# FUTURES PRICING AND THE COST OF CARRY UNDER PRICE LIMITS

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## INTRODUCTION

Price limits are a common feature associated with futures contracts. Used ostensibly to prevent prices from fluctuating too much in a given trading session and thereby to forestall or prevent defaults, they have been shown to play an important role in the design of a futures contract.[1] Their effects on the statistical properties and behavior of futures prices

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[1]For example, Brennan (1986) shows that price limits can be used in conjunction with margins to reduce the cost of futures contracting. Similarly Ackert and Hunter (1989) provide empirical evidence consistent with the establishment of price limits at levels that maximize benefits and minimize costs of futures contracting. In addition Kodres and O'Brien (1991) show that under certain conditions about trader characteristics and behavior and the sources of stochastic shocks, price limits can be Pareto-superior.

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have been studied by other researchers.[2] The objective of this article is to develop a model of futures pricing in the presence of price limits.

The model developed here does not address the welfare benefits of price limits. In their absence, the futures price would be affected by default risk and other costs that price limits are designed to control. Those costs are difficult to capture and the effect of price limits is generally ignored in futures pricing models, such as the cost of carry model. The model in this article is designed to determine the extent to which the futures price in the presence of price limits differs from the traditional cost of carry price. In so doing, it provides an intuitive and economic interpretation of price limits.

Consider a simple world in which the futures price at zero is $f_0$. The price moves to $f_1$ at time 1 and $f_2$ at time 2. Let $f_1$ be significantly below $f_0$ and $f_2 = f_0$.[3] The holder of a futures contract purchased at $f_0$ and sold at $f_2$ has achieved no gain and, in fact, has incurred a loss because the amount $f_0 - f_1$ must be financed to replete the margin account, which is marked-to-market at time 1. Suppose a limit price of $L < f_0 - f_1$ is imposed such that the settlement at time 1 is at $L$ and the loss is only $f_0 - L$. Then this amount, and not $f_0 - f_1$, must be financed the next day. On the upside, a limit move reduces the gain that can be reinvested. Although the contract price ultimately ends up where it started, the path taken in a world with limits is potentially different from a world without limits. This path dependency can have an effect on the futures price.[4]

The purchaser of a futures contract at a relatively low price, i.e., one in which a limit down move is more likely, has the benefit of the lower limit and, thus, the futures price should be higher. The purchaser of a futures contract at a relatively high price, i.e., one in which a limit up move is more likely, has the disadvantage of the upper limit and, thus, the futures price would be somewhat lower. There is one and only one price at which these effects offset such that there would be no difference between the futures price and the cost of carry price. At all other prices, however, a price limit effect would be observed.

[2]See Monroe (1983); Khoury and Jones (1984); Gordan et al. (1988); Kodres (1988, 1993); Ma, Rao, and Sears (1989); and Ma (1991). Chen (1993); Chiang, Wei, and Wu (1990); and Chung (1991) have studied the effect of price limits on spot markets in several Pacific countries.

[3]The assumption that the futures price returns to its original level is not necessary but simplifies the presentation somewhat.

[4]While path dependency is the source of price limit effects in this model, alternative explanations of a more behavioral nature are provided by Isaac and Plott (1981); Smith and Williams (1981); and Coursey and Dyl (1990).

Thus, any model designed to measure the impact of price limits should capture the relative effects of the two possible limit moves. It is tempting to dismiss this matter as trivial. That, however, would be like dismissing the forward-futures price differential in a world without price limits. Clearly, that differential disappears under certain conditions, but under others, the difference is no less significant than transaction costs. That, too, is the case for price limits.

It is important to emphasize that it is assumed that price limits are imposed by the exchange to enable it to guarantee that all payoffs will be made. Suppose the price goes through the limit and a buyer and seller agree to trade either off the exchange at the cost of carry price or at the limit with a side payment from one party to the other. Then the limits have no effect. However, by introducing side payments to get around price limits, traders would be effectively arguing that price limits are not necessary to insure against default. This issue is an interesting one and is similar to the controversy that swaps are simply a market of side payments designed to squeeze the default premium from institutional lenders. The possibility of side payments or off-exchange contracting is not addressed here. This article focuses on the question of how exchange-traded futures are priced in the presence of price limits.

The next section develops a two-period model of a futures market in which price limits are removed at the end of the first period. The third section analyzes several aspects of the model while the fourth provides some numerical estimates of the magnitude and expected frequency of price limit biases. The last section provides a summary and conclusions.

## A TWO-PERIOD MODEL OF FUTURES PRICES UNDER PRICE LIMITS

Assume an exogenous spot market in which the price of the spot asset is determined by some general equilibrium process with unconstrained prices. The spot asset has no cash flows such as dividends or coupons, nor a convenience yield and no physical cost of storage. This assumption can be easily relaxed.[5] Assume no taxes or transaction costs and that all investors can borrow or lend at the constant continuously compounded risk-free rate $r$.

Let there be a futures contract expiring at time $T$. The life of the futures is divided into time periods, which are called settlement intervals,

[5]Under the assumption that these factors could be expressed as a continuous yield, they could be easily added. However, stochastic yields would impose extreme difficulties even for models without price limits.

of length $i$. At the end of each settlement interval, the settlement price is determined and profits and losses are distributed.

Current institutional practices define the length of the settlement interval as one day. Such a restriction is avoided in the general model, for there is no special reason why a settlement interval could not be longer or shorter than a day.[6] It will be shown later that the length of the settlement interval will have an effect on the futures price.

The final settlement interval begins at time $T-i$ and ends at time $T$. Assume price limits are removed for trading during the final settlement interval. This assumption is somewhat analogous to the real-world practice of removing price limits in the expiration month so as to permit the convergence of futures and spot prices. Thus, in this model $S_T = f_T$ where $S$ and $f$ represent spot and futures prices, respectively.

The next to last settlement interval begins at $T-2i$ and ends at $T-i$. Over this interval, price limits are in effect. The settlement price at $T-2i$, which is called $f^S_{T-2i}$, is established and limits of $\alpha_H$ and $\alpha_L$ are added to and subtracted from the settlement price.[7] Current institutional practices have symmetric limits ($\alpha_L = \alpha_H$), but that assumption is not yet made. The limit prices for the settlement interval $T - 2i$ through $T - i$ are $H_{T-i} = f^S_{T-2i} + \alpha_H$ and $L_{T-i} = f^S_{T-2i} - \alpha_L$. During this interval no trades can take place above $H_{T-i}$ (the high) or below $L_{T-i}$ (the low). In addition the settlement price at $T - i$, $f^S_{T-i}$, must be within or at the limits.

With price limits removed during the final settlement interval, any futures price during that interval is easy to determine. Let $j$ represent a time point in the interval $(T - i, T]$. Then the futures price is given as

$$f_j = S_j e^{r(T-j)} \quad \forall \quad T - i < j \leq T \tag{1}$$

This is the well-known cost of carry formulation that must hold to prevent profitable arbitrage.

Of primary interest, however, is how futures prices are affected by limits. Thus, a time point in the settlement interval $[T - 2i, T - i]$ must be examined. Let $t$ be such a time point. The time to expiration of the futures contract is $T - t$, while the time remaining in the settlement interval is $(T - i) - t = \tau$.

If there were no price limits, the futures price would again be given by the cost of carry formulation. This futures price will play an important

[6]Intraday margin calls are in fact employed by futures clearinghouses particularly in fast markets.
[7]This is the current institutional procedure for establishing the limit prices. However, this is but one of many possible schemes. As, it will be shown later, it is a particularly complex one.

role in the model. For expositional distinction, designate it as the cost of carry futures price, $\overline{f}_t$, where

$$\overline{f}_t = S_t e^{r(T-t)} \quad \forall \quad t \tag{2}$$

The payoff at $T - i$ to a holder of a long futures contract is $f^S_{T-i} - f_t$, which is given as

$$\begin{aligned} L_{T-i} - f_t \quad &if \quad \overline{f}_{T-i} \le L_{T-i} \\ \overline{f}_{T-i} - f_t \quad &if \quad L_{T-i} < \overline{f}_{T-i} < H_{T-i} \\ H_{T-i} - f_t \quad &if \quad \overline{f}_{T-i} \ge H_{T-i} \end{aligned}$$

## A Pricing Equation

The next step solves for the futures price at $t$. Assume that European options on futures can be created. Individuals are permitted to enter into private option contracts, or essentially over-the-counter options, with any desired exercise price and expiration. These contracts can be written on the cost of carry futures price. A call at exercise price $X$ and expiring at $T - i$ will have a payoff of $\text{Max}(0, \overline{f}_{T-i} - X)$. A put at exercise price $X$ and expiring at $T - i$ will have a payoff of $\text{Max}(0, X - \overline{f}_{T-i})$. Note that the option payoffs are unaffected by limits because the parties to the contract simply agree that since the cost of carry price can always be determined, the payoffs are identifiable and can be made in cash. The option prices are written as $c(\overline{f}_t, H_{T-i}, \tau)$ and $p(\overline{f}_t, L_{T-i}, \tau)$. For now, it is not necessary to specify the exact functional form of the option price. Some additional assumptions made later will enable the option pricing function to be determined.[8]

The futures can be priced by establishing an arbitrage portfolio containing the futures. Since the futures cannot trade outside the limits, the current futures price is assumed to be within the limits.[9] Consider the following portfolio. At time $t$: (a) Buy $1/K$ units of the spot asset at price $S_t$ where $K = e^{-ri}$, (b) Sell one futures at price $f_t$, (c) Buy $1 - K$

[8]The assumptions in this derivation require options on the cost of carry futures price. It has been observed [see Burghardt (1988)] that the actual futures options often continue to trade when futures prices are locked limit. This suggests that the actual futures options might track the cost of carry futures price rather than the actual futures price. However, it seems likely that the actual futures options would track the actual, rather than the cost of carry, futures price when the futures price is within the limits. Thus, it is not clear whether the actual options would suffice for the purposes of this model. If so, then the private futures options can be the actual futures options. In the next section, however, it is shown that private options are not needed if one is willing to make some further assumptions.

[9]If the price is beyond the limits, no trade can take place and the futures price is artificially set at the limit price.

of the aforementioned puts with exercise price $L_{T-i}$ expiring at $T - i$, (d) Sell $1 - K$ of the aforementioned calls with exercise price $H_{T-i}$ expiring at $T - i$, and (e) Sell discount bonds maturing at $T - i$ with a face value of $f_t$. At time $T - i$ the futures are marked-to-market at the settlement price, which will be either between or at the limits. The puts, calls, and bonds mature and the arbitraguer should liquidate $((1/K) - 1)$ units of the spot asset, leaving one unit of the asset and one futures going into the second period. $K$ new bonds are issued with a present value equal to the settlement price. Table I presents the payoffs of this portfolio. The portfolio is perfectly hedged at expiration and promises a payoff of zero. Thus, the portfolio must be worth zero today. Thus,

$$(1/K)S_t + (1 - K)p(\overline{f}_t, L_{T-i}, \tau) - (1 - K)c(\overline{f}_t, H_{T-i}, \tau) - e^{r\tau} f_t = 0$$

Rearranging and solving for $f_t$ gives

$$f_t = S_t e^{r(T-t)} + e^{r\tau}(1 - K)[p(\overline{f}_t, L_{T-i}, \tau) - c(\overline{f}_t, H_{T-i}, \tau)] \quad (3)$$

or in terms of the cost of carry futures price,

$$f_t = \overline{f}_t + e^{r\tau}(1 - K)[p(\overline{f}_t, L_{T-i}, \tau) - c(\overline{f}_t, H_{T-i}, \tau)] \quad (4)$$

Thus, the futures price will deviate from the equilibrium futures price by the term,

$$e^{r\tau}(1 - K)[p(\overline{f}_t, L_{T-i}, \tau) - c(\overline{f}_t, H_{T-i}, \tau)]$$

which is referred to as the bias.[10] Since the forward price is the cost of carry price, the futures price after accounting for price limits will differ from the forward price if the bias is non-zero.[11]

As discussed in the introduction, the existence of price limits creates an advantage on the downside and a disadvantage on the upside for the holder of the futures. Price limits are, thus, similar to collars.

[10]Here the bias is the difference between the actual futures price (subject to limits) and the cost of carry price. It should not be confused with the bias in which futures prices deviate from expected future spot prices (the normal backwardation/contango).

[11]The approach used here to model price limits is similar to that employed by Ma (1991) to estimate the information content of limit moves. However, Ma's model looks at only the current settlement period. This overlooks the fact that if a limit move occurs, the arbitrageur cannot capture gains or cover losses and remove the obligation that goes with having the futures contract still outstanding. To reflect this contingency, a two-period model is required. In addition, Ma assumes that this approach yields the true futures price while this study assumes that the actual futures price reflects investors' full awareness of the limits and the hedge portfolio. While Ma uses the model to design some empirical tests, the focus here is on other issues.

**TABLE I**

Payoffs of Portfolio Incorporating a Futures Contract Under Price Limits

| | *Payoffs Given Settlement Price at* T − i | | |
|---|---|---|---|
| | $\bar{f}_{T-i} \le L_{T-i}$ | $L_{T-i} < \bar{f}_{T-i} < H_{T-i}$ | $\bar{f}_{T-i} \ge H_{T-i}$ |
| Spot[a] | $((1/K) - 1)S_{T-i}$ | $((1/K) - 1)S_{T-i}$ | $((1/K) - 1)S_{T-i}$ |
| -Futures | $-(L_{T-i} - f_t)$ | $-(\bar{f}_{T-i} - f_t)$ | $-(H_{T-i} - f_t)$ |
| $1 - K$ Puts | $(1 - K)(L_{T-i} - \bar{f}_{T-i})$ | 0 | 0 |
| $-(1 - K)$ Calls | 0 | 0 | $-(1 - K)(\bar{f}_{T-i} - H_{T-i})$ |
| -Bonds | $-f_t$ | $-f_t$ | $-f_t$ |
| Total | $-KL_{T-i}$ | $-K\bar{f}_{T-i}$ | $-KH_{T-i}$ |

| | *Payoffs at T Contingent upon Outcome at* T − i | | |
|---|---|---|---|
| | $\bar{f}_{T-i} \le L_{T-i}$ | $L_{T-i} < \bar{f}_{T-i} < H_{T-i}$ | $\bar{f}_{T-i} \ge H_{T-i}$ |
| Spot | $S_T$ | $S_T$ | $S_T$ |
| -Futures | $-(S_T - L_{T-i})$ | $-(S_T - \bar{f}_{T-i})$ | $-(S_T - H_{T-i})$ |
| -Bonds | $-KL_{T-i}(1/K)$ | $-K\bar{f}_{T-i}(1/K)$ | $-KH_{T-i}(1/K)$ |
| | 0 | 0 | 0 |

Note: The transaction begins at time $t$ and terminates at time $T$ (expiration). The portfolio is set up with $1/K$ units of the asset with price $S_t$, one short futures contracts priced at $f_t$, $1 - K$ long futures put options priced at $p(\bar{f}_t, L_{T-i}, \tau)$ and $1 - K$ short futures call options period at $c(\bar{f}_t, H_{T-i}, \tau)$ where $\tau$ is the length of the holding period, and a discount bond with a face value of $e^{-r\tau}f_t$. $L_{T-i}$ and $H_{T-i}$ are the low and high limits on the settlement price at $T - i$ and $\bar{f}_t$ is the cost of carry futures price given as $S_t e^{r(T-t)}$.

[a]At $T - i$, liquidate $(1/K) - 1$ units of the spot asset leaving one unit and issue $K$ bonds with present value equal to the settlement price. Note that these payoffs make use of the fact that $K$ times the cost of carry price is the spot price.

Equation (4) shows that a futures subject to price limits is equivalent to a futures without price limits plus an adjustment term. The adjustment term is $1 - K$ times $e^{r\tau}$ long puts and $e^{r\tau}$ short calls. Since, under the model's assumptions, an option price is the discounted expected payoff at expiration, the put price is $e^{-r\tau}$ times $E[\text{Max}(0, L_{T-i} - \bar{f}_{T-i})]$ and the call price is $e^{-r\tau}$ times $E[\text{Max}(0, \bar{f}_{T-i} - H_{T-i})]$. Thus, the futures price can be written as

$$f_t = \bar{f}_t + (1 - K)[E(\text{Max}(0, L_{T-1} - \bar{f}_{T-i})) - E(\text{Max}(0, \bar{f}_{T-i} - H_{T-i}))]$$

which provides a much clearer interpretation of price limits. The term $E[\text{Max}(0, L_{T-i} - \bar{f}_{T-i})]$ is the expected reduction in the loss from a limit down move while the term $E[\text{Max}(0, \bar{f}_{T-i} - H_{T-i})]$ is the expected reduction in the gain from a limit up move. This is then multiplied by the term $1 - K$, which is the interest for the next day. Thus, a futures without price limits is equivalent to a futures with price limits plus the expected net gain or loss in interest on the settlement from a limit move. Clearly price limits are contingent upon positive interest rates for if $r = 0$, then $K = 1$ and the bias disappears.

The futures exchanges and the financial press always report a futures price, regardless of whether the price is constrained by the limits. Therefore, denote the reported price as $f_t^*$ and let $\Phi_t$ represent the right-hand side of eq. (4). Then $f_t^* = L_{T-i} + \text{Max}(0, \Phi_t - L_{T-i}) - \text{Max}(0, \Phi_t - H_{T-i})$. This would set the reported price to either the cost of carry price or the upper or lower limit. The reported price will accurately reflect the true price of the futures unless the limit has been hit. Then, of course, no transactions can be made so it is impossible to determine the true price of the futures. Thus, it is important to remember than when the reported price is at the limits, the reported price is an artificial price. The model price must be between the limits and when it is not, the model cannot be indiscriminately used since the assumption that the futures can be traded is violated. But that condition is no more restrictive than simply acknowledging that a market in which trading at the equilibrium price is forbidden is a market in which the equilibrium price is unobservable. Of course, as noted above, participants could bypass the futures market and trade directly but that would induce the possibility of default risk, which would merit consideration in an equilibrium model.

When there are no limits, $L_{T-i} \rightarrow 0$ and $H_{T-i} \rightarrow \infty$, the put and call prices are worthless and the futures price becomes the cost of carry futures price. When there are limits, a futures price that is approaching the limit does so smoothly and progressively. This is due to the time value decay on the implicit options. When the cost of carry futures price approaches the lower (upper) limit, the put (call) will be nearly at-the-money and the call (put) will be out-of-the-money. The bias will be positive (negative) reflecting a higher probability of a limit move that benefits (hurts) the holder of the futures. Thus, at low (high) spot prices, the futures price will tend to exceed (be less than) the cost of carry futures price. As such, the time value decay on the options will cause the futures price to progressively fall (rise) toward the lower (upper) limit. The deviation of the futures price from the cost of carry price will depend on the factors determining the put and call prices. Those factors are examined in the next section.

## An Alternative Derivation

Equations (3) and (4) are extremely general results in that they do not specify the functional forms of the option prices. A mathematical specification of these prices is necessary in order to perform comparative statics and to generate numerical values. Usually, options are priced by assuming continuous trading [Black–Scholes (1973); Black (1976)], or discrete trading with a representative investor with constant proportional risk aversion and a lognormally distributed market factor [Rubinstein (1976)].

A reasonable approach is to assume continuous trading of the spot asset. Let the spot price dynamics be described by the familiar stochastic differential equation,

$$dS_t/S_t = \mu\, dt + \sigma\, dz$$

where $dz$ is a standard Gauss–Weiner process. A portfolio consisting of $h$ units of the spot asset and one futures contract priced at $f_t$ is formed. Since the value of the futures contract when written is zero, the value of the portfolio is initially $V_t = hS_t$. The change in $V_t$ is expressed as $dV_t = hdS_t + df_t$. Using Itô's Lemma, the change in the futures price can be written as

$$df_t = \frac{\partial f_t}{\partial S_t} dS_t + \frac{\partial f_t}{\partial t} dt + \frac{1}{2}\frac{\partial^2 f_t}{\partial S_t^2} S_t^2 \sigma^2$$

With the appropriate choice of $h$, the portfolio is risk-free leading to the following well-known non-stochastic partial differential equation,

$$\frac{\partial f_t}{\partial S_t} rS_t + \frac{\partial f_t}{\partial t} + \frac{1}{2}\frac{\partial^2 f_t}{\partial S_t^2} S_t^2 \sigma^2 = 0 \tag{5}$$

the solution of which is the futures price.[12] The appropriate condition for the futures price is $L_{T-i} \leq f_t \leq H_{T-i} \;\; \forall \;\; T - 2i < t \leq T - i$. The terminal condition is, of course, $f_T = S_T$.

The solution is known from the previous section. However, it should be expressed in terms of its fundamental components rather than its embedded options. The solution, adapted from Black (1976), is

$$\begin{aligned} f_t &= K\overline{f}_t + (1 - K) \\ &\quad \times [L(1 - N(d_{2L})) - \overline{f}_t(1 - N(d_{1L})) - \overline{f}_t N(d_{1H}) + HN(d_{2H})] \end{aligned} \tag{6}$$

[12]Equation (5) differs from the standard partial differential equation for the valuation of an option in that it does not include a term reflecting the opportunity cost for the investment in the contingent claim. In this case the contingent claim, a futures, requires no initial outlay. Details of the derivation of the standard partial differential equation for a derivative contract can be found in many places such as Hull (1993), Chs. 11 and 12. Equation (5) is obtained by applying the standard procedure to the hedge portfolio consisting of $h$ units of the spot asset and one unit of the futures where $h$ is the partial derivative of the futures price with respect to the spot price.

where the $T - i$ subscript is dropped from the limit prices, $N(\bullet)$ is the cumulative normal probability and

$$d_{1L} = \frac{\ln(\overline{f}_t/L) + (\sigma^2/2)\tau}{\sigma\sqrt{\tau}}, \qquad d_{2L} = d_{1L} - \sigma\sqrt{\tau}$$

$$d_{1H} = \frac{\ln(\overline{f}_t/H) + (\sigma^2/2)\tau}{\sigma\sqrt{\tau}}, \qquad d_{2H} = d_{1H} - \sigma\sqrt{\tau}$$

Of course, this specification holds only when the futures price is not at the limits, because the hedge cannot be constructed when the price is beyond a limit. In that case, the futures price is simply considered to equal the limit price, though it must be recognized that it is not truly an equilibrium price.

Thus, if continuous trading is not permitted, the pricing equation can be expressed only in the general form of (3) or (4). Also, the existence of privately traded options, which must be valued by some unspecified process, are required here. With continuous trading, investors can form risk-free hedges, which leads to eq. (6), a specific version of the general pricing equation. Thus, the practical effect is that even though these private options are not required, one can proceed as if the options exist, are continuously traded, and are priced according to the Black model.

It is tempting to believe that the effects of upper and lower price limits is offsetting. Yet the model now indicates that the only way this can occur is if the put and call have equal values. It is well known that there is but a single underlying price that forces a put and call with equivalent characteristics to have the same price. Using Black's (1976) futures option pricing model, such a price can be found implicitly and is given as

$$\overline{f}_t = \frac{L[N(d_{2L}) - 1] - HN(d_{2H})}{N(d_{1L}) - N(d_{1H}) - 1} \tag{7}$$

from which the underlying spot price that satisfies this condition can be easily found. Thus, from among the infinite number of possible states of nature, there is one and only one in which the effect of price limits disappears.

## SOME IMPLICATIONS OF THE THEORETICAL MODEL

### Comparative Statics

This section discusses the implications of changing certain variables in the model. In the absence of price limits, it is simple to analyze the effects of changes in variables on the cost of carry futures price. The hedge ratio, the derivative with respect to the underlying asset price, is $e^{r(T-t)}$. The effect of time to expiration is $-tS_te^{r(T-t)}$, which is simply $-t$ times the futures price. The latter result means that as time progresses, the futures price declines smoothly toward the spot price. The derivative with respect to the interest rate is simply $rS_te^{r(T-t)}$, which is simply the interest rate times the futures price. This means that the futures price will increase with the interest rate, which is simply the cost of carry effect. There is no impact of volatility.

With price limits the effects are considerably more complex. The Appendix provides the mathematical details. The hedge ratio is now more than what it would be without limits. Limits have the effect of dampening futures price moves for a given spot price move. Thus, a hedge should require more futures for a given position in the spot asset. The effect of time to expiration is ambiguous. When the spot price is high (low), the futures price will be below (above) the cost of carry price and will rise (fall) as it moves through time. The effect of interest rates remains strictly positive, but now volatility has an impact on the futures price. When the spot price is high, futures prices will be near the upper limit. Then a change in volatility will affect the implicit short call more than the implicit long put. An increase in volatility will then reduce the futures price. The intuition here is straightforward. When the spot price is high, the futures is approaching the upper limit. Increased volatility makes it more likely that the upper limit will be reached soon. Since hitting the upper limit hurts holders of long futures, they will pay less for the contract. A similar argument can be made on the downside, where increased volatility makes the futures price higher, a result of the increased probability of hitting the lower limit.

Once price limits are added, additional variables can affect the futures price. Although limits are rarely changed, if they were changed, an increase in the limits would raise the futures price on the upside and lower it on the downside. This is because the negative (positive) effect of hitting the upper (lower) limit is less likely with wider limits. The derivative with respect to the settlement price is positive. The higher the settlement price, the higher the allowable trading range. The exchanges can also change the length of the settlement interval. The longer the settlement interval, the lower the futures price on the upside and the higher the futures price on the downside. The implication is that if, for example, the exchanges elected to go to two-day settlements, it would reduce high futures prices and raise low futures prices. The reason is that the options would have more value since their lives would be longer.

The effect of stochastic interest rates in a world without price limits is a topic of much interest and complexity that has received considerable attention in the literature. A complete development of a model of price limits under stochastic interest rates is deferred for future research. However, some intuitive insights can be easily discerned. Recall that when up (down) limit moves are more likely, price limits are disadvantageous (advantageous) and lead to a lower (higher) futures price. Now assume that futures prices are positively correlated with interest rates. When up (down) limit moves are more likely, the disadvantage (advantage) of an up (down) limit move is increased (decreased) by the fact that the holder of the long futures contract forgoes gains (avoids losses) in a higher (lower) interest rate environment. This would tend to lower the futures price. If the covariance between futures prices and interest rates is negative, the effect is the opposite.

To summarize these results, price limits can affect not only the way futures are priced, but the way they are hedged, their rate of time decay or growth, the effect of interest rates and the effect of volatility. In addition, the exchanges have the power to change certain variables that can further effect the futures price, although they rarely institute such changes.

## Behavior of Futures Prices

Price limits clearly affect the way futures behave. The comparative statics discussed in the previous subsection attest to this fact. An additional question of interest is how futures prices behave when near the limits. It is known that near the upper limit the greater moneyness of the implicit short call relative to the implicit long put causes the futures prices to be biased downward relative to the cost of carry price. As noted above, holding the spot price constant would result in the futures price rising with time toward the cost of carry price. This occurs as a result of the time decay in the implicit options. It is known that options lose value in a smooth, progressive manner. Thus, the futures price would tend to rise progressively toward the limit. When the contract is marked-to-market, if the spot price were sufficiently high such that the cost of carry price were above the upper limit, the futures price would converge to the upper limit. If the spot price were such that the cost of carry price were below the upper limit, the futures price would converge to the cost of carry price. A similar result occurs on the downside as the futures moves progressively downward either toward the cost of carry price or the lower limit.

As noted, these effects occur as a result of the time decay on the implicit options. Thus, the important result is that without price limits, futures contracts in which the cost of carry is positive always fall in price through time. With price limits, they may rise or fall in the short run according to their location relative to the price limits. Thus, price limits can act like a magnet.

Price limits can also affect the distribution of futures prices. If the spot price is lognormally distributed with time-homogeneous parameters, the futures price in the absence of price limits will do likewise. Once price limits are imposed, futures prices will follow a chi-square distribution with time-heterogenous parameters. The mathematical details are provided in the appendix. The time heterogeneity results from the fact that options themselves have time-heterogenuous parameters. The obvious implication is that traditional statistical tests, often based on normal probability theory and time-homogenuous parameters, may be quite misleading.

Without understanding the underlying economic effect of price limits as demonstrated in this model, it is tempting to view price limits as truncating gains and losses. Yet this model shows that price limits that are ultimately removed do not truncate gains and losses. Holders of futures contracts cannot avoid losses and do not forego gains by having price limits. This is because when a limit move has occurred, the holder of the futures cannot cover losses or remove gains and close out the position. Further gains and losses may occur when the limits are removed. Limits do, however, reduce both financing charges on losses and interest income on gains. By altering the timing of the stream of cash flows, they introduce path dependency into an otherwise perfect market problem. It should not be surprising that they can, indeed, affect the futures price. It is tempting to dismiss this effect as trivial. In a two-period world with short settlement intervals, it is almost surely a small amount. But there is a difference, however small, when futures prices move significantly up or down toward the limits. This suggests that in a multi-period world there might be a more significant effect.

## SOME ESTIMATES OF THE EFFECT OF PRICE LIMITS ON FUTURES PRICES

As noted in the second section, the theoretical price limits model is restricted to a two-period setting in which the limits are removed at the end of the period. Because the limits are removed in one day or less, the time premia on the imbedded options are small. If the model were extended, the cumulative effects could be larger.

Unfortunately, extending the theoretical model is a difficult, if not impossible, task. For example, consider a three-period world, consisting of days 1, 2, and 3, where day 3 is the expiration day and has no price limits. Suppose one is currently in day 1 and its limits are known. To hedge against limit moves on day 2, however, it is necessary to know the limits on day 2 so that appropriate exercise prices can be set. These limits will be unknown until the settlement price at the end of day 1 is determined. There are, in fact, an infinite number of pairs of possible price limits on day 2.

While a strict derivation of the risk-free hedge model is not possible, it is possible to obtain some estimates of the multi-period effects of price limits by invoking the Cox–Ross–Rubinstein (1979) (CRR) risk neutrality approach. This will make it possible to derive a numerical solution.[13]

## The Numerical Procedure

Numerical methods, which in finance are based on a finite difference approximation of the partial differential equation that describes the contingent claim price, have been used extensively in option pricing. They are seldom used in futures pricing primarily because they are rarely needed. If the futures price is given by the partial differential eq. (5), the terminal boundary condition, $S_T = f_T$, and there are no price limits, the futures price is simply the cost of carry eq. (2) and a direct solution is easily obtained. The assumptions of continuous trading and Itô price dynamics are sufficient but not necessary conditions for the cost of carry model to hold.

Given the existence of price limits, one can obtain a numerical solution using the price limits as bounds on the computed prices. This is similar to using a numerical approach to price an American put. The numerical procedure would generate a put price that would be replaced by the exercise value if the latter were higher. Here, the computed price is replaced with the upper (lower) limit if the computed price exceeds (falls below) the upper (lower) limits.

Numerical methods, however, solve the differential equation by starting at the expiration and working backward toward the present.

[13]Brennan and Schwartz (1978) show that numerical finite difference methods are equivalent to risk neutral valuations.

In keeping with current practices and to permit convergence of the futures and spot prices at expiration, the limits are removed the last 20 days. During that period a numerical solution should be equivalent to the cost of carry price, subject to some approximation error. On any day in which limits are in place, it is necessary to know the limits to price the futures. Current practices, however, specify that the limits are determined by adding and subtracting $\alpha$ to and from the previous day's settlement price. To operationalize this practice in a numerical procedure, would require information (the previous settlement price) that has not yet been derived. Numerical procedures do not specify the path the spot price follows. Rather, they approximate the solution to the differential equation at all possible combinations of the spot price and time to expiration. The practice of establishing the limit prices based on the settlement price imposes severe path dependencies. For example, the same price could be above, between, or below the limits depending on the path followed to get to that price. Since the limits are a function of the settlement price and the settlement price is a function of the limits, the problem is convoluted and a true solution based upon current institutional practices is impossible.

In spite of this limitation, it is worthwhile to question the effect price limits, in a more generic sense, have on futures prices. That is, some insights on the effects of price limits can be obtained without having to specify limits that evolve the same way they do in the real world. One could arbitrarily choose any different set of limits, but it would be advantageous to let the limits evolve according to a process that is generally consistent with their actual process.

Given that the numerical approach to pricing contingent claims is consistent with the risk neutral approach of Cox, Ross, and Rubinstein (1979), it seems reasonable to let the limits be a function of an expected price each day in the futures' life. The limits cannot be a specific function of the futures price for that would require knowledge that would be sufficient to give the futures price and obviate this procedure. An alternative would be to tie the limits to some variable that behaves like the futures price, such as the spot price. Its process is well defined and its expected value at $T$ is $S_t e^{(\mu+\sigma^2/2)(T-t)}$ where $\mu$ is the drift of the log return. Thus, one can trace the path of the expected spot price. A given day's limits are then defined as the expected closing spot price of the previous day, plus or minus $\alpha$. To be consistent with CRR risk neutral approach, the expected spot return is set to the risk-free rate. This procedure will impose a positive drift on the range of acceptable prices and its expected evolution will generally behave like the expected evolution of the futures price, and thus, like price limits in reality. While this approach will not perfectly replicate the behavior of price limits, it should serve as a reasonable proxy.

## The Results

The bias, defined as the futures price under price limits minus the cost of carry price, is calculated for a 60-day futures contract, approximately half-way through the first day.[14] The price limits are 97 and 103, which are based on the assumption that the spot price closed the previous day at 100.[15]Standard deviations of 10, 20, and 30% are chosen. These span most of the range of volatilities obtained by Fama and French (1987) in their study of commodity futures prices. Under the assumption that the spot price drifts at the risk-free rate, the probability of a minimum move from 100 at the end of the previous day to various prices around 100 in one-half day is calculated. These probabilities are then multiplied by 250, the approximate number of trading days in a year, to obtain an estimate of the frequency of a bias of a given magnitude or greater in an absolute sense.

Table II presents the results for three assumptions about the risk-free rate with a 20% volatility. Consider a spot price of 98 and a risk-free rate of 0.06. The bias is 0.98 and the expected frequency is 3. This means that a bias of at least 0.98, observed in the middle of the trading day, would be expected to occur fewer than three times a year, under these assumptions. Clearly the larger biases occur quite rarely, yet biases of up to 1% can occur quite often. The effect of the risk-free rate on the frequency of a given bias is somewhat small. However, the assumption that the spot price drifts with the risk-free rate means that when a higher risk-free rate is assumed, the bias for spot price increases should be greater in an absolute sense and that is indeed the case.

Table III presents the results for three assumptions about the volatility, under the assumption of a 6% risk-free rate. Differences in volatility produce rather large differences in the magnitude and frequency of the bias. Greater volatility obviously causes more frequent occurrences of bias and also increases the magnitude. Consider the

[14]Before imposing price limits on the computational procedure, a sample case is tested to determine if the numerical procedure gives the cost of carry price, which it does to the third decimal place.
[15]Obviously the choice of the limit size does affect the magnitude of the bias. Three-point limits based on a previous closing spot price of 100 are chosen somewhat arbitrarily but they are reasonable for instruments like the widely traded Treasury bond futures. The implications of changing the limit sizes are discussed later.

**TABLE II**
Estimates of Price Limit Bias Under Alternative Risk-Free Rates

| Spot Price | r = 0.04 | | r = 0.06 | | r = 0.08 | |
|---|---|---|---|---|---|---|
| | *Bias* | *Frequency* | *Bias* | *Frequency* | *Bias* | *Frequency* |
| 103.0 | −2.55 | <1 | −2.71 | <1 | −2.90 | <1 |
| 102.5 | −2.17 | <1 | −2.34 | <1 | −2.51 | <1 |
| 102.0 | −1.80 | 3 | −1.96 | 3 | −2.12 | 3 |
| 101.5 | −1.43 | 12 | −1.58 | 12 | −1.74 | 12 |
| 101.0 | −1.07 | 33 | −1.21 | 34 | −1.36 | 34 |
| 100.5 | −0.70 | 73 | −0.84 | 73 | −0.99 | 73 |
| 100.0 | −0.34 | 125 | −0.48 | 124 | −0.62 | 124 |
| 99.5 | 0.02 | 72 | −0.11 | 71 | −0.25 | 71 |
| 99.0 | 0.38 | 32 | 0.25 | 32 | 0.12 | 32 |
| 98.5 | 0.74 | 11 | 0.62 | 11 | 0.49 | 11 |
| 98.0 | 1.10 | 3 | 0.98 | 3 | 0.86 | 3 |
| 97.5 | 1.46 | <1 | 1.35 | <1 | 1.23 | <1 |
| 97.0 | 1.82 | <1 | 1.71 | <1 | 1.60 | <1 |

Note: The bias is the difference between the futures price under price limits and the cost of carry price. The frequency is the number of days on which a bias of the given magnitude or greater in an absolute sense would occur during a year. For example, when $r = 0.04$, a bias of −1.07 or greater would be expected to occur on 33 trading days during a year. The spot price indicated is the spot price that would induce such a bias. The calculations are half-way through the first trading day of a futures contract with a 60-day life. The price limits are 97 and 103, the risk-free rate is in discrete form, and the standard deviation is .20. The cost of carry price and the futures price under price limits are computed using a numerical finite difference procedure with 150 spot price steps and 15 time steps per day.

move of the spot price from 100 to 100.5 in one-half day. Under a 10% volatility, a bias of −0.56 or greater (in an absolute sense) would be expected to occur 34 times during a year. Under a 20% volatility, a bias of −0.84 or greater (in an absolute sense) would be expected to occur 73 times during a year. Under a 30% volatility, a bias of −0.98 or greater (in an absolute sense) would be expected to occur 89 times during a year.

It should be noted that the results shown in these tables are open-ended. The range of spot prices shown (97–103) is arbitrary. There are positive probabilities of even greater biases but those probabilities are extremely small. Biases of 2–3% are difficult to accept. Such biases should indeed be viewed with skepticism. As the table indicates, biases of that magnitude are extremely rare events. Nonetheless, smaller but not insignificant biases are common.

Additional tests are conducted to examine the effects of differences in the limit sizes as well as maturity of the contract. The results are simply summarized here. When the limits are widened to four points, the bias is slightly reduced. For example, with the interest rate at 6% and the volatility at 20%, a spot price move to 101 or higher, which would

**TABLE III**
Estimates of Price Limit Bias Under Alternative Volatilities

| Spot Price | $\sigma = 0.10$ Bias | $\sigma = 0.10$ Frequency | $\sigma = 0.20$ Bias | $\sigma = 0.20$ Frequency | $\sigma = 0.30$ Bias | $\sigma = 0.30$ Frequency |
|---|---|---|---|---|---|---|
| 103.0 | −2.00 | <1 | −2.71 | <1 | −3.06 | 3 |
| 102.5 | −1.68 | <1 | −2.34 | <1 | −2.64 | 8 |
| 102.0 | −1.37 | <1 | −1.96 | 3 | −2.23 | 18 |
| 101.5 | −1.08 | <1 | −1.58 | 12 | −1.81 | 34 |
| 101.0 | −0.81 | 3 | −1.21 | 34 | −1.40 | 57 |
| 100.5 | −0.56 | 34 | −0.84 | 73 | −0.98 | 89 |
| 100.0 | −0.31 | 123 | −0.48 | 124 | −0.57 | 125 |
| 99.5 | −0.07 | 32 | −0.11 | 71 | −0.16 | 88 |
| 99.0 | 0.17 | 3 | 0.25 | 32 | 0.25 | 57 |
| 98.5 | 0.41 | <1 | 0.62 | 11 | 0.66 | 33 |
| 98.0 | 0.65 | <1 | 0.98 | 3 | 1.07 | 16 |
| 97.5 | 0.90 | <1 | 1.35 | <1 | 1.48 | 7 |
| 97.0 | 1.17 | <1 | 1.71 | <1 | 1.89 | 3 |

Note: The bias is the difference between the futures price under price limits and the cost of carry price. The frequency is the number of days on which a bias of the given magnitude or greater in an absolute sense would occur during a year. For example, when $\sigma = 0.10$, a bias of $-0.56$ or greater would be expected to occur on 34 trading days during a year. The spot price indicated is the spot price that would induce such a bias. The calculations are half-way through the first trading day of a futures contract with a 60-day life. The price limits are 97 and 103, the risk-free rate of 0.06 is in discrete form. The cost of carry price and the futures price under price limits are computed using a numerical finite difference procedure with 150 spot price steps and 15 time steps per day.

occur about 33 times a year, would impose a bias of $-1.34$ if the limits were two points, $-1.21$ if the limits were 3 points, and $-1.09$ if the limits were 4 points. These differences are not large when one considers that the limits are contracted or expanded by a third. A longer contract has a greater bias for almost all spot prices. For example, a spot price move to 101 shows a bias of $-1.33$ if the contract has 80 days to go, $-1.21$ if the contract has 60 days to go, and $-0.97$ if the contract has 40 days to go. At prices in the range of 98–99, the biases are essentially the same but then at lower prices, the biases are again larger the longer the time to expiration. Thus, price limit effects do generally accumulate over the life of the contract.

In conclusion, these model cannot tell precisely how price limits (as imposed in the real world) have an impact. The imposition of price limits seems to have an effect that appears to be on the order of transaction costs.

## CONCLUSIONS AND IMPLICATIONS

This study attempts to model the impact of price limits on the cost of carry futures pricing model. Prior research generally inferred that price limits probably have an insignificant effect on the futures price. The advantage of down limit moves should be offset by the disadvantage of up limit moves. This model shows that this way of thinking about price limits is incorrect on two counts.

First, as long as the limits are ultimately removed, price limits do not truncate gains and losses. Rather, they induce a path dependency in the stream of mark-to-market cash flows. Up limit moves reduce the interest gained when long futures accounts are marked-to-market. Down limit moves reduce the interest cost of financing losses incurred when long futures positions are marked-to-market. These interest gains and losses can be modeled as options that reflect the relative probabilities of up and down limit moves. The cumulative effects of price limits over a long-lived futures contract cannot be modeled precisely, but estimates reveal that they would often have an effect of up to 1% of the futures price. Thus, the reason why price limits affect futures has nothing to do with reducing gains and losses; it deals strictly with the interest on the marking-to-market.

Second, even those effects should not be viewed as offsetting. From among the infinite number of states of nature, there is one and only one in which the effects of up and down limit moves offset. Certainly there are many states in which the effects are small, but as shown here, there is a significant probability that price limits will at times have an effect on the order of transaction costs.

There are a number of implications of these results. The most obvious one is that if futures prices are estimated using the cost of carry model, errors may be induced. These errors could result in wasteful arbitrage strategies. An exact multi-period formula is, however, impossible to obtain. Thus, what should a trader do? It seems apparent that one should at least consider that expectations of a strong bull or bear market should be tempered by the effect of price limits, which will cut into mark-to-market gains and reduce mark-to-market losses for holders of long futures. In addition, traders should consider that traditional hedge ratios may not work as well in the presence of price limits. Futures contracts that have negative carry, and, thus, should fall through time, may exhibit the perverse tendency to rise in the short run even when the spot price is not changing.

For students of the markets who conduct empirical research, price limits are shown to have a potentially significant impact on the distribution of futures price changes. They transform what should be a lognormal distribution to a chi-square. In addition, they turn what should be stable parameters into time-varying parameters.

For regulators and exchanges, the implication of these results is that price limits, imposed apparently to reduce the probability of default (among other reasons) have a cost—the cost of price distortion. Although the benefits may ultimately prove to be worth the cost, it is, nonetheless, troublesome that a multi-period pricing model that properly accounts for the effect of price limits cannot be obtained. As noted earlier, that result is because price limits are endogenized in a circular manner. Today's price determines the price limits, which turn around and affect today's price. Exchanges and regulators should study the effect of alternative price limit schemes such as tying the limits to the spot price, asymmetric price limits, multiplicative price limits, or announcing the entire range of price limits that will be in effect throughout the life of the futures. While there is no guarantee that any of these methods would lead to exact or even simpler pricing models, it seems obvious that alternative price limit systems should be studied.

## APPENDIX

### Comparative Statics

*Spot Price*

$$\partial f_t/\partial S_t = e^{r(T-t)}(K + (1 - K)[N(d_{1L}) - N(d_{1H})]) \geq 0 \qquad \text{(A1)}$$

Because the bracketed term ranges from zero to one and $K$ is less than one, the hedge ratio is less than or equal to $e^{r(T-t)}$, the hedge ratio without price limits.

*Volatility*

Of course, volatility has no effect in the absence of price limits, but now the effect is

$$\partial f_t/\partial \sigma = (1 - K)\overline{f}_t\sqrt{\tau}[N'(d_{1L}) - N'(d_{1H})] \gtreqless 0 \qquad \text{(A2)}$$

The sign can be determined by letting the spot price be large or small. With the spot price large (small), the sign is negative (positive).

*Time to Expiration*

$$\begin{aligned}\partial f_t/\partial t = &-((1 - K)(\overline{f}_t\sigma/2\sqrt{\tau})[N'(d_{1L}) - N'(d_{1H})] \\ &+ rK\overline{f}_t + r\overline{f}_t[N(d_{1L}) - N(d_{1H})]) \gtreqless 0 \qquad \text{(A3)}\end{aligned}$$

Because the first bracketed term can be positive or negative, it is difficult to unambiguously determine the sign of the expression. However, when the spot price is high (low), the bias is negative (positive) and the futures price will tend to rise (fall) as it moves through time.

*Interest Rate*

$$\partial f_t/\partial r = \overline{f}_t K(T - t - i) + \overline{f}_t[N(d_{1L}) - N(d_{1H})][(1 - K)(T - t) + iK] - iK[LN(d_{2L}) - HN(d_{2H})] + LiK > 0 \tag{A4}$$

Since the cost of carry futures price is a component of the futures price under price limits, an increase in the interest rate will tend to increase the futures price under price limits. However, it also affects the imbedded options by raising the call price and lowering the put price. The net effect can be shown to lead to an overall positive sign.

*Limit Size*

We assume symmetric limits such that $\alpha_L = \alpha_H = \alpha$.

$$\partial f_t/\partial \alpha = (1 - K)[N(d_{2L}) + N(d_{2H}) - 1] \gtreqless 0 \tag{A5}$$

When the spot price high (low), wider limits reduce (increase) the bias by raising (lowering) the futures price.

*Previous Settlement Price*

$$\partial f_t/\partial f^S_{T-i} = (1 - K)[1 - N(d_{2L}) + N(d_{2H})] > 0 \tag{A6}$$

The higher the previous settlement price, the wider the allowable trading range.

*Length of the Settlement Interval*

$$\partial f_t/\partial i = (1 - q)((1 - K)(\overline{f}_t\sigma/2\sqrt{\tau})[N'(d_{1L}) - N'(d_{1H})] + rK\overline{f}_t + r\overline{f}_t[N(d_{1L}) - N(d_{1H})]) \gtreqless 0 \tag{A7}$$

This sign is difficult to determine but calculations of sample values reveal that at high (low) spot prices, a longer settlement interval lowers (raises) the futures price.

## The Distribution of Returns on Futures

This section examines the futures percentage price change, referred to somewhat loosely as the "return." Assume that the spot price follows the discrete stochastic process,

$$\frac{\Delta S_t}{S_t} = \mu \Delta t + \sigma \Delta Z$$

where $\Delta Z = \epsilon\sqrt{\Delta t}$ and $\epsilon \sim N(0, 1)$. Now consider the process followed by the cost of carry futures price. Expanding it by a Taylor series, ignoring higher powers of $\Delta t$, and substituting for $\Delta S_t^2$ and $\Delta S_t$, one obtains

$$\frac{\Delta \overline{f}_t}{\overline{f}_t} = (\mu - r)\Delta t + \sigma \Delta Z$$

a result that is already well known. The return on the futures is log-normally distributed with $E(\Delta \overline{f}_t)/\overline{f}_t = (\mu - r)\Delta t$ and $\mathrm{Var}(\Delta \overline{f}_t/\overline{f}_t) = \sigma^2 \ \mathrm{Var}(\Delta Z^2)$. In the limiting case where $\Delta t \to 0$, it is well known that $\Delta Z^2 \to dt$ and the variance is $\sigma^2 dt$.

To evaluate the actual futures price process, we reconsider eq. (4) in slightly simplified notation,

$$f_t = K\overline{f}_t + (1 - K)(C_L - C_H)e^{r\tau} + L(1 - K) \qquad \text{(A8)}$$

where $C_L = C(\overline{f}_t, L_{T-i}, \tau)$ and $C_H = C(\overline{f}_t, H_{T-i}, \tau)$, a substitution made possible by appealing to put–call–futures parity. Expanding (A8) with a Taylor series gives

$$\Delta f_t = K\Delta\overline{f}_t + (1 - K)e^{r\tau}\Bigg[ (\Delta\overline{f}_t\left(\frac{\partial C_L}{\partial \overline{f}_t} - \frac{\partial C_H}{\partial \overline{f}_t}\right) + \Delta t\left(\frac{\partial C_L}{\partial t} - \frac{\partial C_H}{\partial t}\right)$$
$$+ \frac{1}{2}\sigma^2\overline{f}_t^2\Delta Z^2\left(\frac{\partial^2 C_L}{\partial \overline{f}_t^2} - \frac{\partial^2 C_H}{\partial \overline{f}_t^2}\right) + O_L(\Delta^{3/2}) + O_H(\Delta t^{3/2})\Bigg] \qquad \text{(A9)}$$

The comparative statics of the Black model can be substituted into (A9) to give

$$\Delta f_t = \Delta \overline{f}_t[K + (1 - K)(N(d_{1L}) - N(d_{1H}))] - \Delta t(1 - K)\overline{f}_t\sigma \times [N'(d_{1L}) - N'(d_{1H})]/(2\sqrt{\tau}) - \Delta t(1 - K)r[LN(d_{2L}) - HN(d_{2H})] + \sigma(1 - K)\overline{f}_t\Delta Z^2[N'(d_{1L}) - N'(d_{1H})]/(2\sqrt{\tau}) + e^{r\tau}(1 - K) \times [O_L(\Delta t^{3/2}) + O_H(\Delta t^{3/2})] \quad \text{(A10)}$$

Define the following values: $\delta_t = N(d_{1L}) - N(d_{1H})$, $\delta_t' = N'(d_{1L}) - N'(d_{1H})$, $\lambda_t = LN(d_{2L}) - HN(d_{2H})$, and $k_t = (1/2)\tau^{(1/2)}$ where the $t$ subscripts indicate that these values change with time. By ignoring higher order powers of $\Delta t$, (A10) can be written as

$$\Delta f_t/f_t = [(\overline{f}_t\mu\delta_t - (1 - K)(\overline{f}_t k_t\sigma\delta_t' + r\lambda_t))\Delta t + (1 - K)\overline{f}_t\sigma\delta_t\Delta Z + (1 - K)\overline{f}_t k_t\sigma\delta_t'\Delta Z^2]/f_t \quad \text{(A11)}$$

In discrete time (A11) can be written as $\Delta f_t/f_t = \theta_0 + \theta_1\Delta Z + \theta_2\Delta Z^2$, which is equivalent to $\Delta f_t/f_t = (\theta_0 - \theta_2\theta^2) + \theta_2(\Delta Z + \theta)^2$ where the $\theta$'s represent known values. Since $\Delta Z + \theta$ is normally distributed, the discrete return $\Delta f_t/f_t$ is distributed as a chi-square with one degree of freedom. In the limit $\Delta Z \to 0$ and $\Delta Z^2 \to \Delta t \to dt$. Then (A11) becomes

$$df_t/f_t = [(\overline{f}_t\mu\delta_t - (1 - K)r\lambda_t)dt + (1 - K)\overline{f}_t\sigma\delta_t dZ]/f_t \quad \text{(A12)}$$

This process is still Geometric Brownian motion, with instantaneous expected return and variance of $E(\Delta f_t/f_t) = (a_t/f_t)dt$ and $\text{Var}(\Delta f_t/f_t) = (b_t^2/f_t^2)dt$ where $a_t$ and $b_t$ are defined from (A12).

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