. (1993): "Hedging with Stock Index Futures: Estimation and Forecasting with Error Correction Model," *The Journal of Futures Markets*, 13:743–752.

Grossman, S., and Shiller, R. (1981): "The Determinants of the Variability of Stock Market Prices," *American Economic Review*, 71:222–227.

Hasza, D., and Fuller, W. (1979): "Estimation for Autoregressive Processes with Unit Roots," *The Annals of Statistics*, 7:1106–1120.

Kroner, K.F., and Sultan, J. (1991): "Exchange Rate Volatility and Time Varying Hedge Ratios," Rhee, S.G., and Chang, R.P. (ed.), *Pacific-Basic Capital Market Research*, Vol II, pp. 397–412.

Kroner, K.F., and Sultan, J. (1993): "Time Varying Distribution and Dynamic Hedging with Foreign Currency Futures," *Journal of Financial and Quantitative Analysis*, 28:535–551.

Lee, C.F., Bubnys, E.L., and Lin, Y. (1987): "Stock Index Futures Hedge Ratios: Tests on Horizon Effects and Functional Form," in Fabozzi, F.J. (ed.), *Advances in Futures and Options Research*, 2:291–311.

Myers, R.J. (1991): "Estimating Time Varying Optimal Hedge Ratios on Futures Markets," *The Journal of Futures Markets*, 11:39–53.

Phillips, P.C.B. (1987): "Time Series Regressio with a Unit Root," *Econometrica*, 55:277–301.

Phillips, P.C.B., and Ouliaris S. (1990): "Asymptotic Properties of Residual Based Tests for Cointegration," *Econometrica*, 58:165–193.

Phillips, P.C.B., and Perron, P. (1988): "Testing for a Unit Root in Time Series Regression," *Biometrika*, 75:335–346.

Santoni, G.J., and Liu, T. (1993): "Circuit Breakers and Stock Market Volatility," *The Journal of Futures Markets*, 13:261–277.

Wahab, M., and Lashgari, M. (1993): "Price Dynamics and Error Correction in Stock Index and Stock Index Futures Markets: A Cointegration Approach," *The Journal of Futures Markets*, 13:711–742.