ON THE ARBITRAGE-FREE PRICING RELATIONSHIP BETWEEN INDEX FUTURES AND INDEX OPTIONS: A NOTE

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INTRODUCTION

The objective of this article is to extend Tucker's (1991, p. 351–353) general put–call–futures parity. In particular, the article studies the relationship between stock index futures and stock index options with the consideration of continuous dividend yield. Some empirical results are presented.

Arbitraging between stock index futures and stock index options (A similar strategy of arbitraging index futures and the underlying stock index portfolio is called index arbitrage) is quite possible because futures and options written on the same index are available for the Standard and

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Poor 500 index (SP 500), the Standard and Poor 100 index (SP 100), and the Major Market Index (MMI). Furthermore, index options and index futures have common expiration days (the third Friday of March, June, September, and December).

Consider the following hypothetical situation: the December SP 500 futures is trading at \$430, the December 430 SP 500 call is trading at \$10, and the December 430 SP 500 put is trading at \$8.[1] Note that the options have a multiplier of \$100 while the SP 500 futures has a multiplier of \$500 per contract. Consider the following strategy:

i. Buy 10 contracts of December SP 500 futures at \$430, short 50 contracts of April 430 SP 500 calls, and receive a total premium of \$50,000 (\$10 times 100 times 50);
ii. Long 50 contracts of April 430 SP 500 puts and pay a total premium of \$40,000 (\$8 times 100 times 50).

Consequently, there is a \$10,000 net cash to the arbitrager. Ignoring marking-to-market and the margin requirements, the payoff from the futures position and that from the options portfolio exactly cancel each other on their common expiration day. This means that, the above situation allows the arbitrager to reap a riskless profit of \$10,000 before commission. In competitive futures and options markets, there should be no abnormal riskless arbitrage profit.

This article examines the arbitrage-free pricing relationship between stock index futures and index options written on the stock index. The study applies Tucker's (1991) general put–call–futures parity relationship to SP 500 index futures and options. Since there are a few empirical studies on the general futures-options parity, this article hopes to shed some light on the arbitrage-free relationship between futures and options markets. Acknowledging the limitations of a small sample and using daily closing prices, the empirical results suggest that the arbitrage-free pricing relationship between SP 500 stock index futures and index options holds fairly well.

[1]There are three types of SP 500 options contract, i.e., SPX, SPL, and NSX. The SPX contract has two nearby expiration months and as many expiration months as four months from the March–June–December cycle. The final settlement price is based on the closing value of the index on the third Friday of the expiration month. The SPL contract has June and December as the expiration months and up to two years of maturity. The NSX contract is the same as the SPX contract except that the settlement price is based on the opening price of the index on the third Friday of the expiration month. For details, see Dubofsky (1992, p. 246–247). For simplicity, this discussion concentrates on SPX.

A GENERAL PUT–CALL–FUTURES PARITY

Tucker (1991, p. 351–353) provides a general put–call–futures parity. Essentially, Tucker's parity is put into a vigorous theorem and we provide a corollary that explicitly considers a continuous dividend yield. The assumptions of the parity are:

A1. The index futures and the index options have the same expiring dates;

A2. The stocks have no dividend payouts (this assumption is relaxed later);

A3. Transaction costs are zero;

A4. The index stock options are not exercised early, i.e., the index options are European options;

A5. The futures position is held until its expiration;

A6. The markets are arbitrage-free.

Theorem. *The arbitrage-free pricing relationship between index futures and index call and put options is given as:*

$$Fe^{-rt} = C - P + Xe^{-rt} \tag{1}$$

where

F = stock index futures price;

r = risk-free interest rate;

t = the time until the futures and options contracts expire;

C = index call option price with any exercise price;

P = index put option price with the same exercise price as the index call option;

X = exercise price of the index options.

Proof: (i) Consider a portfolio with a long position of index futures contract at a futures price of F and invest Fe^{-rt} in a riskless security at r for a term t. Payoff from the futures position is equal to $(S_t - F)$; where S_t is the index level at time t. Payoff from the riskless investment at time t is F. Thus, the total payoff from the long position of index futures and the riskless security is $S_t - F + F = S_t$. The index futures contract has no value at inception. The beginning value of the portfolio is Fe^{-rt}. (ii) Consider another portfolio that contains one long index call option, one short index put option (both have the same exercise price) and invest Xe^{-rt} in a riskless security. At time t, there are two possibilities:

> Case 1—if stock index level, S_t, is greater than the exercise price, then the index call option has a value of $S_t - X$, the index put options has no value, and the riskless security has a value X. The portfolio value becomes S_t.
> Case 2—if stock index level, S_t, is smaller than the exercise price, then the index call option has no value, the index put option has a value of $-(X - S_t)$, and the riskless security has a value X. The portfolio value also becomes S_t.

The two portfolios in (i) and (ii) have the same value at time t. In arbitrage-free markets, they must have the same value today. Q.E.D.

Corollary. *Dividends do not have any impact on the above theorem, i.e., assumption A2 is not required to derive the Theorem.*

Proof: Assume that there is a continuous dividend yield, q, before the index futures and options expire. With the cost-of-carry relation in the index futures market, one obtains:

$$Fe^{-(r-q)t} = S \tag{2}$$

where S is the current stock index level.

In the index options market, assume put–call parity [Stoll (1969)] holds, one obtains:

$$C - P + Xe^{-rt} = Se^{-qt}$$

or

$$(C - P + Xe^{-rt})e^{qt} = S \tag{3}$$

Compare eqs. (2) and (3) together:

$$Fe^{-(r-q)t} = (C - P + Xe^{-rt})e^{qt}$$

or

$$Fe^{-rt} = C - P + Xe^{-rt}$$

Hence, the continuous dividend yield, q, does not affect the theorem in (1). Q.E.D.

METHODOLOGY AND EMPIRICAL RESULTS

The June and September 1993 contracts of SP 500 index futures, index options prices (SPX call and put options with same exercise prices), and 90-day Treasury Bill rate are from the *Wall Street Journal*. There are 337 and 364 usable observations in June and September contracts, respectively.

To test eq. (1), a regression model is run with Fe^{-rt} as dependent variable and $(C - P + Xe^{-rt})$ as independent variable. The model is as follows:

$$(Fe^{-rt})_T = \alpha + \beta(C - P + Xe^{-rt})_T + \epsilon_T$$

where T denotes the observations and ϵ_T is the random error term. The null hypothesis is that $\alpha = 0$ and $\beta = 1$. The results are:

(i) June, 1993 contract:

$$(Fe^{-rt})_T = \underset{(6.8234)}{12.6121} + \underset{(0.0154)}{0.9717}(C - P + Xe^{-rt})_T$$

standard errors are in parenthesis;
adj. $R^2 = 0.9219$;
F-statistics (for $\alpha = 0$ and $\beta = 1$) $= 3.3721$.

(ii) September, 1993 contract:

$$(Fe^{-rt})_T = \underset{(3.1387)}{5.1260} + \underset{(0.0069)}{0.9888}(C - P + Xe^{-rt})_T$$

standard errors are in parenthesis;
adj. $R^2 = 0.9826$;
F-statistics (for $\alpha = 0$ and $\beta = 1$) $= 2.6327$.

The α's are not significantly different from zero and the β's are not significantly different from 1. The F-statistics for the null hypothesis of $\alpha = 0$ and $\beta = 1$ are not statistically significant. The empirical results suggest that the arbitrage-free pricing relationship between SP 500 index futures and options holds fairly well.

Recognizing the limitation of using daily closing prices, a simulation is conducted to examine if there are abnormal trading profits from any violations of the pricing relationship for each observation. Following Chung (1991), a 0.5% transaction cost is used.[2] For the September contract, there are no violations of the arbitrage-free pricing relationship. For the June contract, there is only one violation in 377 observations.[3]

[2]Chung (1991) also used 0.75% and 1% transaction costs.
[3]Detail results are available upon request.

SUMMARY

This article extends Tucker's (1991) general put–call–futures parity by considering a continuous dividend yield. In addition, a small sample empirical test is performed using SP 500 index futures and index options. It is shown that the discount index futures price must be equal to that of the call/put option-riskless asset portfolio. The empirical results suggest that such a pricing relationship holds fairly well and arbitrage looks unattractive.

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