A Nonstationary Trinomial Model for the Valuation of Options on Treasury Bond Futures Contracts

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INTRODUCTION

This article illustrates the pricing of one of the most complex of modern derivative securities, options on Treasury Bond futures contracts. The futures contract—the asset which directly determines the option's

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intrinsic value—is itself a derivative asset whose value depends on not one, but several, potentially deliverable bonds. Each deliverable bond's value is in turn dependent on an underlying interest rate process and may contain additional embedded options. Lastly, there is a delivery interval rather than a single fixed date (as in European options) or continuous delivery (as in American options).

This article utilizes the Bliss and Ronn (1989) interest rate process to illustrate the details of the process of employing a discrete-time model to evaluate options on Treasury Bond futures and to derive and test a trading rule based on the implied theoretical values. The Bliss–Ronn model is selected because it explicitly incorporates time-varying parameters. There is considerable casual evidence, as well as statistical evidence in Bliss and Ronn (1989), suggesting that important characteristics of the interest rate process change through time. Since the empirical application in this article uses an extremely coarse time grid, the trinomial nature of the Bliss–Ronn model is an added advantage relative to a binomial grid.

As in all models of financial economics, the validity of a model is examined not by the applicability of each of the model's assumptions, but rather by the empirical reliability of the model's predictions. Thus, model values are subjected to two tests. In the first, the model is tested to see whether it gives rise to unbiased predictors of market prices. In the second, "paper trading" is used to test whether the model is able to detect arbitrage opportunities in existing market prices.

Valuation of Treasury Bond futures options begins with an assumed model of the underlying interest rate process. Arbitrage arguments then lead to bounds on the values of futures contracts and related options. The interest rate process used to derive the valuation process is something of a matter of taste. For instance, Heath, Jarrow, and Morton (1990) used the discrete-time version of their model.[1] Continuous-time models, pioneered by Cox, Ingersoll, and Ross (CIR) (1985), include Vasicek (1977), Courtadon (1982), Jamshidian (1988), Hull and White (1990), Brennan and Schwartz (1979), and Longstaff and Schwartz (1992), amongst many others. Discrete models include Ho and Lee (1986), Black, Derman, and Toy (1990), and Bliss and Ronn (1989).

Heath, Jarrow, and Morton (1990, 1992a) produced discrete- and continuous-time variations for a class of models and demonstrated the

[1] Unfortunately HJM omitted many of the necessary details of the futures and options valuation process, such as the precise timing sequence of the delivery process.

conditions under which the continuous-time model is a limiting case for a particular discrete-time model.

The advantage of discrete-time models is that they can handle any desired degree of complexity in the derivative security. The disadvantage of these models is that they can be computationally intensive, especially in the case of non-recombining paths, as the refinement of the tree produces an increasing number of nodes to be evaluated. Fortunately, Heath, Jarrow, and Morton (1992b) argued that reasonably useful results can be obtained with a relatively coarse grid.[2] In the academic finance literature, continuous time models appear more frequently, in part because the stylized securities being addressed are amenable to analytic solutions. In practice, however, actual derivative securities are frequently too complex for continuous-time methods, and practitioners, of necessity, prefer answers over elegance.

This article attempts to contribute to the literature by presenting a rigorous theoretical and empirical application of an interest rate model to the pricing of these contracts, as well as outlining the appropriate empirical "hurdles" to which such models should be subjected.[3]

The remainder of the article is organized as follows: It outlines the valuation of the options on the T-Bond futures contract in the context of the Bliss–Ronn model. The valuation is in the form of no-arbitrage bounds that arise from the nature of the contracts' many implicit options and the projected state-dependent values of the underlying deliverable bonds. It implements empirical tests of the set of no-arbitrage conditions applicable to the Treasury Bonds futures option contract. In the final analysis, the question of whether it is the market price of the model value that represents the "fair value" of the option requires an examination of the possible existence of trading opportunities based on the options' model values.

[2] They also noted the empirical advantages of discrete time formulations, stating that "... when computing values for interest rate dependent contingent with early exercise provisions, ..., there are no simple closed form solutions for these financial securities."

[3] Regarding alternative valuation models used by academics or practitioners, the authors are aware of only one and that one considers the pricing of the futures contracts, not the options written on these contracts. Thus, Laraqui (1991) makes the assumption that "bond forward prices are jointly lognormally distributed" (p. 10). Using his model, he reports a reduction in the average prediction errors and the associated standard deviation on the order of 50% and 40%, respectively, relative to a naive implied repo rate (IRR) model which assumes "deliverable-bond-yields and the futures contract IRR shift in parallel."

VALUATION OF OPTIONS ON TREASURY BOND FUTURES CONTRACTS

Sequence of Events for Option Valuation

This section provides for the valuation of call and put options on Treasury Bond futures contracts. Table I presents the time sequence of events for the valuation of the options on Treasury Bond futures contracts. As an example, consider the July 31st valuation date for the options contract on a September T-Bond futures contract.

Valuation Process

In valuating the Treasury Bond futures contract, it is important to explicitly account for the "quality option," involving the choice amongst deliverable bonds, and the "accrued interest option" which permits delivery of the bond throughout the delivery month. The Treasury Bond futures and associated options are modeled under the assumption of two alternative delivery dates for the Treasury Bond: beginning and end of the delivery month.[4]

The following procedure is now utilized to compute the value of the T-Bond futures contract:

Step 1:

A total of 13 term structures of interest rates, each evaluated at maturities $m = 1, 2, \ldots, 30$ years, are generated. These include the time t_0 term structure, the three possible term structures $\{u, n, d\}$ at time t_1, and the 9 possible term structures $\{uu, un, ud\}$, $\{nu, nn, nd\}$, $\{du, dn, dd\}$ at time t_2. Define the term structures of interest rates by the relevant prices of m-period maturity pure discount bonds. $P^0(m)$, $P^s(m)$, and $P^{sq}(m)$. Whereas the $P^0(m)$ values are extracted from the

[4]While it is true that futures can be delivered against anytime during the month, an examination of the delivery months over the period of the T-Bond options' existence—from December 1982 to the end of the sample period—all optimal deliveries are at month's end, time t_2. Thus the model's prediction of this pattern of optimal delivery times provides theoretical support for the work of Hemler (1990), who finds a similar pattern in the actual deliveries against futures contracts in the pre- and post-1982 period.

Further, as noted by Duffie (1989):

> [I]f the optimal bond to deliver has a coupon rate much higher than short-term interest rates, it may be valuable to delay delivery as long as possible, since the accrued interest in the invoice price is accumulating faster than the interest foregone on the cash that would otherwise be received for a delivered bond, other things being equal.

Clearly, if the cheapest-to-deliver bond has a coupon rate significantly lower than the prevailing short term rate of interest, optimal delivery may occur at the beginning of the delivery month.

TABLE I

Sequence of Events for Option Valuation

t_0	t_1	t_2	t_3
July 31	*Aug. 22*	*Sep. 1*	*Sep. 30*

Date	*Value of t*	*Comments*
July 31st	t_0	Date of option valuation
August 22nd	t_1	Option expires: three possible states of nature, $\{u, n, d\}$
September 1st	t_2	First date for delivery against futures contract: same three states of nature as on August 22nd, $\{u, n, d\}$
September 30th	t_3	Second (and final) date for delivery against futures contract. Each state of nature at time t_2 has spawned three possible states of nature, for a total of 9 states: $\{uu, un, ud\}$, $\{nu, nn, nd\}$, $\{du, dn, dd\}$

Note for t_1 (August 22nd): In accordance with the specification of the Chicago Board of Trade, "options stop trading . . . on the last Friday preceding, by a least five business days, the first notice day for the corresponding T-Bond futures contract."

observable prices of bonds and notes at time t_0, the $P^s(m)$ at time t_2 and $P^{sq}(m)$ at time t_3 must be generated using:

1. The values of $P^0(m)$;
2. The values of the state variables implicit in $P^0(m)$; and
3. The estimated, regression-derived, perturbation functions h^s_{tm}.

In addition to the future state-specific term structures, the availability of the perturbation functions permits one to calculate the trinomial risk-neutral probabilities. Thus, using the methodology of Appendix A, one can compute $\{\pi^s_{t1}\}$ for each of the three states at time t_1 and t_2, as well as $\{\pi^{sq}_{t3}\}$ for each of the nine states at time t_3.

Step 2:

Deliverable bonds are those U.S. Treasury Bonds, maturing at least 15 years from delivery data of the futures contract if not callable, and, if callable, not so for at least 15 years from delivery date. The CRSP Government Bonds File is scanned for the set of all bonds satisfying this requirement at date t_0. It is inappropriate to include deliverable bonds which may appear between t_0 and the maturity of the futures contract at t_3 as these issues might not be in the information set at the time the option is evaluated, t_0.

Deliverable bonds include issues such as callable and flower bonds which contain special provisions impacting on their valuation. The future, state-contingent, prices of these bonds cannot be determined solely by discounting their cash flows under the projected, state-contingent, term structures. Even noncallable, nonflower bonds are potentially affected by idiosyncratic liquidity and tax effects as shown in Bliss (1993). In estimating the prices of bonds under projected future term structures, the present value of the cash flows is therefore adjusted by an additive factor designed to capture these affects. This factor is defined to be the spread, observed at time t_0, between the quoted asked price and the present value of the cash flows under the t_0 term structure.

Specifically, let the jth bond's observable market price be $B_j(t_0)$ and calculate the present values of each bond's cash flows as given by $\mathrm{CF}_j(t_0) \equiv \sum_{t=0}^{m_j} c_j P^0(t) + P^0(m_j)$.[5] The difference $e_j(t_0) \equiv B_j(t_0) - \mathrm{CF}_j(t_0)$ is an adjustment factor attributable to tax or liquidity effects, or flower and call provisions.

Step 3:

Proceeding via the method of stochastic dynamic programming, consider the value of the futures contract if delivery takes place at time t_3. Given the state-dependent term structure $P^{sq}(m)$, it is possible to compute the cheapest-to-deliver bond. For each bond at t_3, first calculate the fitted $\mathrm{CF}_j^{sq}(t_3)$, and then "forecast" its market price as $\hat{B}_j^{sq}(t_3) = \mathrm{CF}_j^{sq}(t_3) + e_j(t_0)$.[6] Let $I_j(t_3)$ represent the accrued interest on bond j at time t_3, and C_j the conversion factor of the bond for purposes of delivery against a Treasury Bond futures contract.[7]

Then the value of the futures contract in each of the nine possible terminal states at time t_3 is given by

$$F^{sq}(t_3) = \min_j \frac{\hat{B}_j^{sq}(t_3) - I_j(t_3)}{C_j(t_3)}$$

Step 4:

Now consider the valuation of the futures contract at each of the three states at time t_2: $F^s(t_2)$, $s \in \{u, n, d\}$. In the absence of the early delivery option, in each state, the futures contract price would equal the expected value, under the trinomial risk-neutral valuation, of the cheapest-to-deliver bond at time t_3.

$$F^s(t_2) = \sum_{q \in \{u,n,d\}} \pi^{sq} F^{sq}(t_3) \tag{1}$$

[5] For callable bonds, the usual convention is adopted: if $B_j > 100$, m_j is set to the first call date, and CF_j and e_j calculated accordingly; conversely, if $B_j < 100$, m_j is set to the maturity date.

[6] This assumes that the tax/liquidity effect is a random walk: $e_{jt} = e_{j,t-1} + \eta_{jt}$. In testing this hypothesis, such an adjustment factor succeeds in reducing significantly the deviation of the predicted price $\mathrm{CF}_j^{sq} + e_{j,t-1}$ from the observed market price, B_{jt}. For a detailed analysis of these empirical results, see Appendix B.

[7] As is well known, the conversion factor for a given issue with coupon rate c_j and maturity date m_j is the price per dollar of face value of a bond yielding 8% compounded semi-annually with coupon rate c_j and maturity date m_j.

where $\sum_{q \in \{u,n,d\}} \pi^{sq} F^{sq}(t_3)$ is not discounted since a futures contract does not require any outlay of cash at time of initiation.

The early delivery option, however creates an alternate valuation bound. Specifically, the futures price must recognize the buy-and-deliver arbitrage strategy; hence, the futures price must be no greater than the adjusted price of the then-current cheapest-to-deliver bond:

$$F^s(t_2) \leq \min_i \frac{\hat{B}_i^s(t_2) - I_i(t_2)}{C_i(t_2)} \tag{2}$$

Further, the futures contract price at time t_2 is also constrained by another arbitrage strategy, namely, a policy of buy-now, deliver-at-month's-end. This yields the following inequality constraint:

$$F^s(t_2) \leq \min_i \left\{ \frac{\hat{B}_i^s(t_2)[1 + r_1^s(t_2)] - I_i(t_3)}{C_i(t_3)} \right\} \tag{3}$$

where $r_1^s(t_2)$ is the time t_2 state-dependent deannualized interest rate on a one-month maturity instrument. Combining the constraints implied in eqs. (1)–(3), the futures price at time t_2 must satisfy

$$F^s(t_2) = \min \left\{ \sum_{q \in \{u,n,d\}} \pi^{sq} F^{sq}(t_3), \min_i \frac{\hat{B}_i^s(t_2) - I_i(t_2)}{C_i(t_2)}, \right.$$

$$\left. \min_i \frac{\hat{B}_i^s(t_2)(1 + r_1^s) - I_i(t_3)}{C_i(t_3)} \right\} \tag{4}$$

Step 5:

The current time-t_0 value of the futures contract, $F(t_0)$, is given by the undiscounted, risk-neutral probability weighted value of the futures contract at the end of the current month, time t_2:

$$F(t_0) = \sum_{s \in \{u,n,d\}} \pi^s F^s(t_2) \tag{5}$$

As was the case for the state-contingent futures prices $F^s(t_2)$, the time t_0 futures price must satisfy the no-arbitrage condition that this price be no greater than the minimum value of (1) buy-now, deliver at beginning of delivery month; and (2) buy-now, and deliver at end of delivery month investment strategies. Thus,

$$F(t_0) \leq \min \left\{ \min_i \frac{B_i(t_0)[1 + r_1(t_0)] - I_i(t_2)}{C_i(t_2)}, \right.$$

$$\left. \min_i \frac{B_i(t_0)[1 + r_2(t_0)] - I_i(t_3)}{C_i(t_3)} \right\} \tag{6}$$

where $r_m(t_0)$ are the deannualized interest rates for month m at t_0.

Note that while relations (1) and (5) are driven by the model, the inequalities (4) and (6) follow from additional no-arbitrage conditions outside the framework of the model. An examination of whether the futures model values satisfy these two inequalities thus becomes an additional implicit test of the model.

Step 6:

It is necessary to make one important modification to the futures contract's model values. From the discussion in Bliss and Ronn (1989) and Appendix A it follows that all interest rate-contingent assets must satisfy the no-arbitrage condition

$$P_0(m+1) = P_0(1)\sum_s \pi^s P_1^s(m)$$

In words, an asset's fair value equals its discounted expected value, using the risk-neutral probabilities.

The $F^s(t_2)$ are adjusted to $\epsilon F^s(t_2)$ to satisfy the analogous no-arbitrage constraint (which accounts for the costless initiation of the futures contract) $\sum_s \pi^s \epsilon F^s(t_2) = f(t_0)$. This result is accomplished by setting $\epsilon = f(t_0)/\sum_s \pi^s F^s(t_2)$. This procedure is designed to test the proposition that the option price is correctly priced relative to the futures contract price, even if the futures itself is incorrectly priced.

By assumption, the realized state of nature at time t_1 is the same one realized at time t_2; hence, $F^s(t_2) = F^s(t_1)$ for all s. The payoffs to a call option in state s are $\max\{0, F^s(t_1) - K\}$, and the value of an American call at time t_0 is equal to the maximum of zero, the intrinsic value, and the discounted expected (using the risk-neutral probability distribution) value of the payoffs at time t_1:

$$C^0(K) = \max\left\{0, f(t_0) - K, \frac{\sum_{s\in\{u,m,d\}} \pi^s \max\{0, F^s(t_1) - K\}}{1 + r(t_0)}\right\} \tag{7}$$

where $f(t_0)$ is the futures contract's market price and $r(t_0)$ is the deannualized three-week interest rate.

Analogous formulas, using $\max\{0, K - f(t_0)\}$ and $\max\{0, K - F^s(t_1)\}$, obtain for the put options.

EMPIRICAL RESULTS—OPTIONS ON T-BOND FUTURES

A natural procedure for testing the validity of the option pricing model presented herein is to examine the biasedness of the model value against the observed market price: thus, for the call options, one might examine

$$C_{jt}(K) = \alpha + \beta V_{jt}(K) + \epsilon_{jt}$$

where $C_{jt}(K)$ is the market price, $V_{jt}(K)$ the model value, for a K exercise price option at time t, and ϵ_{jt} is a random prediction error. A regression framework could then be used to test the null hypothesis, H_0: $\alpha = 0$, $\beta = 1$. Such an obvious test unfortunately lacks power, as the (observable) intrinsic value $\max\{0, f(t_0) - K\}$ is common to both $C_{jt}(K)$ and $V_{jt}(K)$, and will yield a high R^2 even if the model had no ability to predict the option's *time value*.

A more rigorous and economically meaningful test of the model is to verify whether the model is able to explain the time value remaining in the call option: $c_{jt}(K) \equiv C_{jt}(K) - \max\{0, f_t(t_0) - K\}$, where $f_t(t_0)$ is the current price of the futures contract. Similarly, computing the model's remaining time value as $v_{jt}(K) \equiv V_{jt}(K) - \max\{0, f_t(t_0) - K\}$, regress c_{jt} on v_{jt} (and a constant). Analogous calculations, with appropriate modifications of the maximization operator, are performed for the put options.

The regression-based tests of unbiasedness are:

$$c_{jt} = a_0 + a_1 v_{jt} + \epsilon_{jt}$$

$$p_{jt} = a_0 + a_1 u_{jt} + \epsilon_{jt}$$

where unbiasedness implies $a_0 = 0$, $a_1 = 1$.

Given that v_{jt} and u_{jt} are measured with error, the standard error-in-variables (EIV) problem may cause the OLS $\hat{a}_1$ to be downward-biased. The instrumental variables approach presented in Johnston (1972) is used to address this EIV problem.[8] The empirical results are summarized in Table II below.

A comparison of the mean c_{jt} and v_{jt} values indicates that the model understates interest rate volatility relative to market prices, resulting in a downward bias of the model's call option values relative to market prices.

[8] Specifically, the procedure calls for sorting the observations by the size of the independent variable v_{jt} (u_{jt}). Then, define

$$Z' = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 3 & \cdots & n \end{bmatrix}$$

$$X' = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \end{bmatrix}$$

where the x variables are the v_{jt} (u_{jt}) deviations ranked by size. This yields the instrumental-variables EIV estimate $\mathbf{b} = (\mathbf{Z'X})^{-1}\mathbf{Z'y}$ with its associated asymptotic variance–covariance matrix $\sigma_u^2(\mathbf{Z'X})^{-1}\mathbf{Z'Z}(\mathbf{X'Z})^{-1}$. The EIV estimator $\mathbf{b}$ is then an unbiased alternative to the biased OLS coefficient estimate.

TABLE II

Comparison of Market Prices and Model Values

	$c_{it} = a_0 + a_1 v_{it} + \epsilon_{it}$				$p_{it} = a_0 + a_1 u_{it} + \epsilon_{it}$			
	$N = 190$		$v_{it} > 0, N = 91$		$N = 185$		$u_{it} > 0, N = 84$	
	OLS	*IV*	*OLS*	*IV*	*OLS*	*IV*	*OLS*	*IV*
R^2	0.501	0.501	0.345	0.343	0.502	0.501	0.320	0.319
$\hat{a}_0$	0.0947	0.0896	0.1566	0.1377	0.1003	0.0930	0.1827	0.1679
	(4.87)	(4.28)	(2.32)	(3.41)	(5.09)	(4.39)	(3.41)	(4.04)
$\hat{a}_1$	0.9535	0.9878	0.8377	0.8963	0.9426	0.9914	0.7898	0.8346
	(−0.67)	(−1.61)	(−1.30)	(−1.04)	(−0.83)	(−0.10)	(−1.65)	(−1.64)
Mean c_{it}	0.235		0.426		0.242		0.443	
Mean v_{it} (u_{it})	0.147		0.321		0.150		0.330	
Std. c_{it}	0.322		0.376		0.322		0.368	
Std. v_{it} (u_{it})	0.239		0.263		0.242		0.264	

Notes: (1) N = number of observations. (2) The numbers under the $v_{it} > 0$ ($u_{it} > 0$) headings contain only those options with strictly positive model time values. (3) Numbers in parentheses are t-statistics. For $\hat{a}_1$, the t-statistic is for $H_0: a_1 = 1$. (4) IV = Instrumental Variable estimation procedure.

This is in part attributable to the results of the time-series empirical tests of the perturbation functions $h_i^*(m)$ in the Appendix (Preliminary Regression Results), where evidence showed that the interest rate variables do not explain 100% of the subsequent shifts in the term structure of interest rates. As is well known, fitting the h and π functions is a smoothing process which necessarily results in the loss of some information.

Although the intercept terms $\hat{a}_0$ in the regressions are significantly positive, the slope coefficients $\hat{a}_1$ do not differ significantly from unity. Moreover, the instrumental-variables methodology does cause the point estimates of the slope coefficients to increase towards unity, as would be expected if an error-in-variables problem existed. Finally, the results demonstrate that the model explains between 30% and 50% of the variation in the observed time values of the option premiums. Thus, the above results are broadly supportive of the model.

DERIVATION OF ARBITRAGE PROFITS FROM "MISPRICED" OPTIONS ON TREASURY BOND FUTURES CONTRACTS

Ultimately, any deviation between the market prices and model values—for the futures contracts as well as the options—must be subjected to an examination of whether such discrepancies give rise to trading opportunities.

The empirical results reported herein can be interpreted in two ways. If there exist economic agents who can trade at Fama–Bliss prices, then this analysis will indicate the existence/absence of arbitrage opportunities. Alternatively, if the results indicate "arbitrage" opportunities, these can be interpreted as indicating that the model's values better approximate "fair values" than do their market prices, perhaps reflecting imperfections in the market or frictions not accounted for in these tests.

Calculation of Option Replicating Portfolios

For the purposes of examining arbitrage opportunities in call and put option prices, a portfolio of three linearly independent interest rate-contingent assets is required. The futures contract is a natural choice for one such asset, to hold constant whatever "mispricing" may exist in that contract.[9] In addition, the three-week bill has a known payoff and is also included. Finally, the five-year annuity $A_1 = \$1$ received for years 1 through $5 \equiv \sum_{m=1}^{5} P(m)$, is included as an asset with a low correlation with the longer-maturity futures and, by definition, zero correlation with the payoffs to the three-week riskless asset. Thus, the replication portfolio is composed of α futures contracts, β units of the $A_{5Yr,1M}$ annuity and γ units of the three-week Bill. At the option's expiration date, in week 3, the replicating portfolio should mimic the payoff to the option. The analysis now proceeds to examine the arbitrage opportunities in call options; naturally, analogous results obtain for the puts. Thus, for a call option, $C^s(t_1) = \max\{0, \epsilon F^s(t_1) - K\}$. This option's replicating portfolio is thus constructed as the solution to

$$
\begin{aligned}
C^u(t_1) &\equiv \max\{0, \epsilon F^u(t_1) - K\} = \alpha[\epsilon F^u(t_1) - f(t_0)] + \beta A^u_{5Yr,1Wk} + \gamma \\
C^n(t_1) &\equiv \max\{0, \epsilon F^n(t_1) - K\} = \alpha[\epsilon F^n(t_1) - f(t_0)] + \beta A^n_{5Yr,1Wk} + \gamma \\
C^d(t_1) &\equiv \max\{0, \epsilon F^d(t_1) - K\} = \alpha[\epsilon F^d(t_1) - f(t_0)] + \beta A^d_{5Yr,1Wk} + \gamma
\end{aligned}
\tag{8}
$$

[9] As noted above, the arbitrage strategy for the call option can be implemented with any three linearly independent interest rate-dependent assets. Thus, if maturities m, m_1 and m_2 has been utilized, the hedge portfolio would be

$$C^0(K) = \alpha P_m + \beta P_{m_1} + \gamma P_{m_2}$$

where $C^0(K)$ is the model's value for the call option, eq. (7). By using the futures contract in the hedge portfolio, one may more confidently abstract from the option of delivering the bond within the maturity month, since this option is likely to affect the long (short) call and the short (long) futures position in offsetting ways.

where

$C^s(t_1)$—is the state-dependent value of the call option at t_1;

$f(t_0)$—is the current t_0 value of the futures contract;

$\epsilon F^s(t_1)$—is the state-dependent value of the futures contract at the option's maturity date t_1;[10] and

$A_{5YR,3WK}$—is the projected value of the state-contingent five-year, one-month annuity three weeks later.[11]

The solution to (8) is denoted by $\{\alpha^*, \beta^*, \gamma^*\}$.

As the futures contract involves no initial cash investment, the time t_0 value of the replicating portfolio—$C^0(K)$, the call option's fitted value at time t_0—is given by

$$C^0(K) = \beta^* A_{5YR,1MO} + \gamma^* P_{3WK}$$

The next step is to calculate the realized profits from a trading strategy based on the call option's current market price, $c(t_0)$, and the market price of the hedge portfolio. If $c(t_0) > C^0(K)$, short the call and buy the replicating portfolio; the realized profits at the three week-hence option maturity date t_1 are:

$$\frac{c(t_0) - (\beta^* A_{5YR,1MO} + \gamma^* P_{3WK})}{P_{3WK}} - \max\{0, f(t_1) - K\} + \{\alpha^*[f(t_1) - f(t_0)] + \beta^* A_{5YR,3WK} + \gamma^*\} \quad (9)$$

where $f(t_1)$ and $A_{5YR,3WK}$ denote the realized values of these assets at week 3.

Conversely, if $c(t_0) < C^0(K)$, buy the call and short the hedge portfolio; the realized profits at the call option's maturity date are the negative of eq. (9). Once again, analogous calculations are used in evaluating arbitrage strategies for put options.

This process is repeated at the end of January, April, July, and October when the shortest T-Bond Futures option has three weeks to expiration. Portfolios are formed for each available striking price for which there is positive open interest and trading volume.

[10] The state's futures prices require alignment with the current price f, so that these projected prices are arbitrage free. Thus, the option is valued relative to the futures contract price by transforming $F^s(t_1)$ to $\epsilon F^s(t_1)$, where ϵ is given by $f(t_0)/\sum_s \pi_s F^s(t_1)$.

[11] The generation of $A_{5YR,3WK}$ requires two important ingredients: first, that these projections account for a three-week, rather than a one-month, projection; second, that they be arbitrage-free under the risk neutral probability measure, π_s. Both objectives are accomplished by contracting the perturbation function. h^s_{3W} is contracted through multiplication by the ratio $P(1)/d_3$, where $P(1)$ is the one-month bill price and d_3 is the three week present value operator.

Empirical Results

The empirical results of these arbitrage-based tests are reported in Table III. These calculations give the average profit for the "All Calls" and "All Puts" categories as well as sub-categories thereof. The choice of these sub-categories is dictated by the desire to demonstrate the model's performance for options with positive time value ($c_{jt} > 0$ and $p_{jt} > 0$), zero model value ($V_{jt} = U_{jt} = 0$) and market price exceeding an arbitrary lower bound ($C_{jt} \geq 1/16, P_{jt} \geq 1/16$). The empirical results demonstrate convincingly that the model values can, at least to market makers trading with close to zero transaction costs, generate arbitrage profits and therefore represent more accurately the "fair value" of these options.

SUMMARY

This article illustrates the application of discrete time term structure models to the problem of valuing Treasury bond futures options. The various explicit and implicit options and contract details that need to be accounted for are carefully laid out. The test procedure is implemented using the Bliss–Ronn trinomial, time-varying parameter model.

The empirical results represented herein display statistically significant power in explaining the time series cross-section prices of Treasury Bond futures contracts and options on these futures contracts. Further, the model appears to have some power to detect "arbitrage" opportunities, but only for low transaction-cost agents able to trade at

TABLE III

Analysis of Arbitrage Profits in Option Contracts

Categories	*N*	*Percent Profitable*	*Av. Profit ($/option)*	*t-stat on Av. Profit*
All Calls	185	78.4%	0.480	1.98
$c_{jt} > 0$	163	75.9%	0.555	2.01
$c_{jt} > 0, C_{jt} \geq 1/16$	115	68.7%	0.770	1.99
$c_{jt} > 0, V_{jt} = 0$	55	96.4%	0.144	0.87
$c_{jt} > 0, V_{jt} = 0, C_{jt} \geq 1/16$	15	86.7%	0.471	0.77
All Puts	179	79.4%	0.545	2.18
$p_{jt} > 0$	173	79.1%	0.563	2.18
$p_{jt} > 0, P_{jt} \geq 1/16$	112	69.6%	0.865	2.18
$p_{jt} > 0, U_{jt} = 0, P_{jt} \geq 1/16$	22	100%	0.15	6.30

Notes. (1) N = number of observations. (2) C_{jt} (P_{jt}) is the call (put) option's market price. (3) $c_{jt} = C_{jt}(K) - \max(0, F_t - K)$ ($p_{jt} = P_{jt}(K) - \max(0, K - F_t)$) is the time value in the call (put) option's market price. (4) V_{jt} (U_{jt}) is the model's value for the call (put) option.

market prices and borrow/lend risklessly; alternatively, one can interpret these results as yielding asset values closer to the arbitrage-free values of these instruments.

The model's successful ability to price Treasury Bond futures contracts and their options is indicative of its more general property as a mechanism for generating hedge ratios for arbitrary interest rate-contingent claims. This nonstationary trinomial model thus constitutes important evidence on intertemporal changes in the riskless term structures of interest rates.

The methodologies developed in this article can be applied using alternative term structure models, thus constituting a test of the usefulness of those models. The methodology is also generally applicable, with appropriate modifications, to other complex interest rate derivative securities such as Treasury Note futures options.

APPENDIX A: THE BLISS–RONN INTEREST RATE MODEL

The interest rate model used in this article is a variation of the Bliss and Ronn (1989) model. This appendix summarizes the model and notes the modifications made to the original methodology for the purposes of this application.

The Trinomial Lattice

Let P_{tm} denote the price of an m-maturity pure discount bond at time t. At time $t-1$, consider an economy in which the term structure of interest rates can adopt one of three alternative forms in the next period (period t). These possible term structures are determined by the current $(t-1)$ term structure and the three possible perturbations h^u_{tm}, h^n_{tm}, and h^d_{tm}; the three states are defined as the "up," "down," and "no change" states. The future pure discount bond prices are determined as follows:

$$P_{tm} = \frac{P_{t-1,m+1}}{P_{t-1,1}} h^s_{tm} \tag{10}$$

where "s" denotes the state realized (u, n, or d). By the definition of a "state of nature," the state realizations are the same across all maturities, m, for any given period t.

If riskless arbitrage opportunities are to be precluded, the perturbation functions h^u_{tm}, h^n_{tm}, and h^d_{tm} must satisfy the conditions:[12]

$$h^u_{tm}\pi^u_t + h^n_{tm}\pi^n_t + h^d_{tm}\pi^d_t = 1 \quad \forall m \tag{11}$$

$$\pi^u_t + \pi^n_t + \pi^d_t = 1 \tag{12}$$

$$\pi^s_t \geq 0, \quad \forall s \tag{13}$$

Derivation of eqs. (11)–(13) consists of constructing a portfolio of three distinct maturities which has the same payoff in each of the three states and is therefore riskless. The solution to this problem is equivalent to solving a system of equations in the π's. Intuitively, note that under no-arbitrage valuation with spanning, the price of any interest rate-sensitive asset is equal to the discounted expected present value of the payoffs, using the risk-neutral probability distribution to generate expectations. Thus, for a time-0 pure discount bill of maturity $m + 1$,

$$\begin{aligned} P_0(m + 1) &= P_0(1)\sum_s \pi^s P^s_1(m) \\ &= P_0(1)\sum_s \pi^s \frac{P_0(m + 1)}{P_0(1)} h^s(m) \end{aligned}$$

which, upon simplification, implies that there exists a maturity-independent parameters, π, such that the cross-sectional constraint, eq. (11), obtains for all maturities m. In words, under the risk-neutral probabilities, the expected value of the perturbation is unity.

State dependency is introduced into the trinomial model by making the perturbation functions dependent on a $K \times 1$ vector of state variables, $\mathbf{x}_t$. This permits time-variation of the perturbations with the level of interest rates (amongst other variables), thus potentially alleviating a situation in which these perturbations give rise to an unreasonably low or high level of interest rates.

The π^s_t are indirect functions of $\mathbf{x}_{t-1}$ through the $h^s_t(\cdot)$ functions:

$$h^u_t(\mathbf{x}_{t-1}, m)\pi^u_t + h^n_t(\mathbf{x}_{t-1}, m)\pi^n_t + h^d_t(\mathbf{x}_{t-1}, m)\pi^d_t = 1$$

and

$$\pi^u_t + \pi^n_t + \pi^d_t = 1$$

[12] For proof, see Bliss and Ronn (1989). The derivative of eqs. (10)–(13) preclude single period arbitrage opportunities. However, these equations also ensure that the price of any bond follows a martingale process defined with respect to the risk neutral probabilities. This condition suffices to guarantee the model is also arbitrage free over multiple periods.

Data

This article implements an extension of the procedure used in Fama and Bliss (1987) to estimate term structures of interest rates. The resulting term structures are used to construct a monthly series of pseudo-discount bond prices for maturities extending to 30 years, for the period 1979:10–1988:12. These pseudo-bond prices in turn are used to calibrate the term structure model for the tests in this article.

The current applications of the Bliss–Ronn interest rate model, designed to provide the value of traded options, requires the specification of the state variables' values in each of the possible different states of nature in future periods. The state variables are, therefore, deliberately chosen to be endogenous to the model, so that their future values, conditional on the states of the world in subsequent periods, can be projected. This permits one to propagate forward the trinomial tree for multiple periods as needed. The variables chosen include the following:

1. The three-month T-Bill rate;
2. The term premium (defined as the 30-year government bond yield less the three-month T-Bill yield);
3. A measure of the "hump" (if any) in the term structure;[13]
4. The one-month change in the three-month T-Bill rate.

The three-month T-Bill rate captures the near-term inflation expectations and the near-term level of the real interest rates, while the spread of the long-term rate over the short-term and the "hump" in the term structure captures the information inherent in the slope and curvature of the term structure of interest rates. Finally, the change in the short-term interest rate measures the volatility in short-term rates of interest.

The use of these variables finds significant support in the literature. Fama and French (1989) note that the "term spread is . . . closely related to the short-term business cycle . . . ." These interest rate variables have also been empirically found to explain a significant portion of the volatility in interest rates. For instance, Litterman, Scheinkman, and Weiss (1991) estimated

> a model in which future short rates depended on (1) today's short rate, (2) the level toward which the short rate was expected, as of today, to converge—which we call the "long" rate—and (3) the

[13] The "hump" is empirically defined as the maximum excess, if any, of the intermediate-term interest rates (of maturities 1, 2, 3, . . . , 27, 28, 29 years) over the maximum of the three-month T-Bill rate and the 30-year spot rate: $\max\{0, \max\{r_1, r_2, \ldots, r_{29}\} - \max\{r_{3M}, r_{30}\}\}$.

> volatility of this "long" rate. . . . [They] confirmed that, historically, variations in these three variables satisfactorily explain the past movements of the yield curve.

In a further study Litterman and Scheinkman (1991) consider three factors, "*level, steepness* and *curvature* [emphasis in original]," and note that these "explain—at a minimum—96% of the variability of excess returns of any zero" coupon bond.

Empirical Tests

Classification of Time Periods

To test the model it is necessary to classify each time-period's term structure realization as either an "up," a "down," or a "no change" state.

The hypothesized (and estimated) perturbations are changing through time, precluding a fixed classification scheme. A maximum likelihood approach, based on the predicted outcomes, is used to assign each observation to one of the three outcomes. The reader is referred to Bliss and Ronn (1989) for details of the procedure used.[14]

Application of this methodology to the 110-month period 1979:10–1988:12 results in 21 "up" months, 57 "no change" months, and 32 "down" months.[15]

Preliminary Regression Results

Consider the following regressions of the realized perturbation, y^s_{tm}, at time t for maturity m assigned by the classification procedure to a given state s_t

$$y^s_{tm} = \alpha^s_{1m} + \sum_{k=2}^{K} \alpha^s_{km} x_{k,t-1} + \epsilon^s_{tm} \tag{14}$$

Note that the regression equation $y^n_{tm} = \alpha^n_{1m} + \sum_{k=2}^{K} \alpha^n_{km} x_{k,t-1} + \epsilon^n_{tm}$ allows the "no change" state to have perturbation functions $h^n_{tm} \neq 1$.

For maturities of one to 29 years, OLS regressions are performed for each state over the period 1979:10–1988:12. Across the individual

[14] Classification obtained vary slightly depending on the maturity, m, used in the algorithm. Bliss and Ronn based their classification on only the five year maturity. To minimize potential problems, the classifications used in this study are obtained using maturities ranging from 1–60 month. The final classification is based on the modal classification across maturities.

[15] It would clearly be desirable to examine the perturbations functions' coefficients by alternate sub-periods, and/or to use an "expanding window" regression analysis to predict next period's perturbations based only on past data. Unfortunately, this possibility is precluded by the relatively small number of months available for the "up" and "down" classifications in the sample period 1979:10–1988:12.

maturities, the R^2's range from 12%–65% for the "up" state, from 23%–58% for the "down" state, and from 57%–84% for the "no change" state. In total, the state variables explain 66.5% of the variation in the realized perturbations, y^s_{tm}, across all states and maturities.[16]

Estimation of the Nonstationary Trinomial Probabilities

The Bliss–Ronn (1989) derivation of the trinomial model for shifts in the term structure of interest rates demonstrated that the π's are defined in terms of $\{h^u, h^d, h^n\}$. In practice, the nonlinearity of the definitions of the π's as given in eq. (20) of that article is extremely cumbersome to implement. Fortunately, another problem yields the same solution for the π's and simultaneously lends itself to a more tractable imposition of the constraints. Using two benchmark maturities, m_1 and m_2, define

$$\mathbf{A} \equiv \begin{bmatrix} h^u_{m_1} & h^n_{m_1} & h^d_{m_1} \\ h^u_{m_2} & h^n_{m_2} & h^d_{m_2} \\ 1 & 1 & 1 \end{bmatrix}$$

$$\pi \equiv [\pi^u, \pi^n, \pi^d]'$$

$$\iota_3 = [1, 1, 1]'$$

The solution to $\mathbf{A}\pi = \iota_3$ is identical to eq. (20) in Bliss and Ronn (1989).

In practice, the parameters of the perturbation functions, h^s_m, are unobservable and must be estimated. Using the regressions eq. (14), set the preliminary estimates of the perturbation functions, ${}_0h^s_{tm}$, equal to the regressions' fitted values $\hat{y}^s_{tm}$. At each point in time t, obtain preliminary estimates of the π's, ${}_0\pi^s_t$, as solutions the weighted least-squares regression

$$1 = {}_0h^u_{tm}\,{}_0\pi^u_t + {}_0h^n_{tm}\,{}_0\pi^n_t + {}_0h^d_{tm}\,{}_0\pi^d_t + \epsilon_{tm}$$

where it is assumed that $\mathrm{Var}(\epsilon_{tm}) = m\sigma^2_t$ to account for the heteroscedasticity of the perturbation functions ${}_0h^s_{tm}$.

The ${}_0\pi^s_t$'s obtained from this regression procedure do not necessarily have the desired properties which allow them to be interpreted as

[16] The relative paucity of data in the region beyond 20 years requires extrapolating to estimate spot interest rates in that maturity region. This causes the estimated regression coefficients in this maturity range to be less reliable.

(risk-neutral) probabilities. To generate a set of π_t^s which are nonnegative and sum unity, these variables are then standardized by setting

$$\pi_t^s = \frac{\max[0, {}_0\pi_t^s]}{\max[0, {}_0\pi_t^u] + \max[0, {}_0\pi_t^n] + \max[0, {}_0\pi_t^d]} \quad \text{for } s \in \{u, n, d\}$$

Using the above methodology, values of the π_t^s can be calculated for the entire sample period. For most months in the overall sample period October 1979–November 1988, all three ${}_0\pi_t^s$ are positive, so that $\max[0, {}_0\pi_t^s] = {}_0\pi_t^s$ and the nonnegativity constraint does not have to be explicitly applied.[17]

Having satisfied the nonnegativity of π_t^s and the equality of $\sum_s \pi_t^s$ to unity, the cross-sectional constraint, eq. (11), is guaranteed to obtain for the adjusted π_t^s by setting h_{tm}^s equal to ${}_0h_{tm}^s / \sum_q \pi_t^q \, {}_0h_{tm}^q$.

APPENDIX B: ANALYSIS OF ADJUSTMENT FACTOR FOR BOND PRICES

In the empirical analysis of futures contract pricing, the projected price of the deliverable bonds is adjusted for a tax/liquidity effect. This appendix documents the results that such an adjustment does indeed succeed in reducing the mean squared forecast error in the bonds' prices.

TABLE BI

Pricing Error, $e_{jt} = B_{jt} - CF_{jt}$ in Bond Prices Per $100 Face Value

	All Issues: Number of Issues = 56			
	Mean Error		*Mean Squared Error*	
	Fitted	*Adjusted*	*Fitted*	*Adjusted*
Min	−1.616	0.1788	0.0049	0.000
Mean	2.481	0.0982	95.3917	1.283
Max	47.896	1.5896	2356.4982	20.764
	Excluding Flower Bonds: Number of Issues = 53			
	Mean Error		*Mean Squared Error*	
	Fitted	*Adjusted*	*Fitted*	*Adjusted*
Min	1.6164	−0.1788	0.0049	0.0000
Mean	0.4246	0.0417	2.1842	0.4370
Max	3.8768	0.4952	17.7111	1.9186

Note: "Fitted" refers to the present values of the cash flows, CF_{jt}, whereas "Adjusted" refers to $CF_{jt} + e_{jt}$.

[17] The nonnegativity constraint is satisfied 79.6% of the time; the violations in the remaining months do not constitute significant evidence against the model, since relatively minor changes in the estimated perturbations ${}_0h_{tm}^s$ give rise to positive ${}_0\pi_t^s$.

It should be noted that a proportional model for e_j: $e_j = B_j/CF_j$ was tested also. This resulted in virtually identical results in terms of reducing the fitted errors for noncallable bonds, but significantly worse results for callable/flower bonds.

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