THE COST OF HEDGING AND THE OPTIMAL HEDGE RATIO

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Hedging benefits offered by the futures market come at a cost. This article develops a concept of hedging costs, shows how it impacts the hedging decision, and derives an optimal hedge ratio in the context of the cost concept. The hedging cost of using futures is comprised of two components. The first component represents the fixed costs of setting up and managing a hedging program. The second component is the result of spot/futures arbitrage and the fact that the futures contract is an imperfect substitute for a commercial transaction.[1] It is shown that arbitrageurs drive the expected futures return equal to the spot risk premium. Thus as hedgers take a short futures position, expected return is reduced by the amount of futures shorted times the spot risk premium.

Hedgers can seldom create a perfect hedge due to mismatches between spot and futures delivery dates and contract specifications. Thus, the hedger faces the situation of paying full cost (i.e., reducing expected return by the amount of the fixed costs plus the spot risk premium) while receiving less than the full benefits (i.e., the elimination of all risk). The hedger, therefore, is required to make a risk/return decision since, as will be demonstrated, the marginal cost of hedging

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[1]Certain costs are included here in the fixed component. Also, some variable costs are ignored. For example, transaction costs may be influenced by the size of the hedge, price, and the place at which the hedge is lifted. It is assumed that the simple equation used in this article adequately captures the salient costs of future hedging.

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rises sharply as the risk minimizing hedge ratio is approached. As a result, the hedger will select a hedge ratio which is less, perhaps substantially less, than the risk minimizing hedge ratio.

The question of whether there is a non-zero spot risk premium, and, in turn, whether there is a cost of hedging has been tested by a number of authors including Telser (1958), Brennan (1958), Cootner (1960), Gray (1961), Dusak (1973), Bodie and Rosansky (1980), Carter et al. (1983), Chang (1985), Baxter et al. (1985), Ehrhardt et al. (1987), Fama and French (1987), Hartzmark (1987), Maddala and Yoo (1990), and Kolb (1992). There is considerable evidence that expected futures returns are indeed positive. Bodie and Rosansky (1980) found that 22 of the 23 futures contracts exhibited positive average returns while Fama and French (1987) found 18 of 21 positive. A recent study by Kolb (1992) came to a similar conclusion. Using 980,800 daily settlement prices for 29 futures contracts over the period 1959–1988 (not all contracts were available for the entire period), Kolb found that 24 of the 29 futures exhibited positive returns. Of those that were positive, 15 were significantly different from zero. The average annual futures return over all 29 contracts was about 4% while one contract (live hogs) averaged over 13% (1969–1988). Thus, for many futures contracts, the expected return is positive and, therefore, hedging costs are an important consideration.[2]

There have been several attempts to include cost or return considerations into the theory of hedging. Among them are Anderson and Danthine (1981) who developed a model which includes both risk and return considerations and requires knowledge of the hedger's utility function. Howard and D'Antonio (1984) and Cecchetti et al. (1988) got around this problem by introducing a risk-free asset and, thus, were able to obtain a solution independent of the hedger's utility function. Unfortunately, these models are based on the assumption that the hedger is free to trade among the risk-free security, the spot market, and the futures market. This assumption obviously does not hold for a number of hedgers who may not be free to trade away their spot position. This article develops a hedging cost model based upon the assumption that the hedger must hold the spot position. The model also incorporates spot/futures arbitrage and the CAPM into the

[2]As pointed out by Gray (1961), the positive return on long futures may be the result of the idiosyncratic nature of the specific futures contract relative to the underlying spot market rather than the results of an imbedded risk premium. Recent studies have not been subjected to Gray's careful analysis as have the earlier studies by Brennan (1958) and Telser (1958). Thus, some of the measured risk premiums in these later studies may not actually be risk premiums.

hedging decision and leads to an optimal hedge ratio which can be estimated in a straightforward way using historical return data. The model eliminates the need for estimating the spot risk premium and the hedger's level of risk aversion, two aspects that bedevil other optimal hedge ratio models.

In the next section a model is developed for measuring both hedging cost and risk reduction for a futures contract. The existence of spot/futures arbitrage plays an important role in the development of the model and is discussed in the second section of the article. In the third section, the optimal hedge ratio is derived and a simple numerical example is presented. The final section summarizes results and discusses future research issues.

A MODEL OF HEDGING COST AND RISK REDUCTION

The futures contract, the spot commodity, and other securities and assets are assumed to be traded in fully integrated markets which are in equilibrium (i.e., all risk-free arbitrage opportunities have been eliminated). The resulting spot risk premium is a positive function of the non-diversifiable risk inherent in holding the spot position, since, in general, some portion of the spot risk will be non-diversifiable. Under these assumptions, the problem facing the hedger is displayed in risk/return Figure 1 where i is the certain return on the risk-free asset, S represents the spot asset, and M represents the all-inclusive market portfolio. The spot return is the sum of the change in the spot price and the convenience yield[3] net of storage costs. Since the spot

[3]The concept of a convenience yield was first introduced by Kaldor (1939) and further refined by Working (1948) to explain the puzzling phenomenon of futures prices at times being lower than cash prices. In a world in which storage costs are positive, how could such a price inversion exist? Working postulated in his Theory of Storage that such a situation could exist if the holders of the commodity received a "convenience yield" for having inventory on hand. Along this line, Working divided inventory into two types:

> Distinction is sometimes drawn between 'surplus stocks' which a merchant or manufacturer will not carry without assurance or expectation of a special return, and 'necessary working stocks' which will be carried for other reasons. (Working, 1948)

The greater the proportion of "necessary working stocks", the greater will be the impact of the convenience yield on futures price determination. Thus, argued Working, the tighter the inventories, the more likely one observes a negative carrying cost between the spot and futures markets. The concept of a convenience yield was further developed by Brennan (1958) and Telser (1958) and has been extensively tested. A recent test by Fama and French (1988) found, among other things, that the convenience yield for metals varies with the business cycle. Brennan (1991) found that the convenience yield was a function of inventories but was poorly modeled as a function of the spot price.

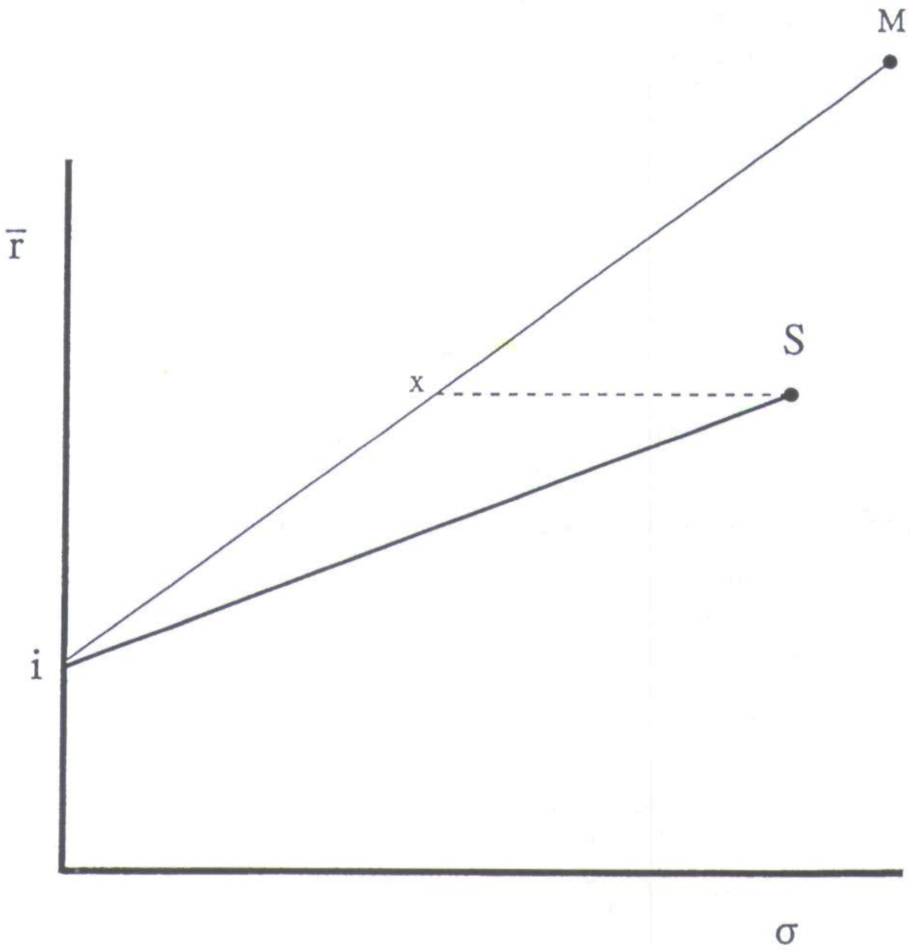

FIGURE 1
Risk/return graph of Hedger's problem.

position includes some non-priced, diversifiable risk, S necessarily falls below the i–M line as shown in Figure 1.

The goal of the potential hedger holding the spot position S is to reduce risk while considering costs. For mathematical convenience, risk is measured as the standard deviation of return. Therefore, in addressing the futures hedging problem, there is a need for a benchmark measure of risk reduction and hedging cost.

The i–S line in Figure 1 is used as this benchmark for two reasons. First, liquidating a portion of the spot position and investing the proceeds in the risk-free asset is a logical alternative to using futures for adjusting the risk/return characteristics of the portfolio. The problem, of course, is that the spot position must be liquidated, at least partially, to implement this alternative strategy. Thus, the vertical difference between a portfolio on the $i-S$ line and the spot/futures portfolio is a logical measure of the incremental cost of maintaining the spot position while at the same time reducing risk through the use of futures. Second, spot/futures arbitraguers trade along the $i-S$ line. Since, as is shown in the next section, arbitrage plays an important role in the determination of the hedging cost, using the $i-S$ line as a benchmark provides an intuitive link between arbitrage and the cost of hedging.[4]

Three possible outcomes from hedging are shown in Figure 2. In the first, the spot/futures portfolio outcomes are located above the $i-S$ line, which means that there is a risk-return benefit of using futures. That is, when compared to the simple spot/riskfree portfolio which lies on line $i-S$, the futures/spot portfolio provides better risk/return characteristics. Outcome 1 occurs rarely due to arbitrage activity between the spot and futures markets.

The second outcome is one where the resulting spot/futures portfolio lies on the $i-S$ line. This indicates that using futures leads to a portfolio which produces the same risk/return characteristics as does a portfolio made up of the risk-free and spot securities. This will be referred to as the "no hedging cost" outcome. Again, outcome 2 occurs infrequently due to arbitrage activity.

The third outcome is one where the resulting spot/futures portfolio lies below the $i-S$ line and, thus, there is a cost to hedging. What is meant by hedging cost is that a firm using the futures market must give up some return, as measured by the vertical difference between the $i-S$ line and the spot/futures portfolio to retain the benefit of continued spot ownership. It will be shown that outcome 3 is the most common.

Risk-Return Equations

Three risk/return equations describe the outcomes shown in Figure 2. These equations are derived under the assumptions that no margin

[4]Several other benchmarks could have been used. One approach would be to not use a benchmark at all and simply measure the change in expected returns as futures are added to the portfolio. This approach has the disadvantage of not capturing as neatly the linkage between arbitrage and hedging cost. Another possibility would be to use the $i-x-S$ line as a benchmark. This has the mathematical disadvantage of the kink at point x. A final possibility would be to use the $i-M$ line. This has the disadvantage of including the distance between the $i-M$ and $i-S$ lines which has little to do with the hedging decision. Since the general conclusions in the following sections are unaltered when different benchmarks are used (even though the equations will change somewhat), the $i-S$ line is used as the hedging cost benchmark for the reasons given above.

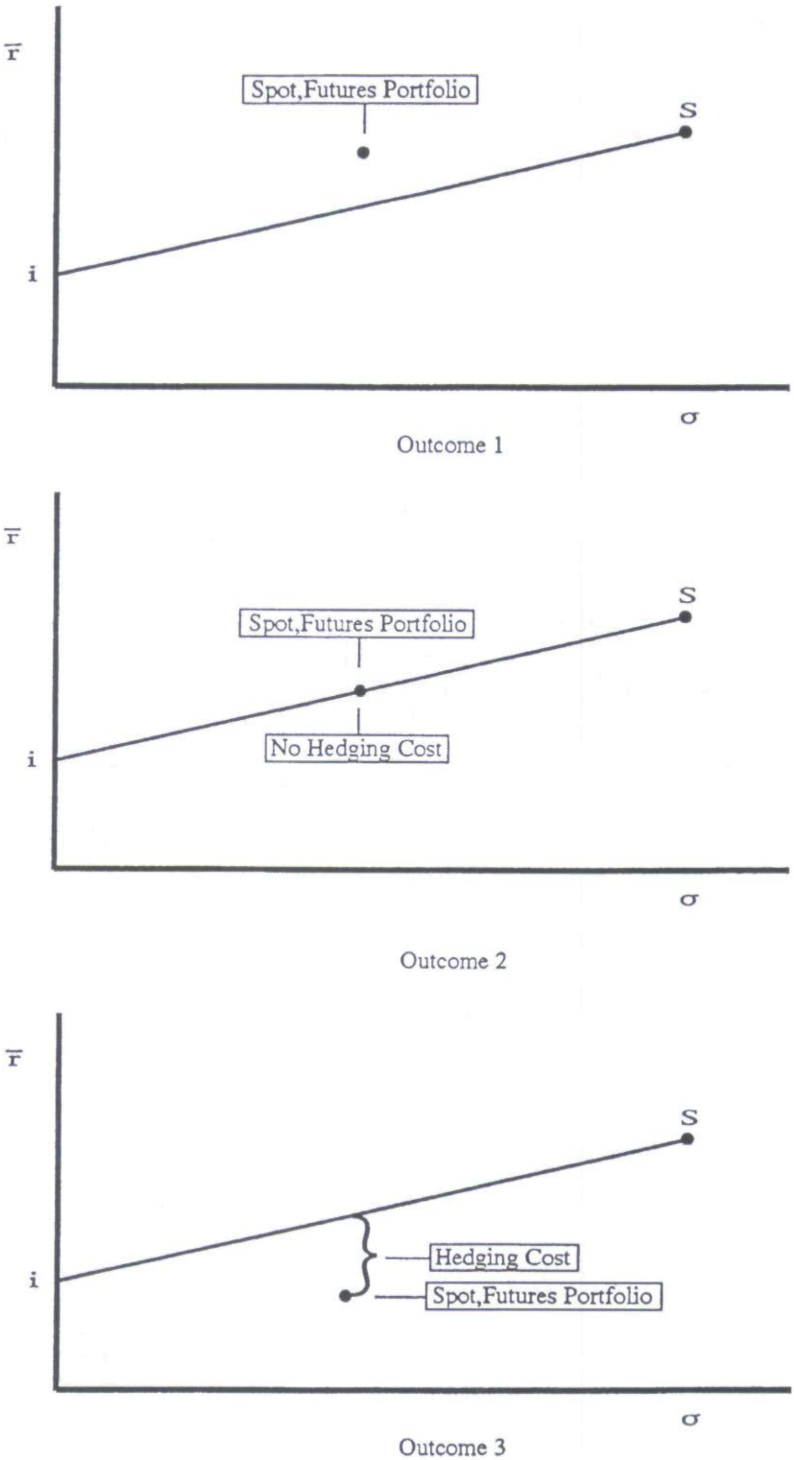

FIGURE 2
Potential hedging results.

is required for a futures position, the market is dominated by short hedgers,[5] and the hedge ratio b represents the fraction of the long spot position hedged with short futures contracts. Equation (1) defines riskfree/spot portfolios which lie on the i–S line.

$$\bar{r}_{is} = i + RP\,\frac{\sigma_{is}}{\sigma_s} \tag{1}$$

where

$\bar{r}_{is}$ = the expected return from the risk-free/spot portfolio;
i = risk-free rate of return;
$\bar{r}_s$ = expected return on the spot position, and is the sum of the expected spot price change and the convenience yield net of storage costs;
σ_s = standard deviation for the spot position;
σ_{is} = standard deviation of return for the spot/risk-free portfolio, where $0 \le \sigma_{is} \le \sigma_s$; and
RP = $\bar{r}_s - i$ = spot risk premium.

Equation (2) defines the expected return equation for the portfolios comprised of spot and futures.

$$\bar{r}_{sf} = \bar{r}_s - (F + b\bar{r}_f) \tag{2}$$

where

$\bar{r}_{sf}$ = expected return on the spot/futures portfolio;
F = fixed cost of setting up and managing a hedging program;
b = short futures hedge ratio; and
$\bar{r}_f$ = expected change in the futures price.

Since a zero margin requirement is assumed, $\bar{r}_f$ is not technically a return but is simply the expected % Δ in the futures price. The final equation is the spot/futures variance equation.

$$\sigma_{sf}^2 = \sigma_s^2 + b^2\sigma_f^2 - 2b\rho\sigma_s\sigma_f \tag{3}$$

where

σ_{sf}^2 = the variance of the spot/futures portfolio;
σ_s^2 = the variance of the spot return;

[5]The equations and the model could also be derived based upon a market dominated by long hedgers. However, the short hedger scenario is the one most often encountered in actual futures markets and so the short hedger scenario is the focus. See Hirshleifer (1988) for a discussion of the determinants of short and long hedging.

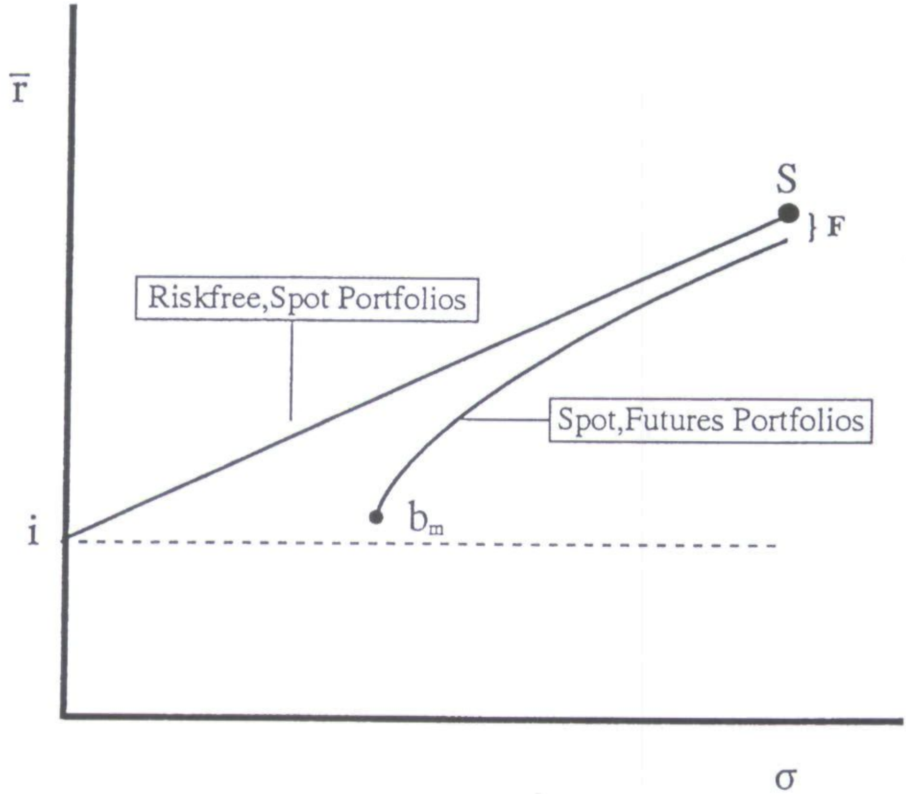

FIGURE 3
Spot/futures portfolios versus risk-free, spot portfolios.

σ_f^2 = the variance of the futures return; and
ρ = the correlation between spot and futures.

The three equations provide a mathematical representation of the spot/futures portfolios and the risk-free/spot portfolios. One graphical representation of these equations is shown in Figure 3, which is drawn under the assumptions that the spot/futures correlation is less than one, that the hedge ratio varies from 0 to the risk minimizing hedge ratio when costs are ignored, and that $\bar{r}_f$ is positive. If the hedger chooses to minimize risk, the hedge ratio is $b_m = \rho/\pi$ where $\pi = \sigma_f/\sigma_s$ [Ederington (1979)]. The result of using the risk-minimizing hedge ratio is indicated as the lower left portfolio in Figure 3. It is obvious that there is a diminishing return to hedging. That is, an additional unit of risk reduction can only be obtained at an increasing marginal cost. The hedger will not go beyond b_m since both risk and cost increase past this point.

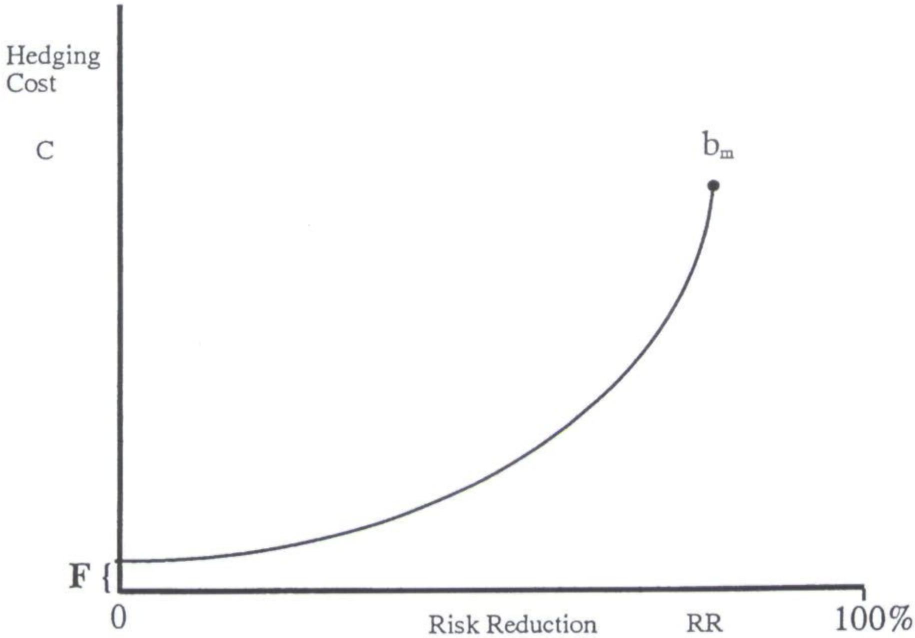

FIGURE 4
Hedging cost/risk reduction graph.

To better visualize this relationship, Figure 4 shows the same results as does Figure 3, but in the form of a hedging cost/risk reduction graph. The origin represents a portfolio with no futures. As the short futures position is increased, both cost and risk reduction increase. The hedging cost is measured as the sum of the fixed cost, F, and the return reduction relative to the i–S line; while risk reduction is measured as the percent reduction in the spot standard deviation. Figure 4 clearly reveals the diminishing benefit of using futures. The reason for this non-linear relationship between cost and risk reduction becomes apparent in the following section.

Hedging Cost and Risk Reduction Equations

Hedging cost, C, is determined by subtracting the return for the spot/futures portfolio from the return for the risk-free/spot portfolio with the same standard deviation. Using eqs. (1) and (2), C can be expressed as:

$$C = i + (RP)\frac{\sigma_{is}}{\sigma_s} - [\bar{r}_s - (F + b\bar{r}_f)]$$

$$C = RP\left(\frac{\sigma_{is}}{\sigma_s} - 1\right) + F + b\bar{r}_f \qquad (4)$$

Since the standard deviations are the same for both the i–S and the spot/futures portfolios, eq. (3) can be substituted for σ_{is} above. Therefore,

$$C = RP\left[\frac{(\sigma_s^2 + b^2\sigma_f^2 - 2b\rho\sigma_s\sigma_f)^{1/2}}{\sigma_s} - 1\right] + F + b\bar{r}_f$$

$$C = RP[(1 + b^2\pi^2 - 2b\rho\pi)^{1/2} - 1] + F + b\bar{r}_f \tag{5}$$

where

$$\pi = \frac{\sigma_f}{\sigma_s} = \text{the futures/spot risk relative}$$

The term in [ ]'s in eq. (5) is the negative of the risk reduction, RR, measure when risk is measured by the standard deviation. From eq. (3), the fraction of risk eliminated by taking a short futures position is

$$RR = \frac{\sigma_s - (\sigma_s^2 + b^2\sigma_f^2 - 2b\rho\sigma_s\sigma_f)^{1/2}}{\sigma_s}$$

$$RR = 1 - (1 + b^2\pi^2 - 2b\rho\pi)^{1/2} \tag{6}$$

Substituting eq. (6) into eq. (5),

$$C = -RP(RR) + F + b\bar{r}_f \tag{7}$$

As has been demonstrated by others [e.g., Siegel and Siegel (1990)] and is verified in the Appendix, spot/futures arbitrage drives the expected futures return equal to the spot risk premium.

$$\bar{r}_f = RP \tag{8}$$

Substituting eq. (8) into eq. (7) and rearranging,

$$C = F + RP(b - RR) \tag{9}$$

Thus C is the sum of a fixed component and a component which varies with the hedge ratio. Equations (6) and (9) define RR and C, respectively, which are graphed in Figure 4.

ARBITRAGE AND THE COST OF HEDGING

Assume that purchasing or selling a futures contract is a perfect substitute for the corresponding commercial transaction, $\pi = \rho = 1$. In this situation, RR becomes

$$RR = 1 - (1 + b^2 - 2b)^{1/2}$$

$$RR = b \tag{10}$$

Substituting eq. (10) into eq. (9),

$$C = F + RP(b - b)$$
$$C = F \tag{11}$$

Thus, the cost of hedging is fixed and the futures market represents a near perfect substitute for selling the spot asset in a commercial transaction. That is, the hedger is able to trade along a line parallel to the i–S line in Figure 1 much as does an arbitrageur.[6]

On the contrary, most hedgers face a situation in which the timing of the commercial transaction does not match the futures delivery date and/or the spot commodity being hedged does not exactly match the deliverable commodity. Thus, the futures contract represents an imperfect substitute for the corresponding commercial transaction. This means that the spot/futures correlation is less than perfect ($\rho < 1$) and the risk relative may be other than unity ($\pi \neq 1$). In this case, C is as defined before.

$$C = F + RP(b - RR) \tag{12}$$

As can be seen, C depends upon both RP and the relationship between b and RR. When ρ is less than one, RR is less than b and, consequently, there is an increasing cost to hedging.[7] What is more, the cost increases

[6]If the arbitrageur's and hedger's fixed cost are the same, then the arbitrageur and the hedger will trade along the same line.

[7]C will exceed F when $b - RR$ is positive, as can be seen in eq. (12), and RP is positive. The latter is assumed. To show the former, focus on the upper limit for b, that is $b = b_m = \rho/\pi$. Thus, if C exceeds F, then

$$b_m - RR_m > 0$$

and

$$\frac{\rho}{\pi} - \left[1 - \left(1 + \frac{\rho^2}{\pi^2}\pi^2 - 2\frac{\rho}{\pi}\rho\pi\right)^{1/2}\right] > 0$$

After much rearranging of this expression, it can be shown that $b_m - RR_m$ will be positive if

$$\rho < \frac{2\pi}{(1 + \pi^2)} \sim 1$$

for plausible values of π ($0.8 \leq \pi \leq 1.1$). Since the right-hand side of the above equation is approximately equal to 1 (π of 0.8 yields 0.98 while a π of 1.1 yields 1.00), a correlation of less than 1 leads to a positive value for $b_m - RR_m$. This does not rule out declining values for b less than b_m. However, since $C = F$ when $b = 0$ (since $RR = 0$) and given the general shape of the C, RR relationship shown in Figure 4, such declining values for $b - RR$ will be of little practical importance.

at an increasing rate as shown in Figure 4. Put another way, when the futures market is an imperfect substitute for the corresponding spot transaction, there is an increasing marginal cost of hedging.

Interestingly enough, this increasing marginal cost of hedging is the result of arbitrage in the futures market. Arbitrageurs focus on those opportunities where there is a perfect match between the spot and futures delivery dates, a situation in which perfectly hedged portfolios can be created. Consequently, the expected futures return ($\overline{r}_f$) is driven to equal the spot risk premium (RP), as is demonstrated in the Appendix. That is, an arbitrageur drives the *hedged* return down to the risk-free rate while, in turn, receiving the full benefit of a perfectly hedged portfolio. However, the hedger who cannot perfectly match the spot position with a futures contract is not quite so fortunate. Instead, the hedger drives the hedged return close to the risk-free rate while eliminating only a portion of the risk. Such a result leads to the increasing marginal cost of hedging depicted in Figure 4.

THE OPTIMAL HEDGE RATIO

In this section an expression for the marginal cost of hedging is derived and used to derive the optimal hedge ratio equation. The marginal cost of hedging, mc, is defined as the derivative

$$mc = \frac{dC}{dRR} \tag{13}$$

the slope of the curve in Figure 4. The marginal cost is the cost of obtaining the next unit of risk reduction. Since both C and RR are functions of the hedge ratio b, an expression for mc can be obtained by taking the derivative of C with respect to b and RR with respect to b and then dividing the former by the latter.

Using eqs. (6) and (9), these derivatives are

$$\frac{dRR}{db} = \frac{\pi^2(b_m - b)}{\sigma_b} \tag{14}$$

and

$$\frac{dC}{db} = RP - \frac{RP\,\pi^2(b_m - b)}{\sigma_b} \tag{15}$$

where

$$\sigma_b = (1 + b^2\pi^2 - 2b\rho\pi)^{1/2} \tag{16}$$

Using these results, the marginal cost of hedging is

$$mc = \frac{dC/db}{dRR/db} = \frac{RP\,\sigma_b}{\pi^2(b_m - b)} - RP \tag{17}$$

The range for mc can be determined by inserting $b = 0$ and $b = b_m$ into eq. (17).

$$b = 0 \text{ means } mc = \frac{RP}{\rho\pi} - RP \sim 0 \tag{18}$$

$$b = b_m \text{ means } mc = \infty \tag{19}$$

[Equation (18) is the result of $\sigma_b = 1$ when $b = 0$ and the definition of b_m]. Thus, as b ranges between 0 and b_m, mc ranges between roughly zero and infinity.

To better understand the optimal hedging decision, Table I contains the cost, marginal cost, and risk reduction results for various hedge ratios (assuming $RP = 2.0\%$, $\pi = 1$, $\rho = 0.95$, and $F = 0.10\%$). As the hedge ratio increases from 0 to 0.95, the cost of hedging increases from 0.10% to 0.62% and risk reduction reaches 69% (90% reduction in terms of variance). But it is the marginal cost of hedging which is most important in choosing the optimal hedge ratio. By examining column 3 in Table I, it can be seen that the marginal cost for hedge ratios of 0.5 and smaller are rather modest. Beyond 0.5, mc increases dramatically. As a result, the hedger will choose a hedge ratio less than the risk minimizing hedge ratio. The cost, mc, and risk reduction columns in Table I point to an optimal hedge ratio of approximately 0.8. Values of greater than 0.8 produce substantially higher costs coupled with only minor reductions in risk. Thus, the mc variable captures the changing relationship between hedging cost and benefits as the hedge ratio is increased.

As the hedger chooses the optimal hedge, there will be some upper limit on mc, which is designated mc^u, beyond which the hedger will simply not go. The upper limit, mc^u, can be thought of as a measure of the hedger's risk aversion. The higher is mc^u, the greater is the hedger's risk aversion. Referring to Table I, a hedger with $mc^u = 2.62$ would choose a hedge ratio of 0.8, while a hedger with $mc^u = 4.56$

TABLE I
Expected Hedging Results for Various Hedge Ratios

b Hedge Ratio	C Hedging Cost	mc Marginal Cost	RR Risk Reduction Percent of Standard Deviation	Variance Reduction Percent of Variance
0.00	0.10%	0.11%	0%	0%
0.25	0.13	0.19	23	41
0.50	0.20	0.43	45	70
0.70	0.30	1.20	60	84
0.80	0.39	2.62	65	88
0.85	0.46	4.56	67	89
0.90	0.53	10.65	68	90
$b_m = 0.95$	0.62	∞	69	90

Assumptions: RP = spot risk premium = 2%
π = risk relative $= \sigma_f/\sigma_s = 1$
ρ = spot, futures correlation = 0.95
F = fixed cost = 0.10%

Equations: $C = F + RP(b - RR)$
$mc = [RP\sigma_b/\pi^2(b_m - b)] - RP$
$\sigma_p = (1 + b^2\pi^2 - 2b\rho\pi)^{1/2}$
$RR = 1 - (1 + b^2\pi^2 - 2b\rho\pi)^{1/2}$
Variance Reduction $= 2b\rho\pi - b^2\pi^2$

would choose a hedge ratio of 0.85. The latter hedger is more risk averse and, thus, is willing to pay a higher marginal cost per unit of risk reduction.

It should be apparent from the preceding discussion that the optimal hedge ratio is a function of mc^u. In fact, eq. (17) can be used to derive an optimal hedge ratio (b^*) equation which is based on mc^u. Recognizing that σ_b is a function of b and rearranging eq. (17),[8]

$$b^* = b_m\gamma \tag{20}$$

where

$$\gamma = 1 - \frac{A}{\rho\pi}\sqrt{\frac{1-\rho^2}{1-\dfrac{A^2}{\pi^2}}} \tag{21}$$

[8]The quadratic solution is used in deriving eqs. (20) and (21). There are two solutions for any given quadratic equation: $+\sqrt{}$ and $-\sqrt{}$. The $+\sqrt{}$ solution would produce hedge ratios for the top half of the parabola which has been left out of Figure 4. Obviously, these hedge ratios are of little interest since they are associated with higher costs and higher risks. Thus, eq. (21) includes only the $-\sqrt{}$ solution.

$$A = \frac{RP}{RP + mc^u} \tag{22}$$

The A variable is the ratio of the spot risk premium and the risk aversion variable, mc^u. mc^u must exceed F, the fixed hedging cost. If it does not, then the decision will be to not hedge at all since fixed costs exceed the maximum amount the hedger is willing to pay per unit of risk reduction. In fact, many potential hedgers face this very situation and, consequently, never take a futures position due to the relatively high entry costs. Therefore, the only situation of interest for an active hedger is

$$0 < F < mc^u \tag{23}$$

and, thus, mc^u in eq. (22) must be positive. On the other hand, RP depends upon the non-diversifiable risk inherent in the spot position and, therefore, will be greater than or equal to zero. Other than the signing arguments just presented, there is no basis for determining the relative sizes of mc^u and RP. Thus,

$$0 \leq A < 1 \tag{24}$$

If $A = 0$ because $RP = 0$, then $\gamma = 1$ and $b^* = b_m$. This is the situation in which arbitrage passes nothing to the futures market and the hedger becomes a risk minimizer since there is no cost to hedging. As A approaches 1, b^* approaches zero. A approaches 1 when RP becomes large relative to mc^u. In other words, the cost of hedging grows large relative to the hedger's risk aversion and the hedger chooses an ever smaller hedge ratio.

The problem with the optimal hedge ratio given by eqs. (20), (21), and (22) is that both RP and mc^u are difficult to estimate. However, if two additional assumptions are made, this problem can be avoided.

Assumption 1: The CAPM holds and

$$RP = \beta\lambda \tag{25}$$

where β is the spot beta and λ is the market price of risk.

Assumption 2: The hedger is willing to pay up to the market price of risk when deciding upon the optimal hedge ratio and

$$mc^u = \lambda\frac{\sigma_s}{\sigma_M} - RP = \lambda\left(\frac{\sigma_s}{\sigma_M} - \beta\right) \tag{26}$$

where σ_M is the market standard deviation.

The first assumption is self-evident.[9] The second assumption needs further elaboration. The market price of risk λ represents an economy-wide, equilibrium price of risk. Thus, it is not unreasonable to assume the hedger will view this market-determined rate as an upper bound for decision purposes. The usual interpretation of λ is that it is the additional return demanded by investors for taking on market risk. In the context of this article, the interpretation is just the opposite. It is the amount of return the hedger is willing to give up to eliminate all risk. Since the spot and market standard deviations differ, λ is scaled by the ratio σ_S/σ_M in eq. (26). Since the i–S line is used as the cost benchmark, RP is already accounted for in calculating mc. Thus, RP is subtracted in eq. (26). Graphically, eq. (26) represents the vertical distance between point S and the i–M line in Figure 1.

Substituting eqs. (25) and (26) into eq. (22) and recognizing that $\beta = \rho_{SM}(\sigma_S/\sigma_M)$ where ρ_{SM} is the correlation between the spot commodity and the market,

$$A = \frac{\beta\lambda}{\beta\lambda + \lambda\left(\dfrac{\sigma_S}{\sigma_M} - \beta\right)}$$

$$A = \rho_{SM} \tag{27}$$

Thus γ becomes

$$\gamma = 1 - \frac{\rho_{SM}}{\rho\pi}\sqrt{\frac{1 - p^2}{1 - \dfrac{\rho_{SM}^2}{\pi^2}}} \tag{28}$$

From eq. (28), $\gamma = 1$ if $\rho_{SM} = 0$ and $\gamma < 1$ if $\rho_{SM} > 0$. What is more, all parameters can be estimated in a straightforward manner using historical spot, futures, and market return data. Thus, it is no longer necessary to estimate the spot risk premium and the hedger's risk aversion.

The optimal hedge ratio using eqs. (27) and (28) for various values of ρ and ρ_{SM} are presented in Table II (assuming $\pi = 1$). In general, as ρ decreases and/or ρ_{SM} increases, b^* falls further below b_m. At one extreme in Table II, $\rho = 0.95$ and $\rho_{SM} = 0.1$ leads to $b_m = 0.95$ while

[9]There is considerable controversy surrounding the CAPM. However, recent empirical tests have shown it to hold up reasonably well against more complex, alternative models such as the Arbitrage Pricing Theory. Fama and French (1992) present empirical evidence that size and the market-to-book ratio predict future stock returns better than does beta. Even if this is the case, size and market-to-book ratios are obviously not appropriate for measuring spot commodity risk premiums.

TABLE II
Optimal Hedge Ratio for Various Values of ρ and ρ_{SM}

		ρ_{SM}				
		0.10	0.20	0.30	0.40	0.50
				b^*		
ρ	b_m					
0.95	0.95	0.92	0.89	0.85	0.81	0.77
0.90	0.90	0.86	0.81	0.76	0.71	0.65
0.85	0.85	0.80	0.74	0.68	0.62	0.55
0.80	0.80	0.74	0.68	0.61	0.54	0.45

Assumption: $\pi = 1 = \sigma_f/\sigma_s$ = risk relative
Equations: $b^* = b_m\gamma$

$$\gamma = 1 - \frac{\rho_{SM}}{\rho\pi}\sqrt{\frac{1-\rho^2}{1-\frac{\rho_{SM}^2}{\pi^2}}}$$

$b_m = \rho/\pi$
ρ = spot/futures correlation
ρ_{SM} = spot/market correlation

$b^* = 0.92$. Thus, hedging cost has a small impact on b^*. At the other extreme in Table II, $\rho = 0.8$ and $\rho_{SM} = 0.50$ leads to $b_m = 0.8$ while $b^* = 0.45$. Hedging cost has a very large impact on b^*. This is quite clear in Figure 5 which is a plot of the optimal hedge ratios reported

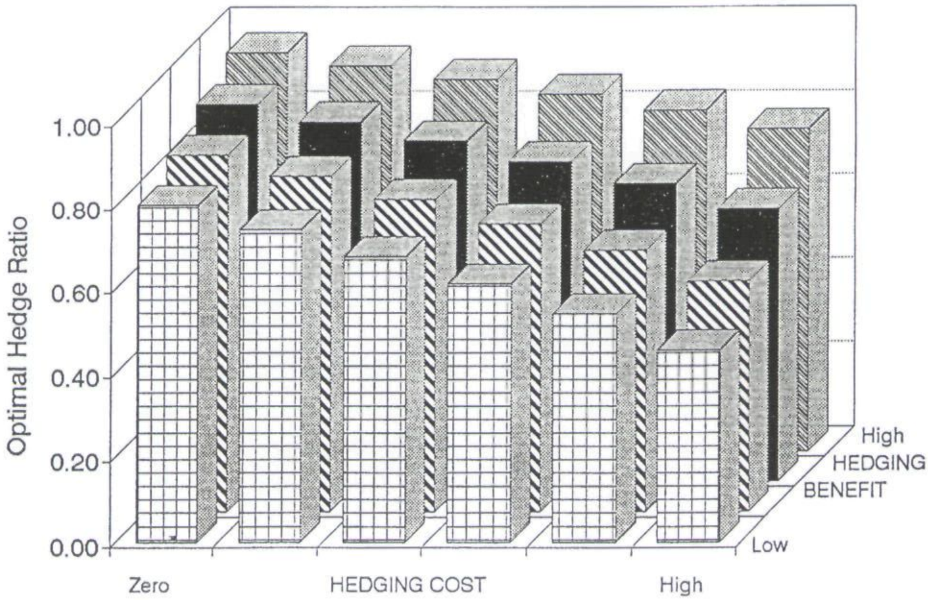

FIGURE 5
Optimal hedge ratios.

in Table II. The optimal hedge ratio is largest in the back left corner due to the fact that hedging costs are zero (i.e., $\rho_{SM} = 0$) and hedging benefits are highest (i.e., $\rho = 0.95$). On the other hand, it is smallest in the front right corner for which cost is highest (i.e., $\rho_{SM} = 0.5$) and benefits are lowest (i.e., $\rho = 0.8$).

The optimal hedge ratio is a positive function of the spot/futures correlation since this implies increasing hedging benefits, while the optimal hedge ratio is a negative function of the spot/market correlation since this implies increasing hedging costs. The interaction between these two correlations is captured by the equation for γ [eq. (28)].

CONCLUSIONS AND FUTURE RESEARCH

Futures markets provide the service of risk reduction without having to forsake the benefits of spot asset ownership, but the futures market imposes a cost for this service. Arbitrage activity between the spot and futures market plays an important role in determining this cost. If the futures market is fully arbitraged and represents an imperfect substitute for the corresponding commercial transaction (which is most often the case), the hedging cost will be a function of the spot risk premium. A hedging cost equation is derived which incorporates both fixed and variable components. Using this equation, it is shown that there is an increasing marginal cost of hedging. Consequently, a hedger will choose a hedge ratio which is less, maybe substantially less, than the risk minimizing hedge ratio which ignores costs. In addition, an equation is derived which reveals that the optimal hedge ratio is a positive function of the spot/futures correlation and a negative function of the spot/market correlation. It is demonstrated that these two correlations proxy for hedging benefits and costs, respectively.

Do actual hedgers operate according to the predictions of the hedging cost model? There is surprisingly little direct evidence on actual hedging activity. Virtually all empirical hedging studies focus on potential rather than on actual results. Two exceptions are Hartzmark (1987) who presents results for actual oat and wheat traders and Peck and Nahmias (1989) who present results for flour millers. Hartzmark, using 1980 data, reports an average b_m of 0.82 with an actual hedge ratio of 0.45 for oat traders. For wheat traders, the corresponding hedge ratios were 0.88 and 0.49, respectively. These results are consistent with the hedging cost model prediction that the actual hedge ratio will be less than the risk-minimizing hedge ratio. Peck and Nahmias, using flour miller data from 1964–1979, report a b_m of 1.05 and an actual hedge ratio of 0.76. Again this result is consistent with the hedging cost model. But, at best, these two studies are supportive only in the most general terms. For without the required parameter estimates, it is impossible to say how closely the hedging cost model describes actual hedging behavior. Resolution of this issue must await further research.

APPENDIX

Arbitrage and the Expected Futures Return

This Appendix is largely based on the results reported in Chapter 2 of Siegel and Siegel (1990). Consider, for example, a spot and futures market with unit prices of P_s and P_f, respectively, and insignificant transactions costs. If the spot asset can be easily purchased as well as sold short, if storage costs are zero, and if convenience yield is zero, then it is easy to show that

$$\frac{P_f}{P_s} = e^{it} \tag{A1}$$

where t equals the time to delivery, i is the constant risk-free rate, and e is the natural base.

If eq. (A1) does not hold, pure arbitrage is possible, thus arbitrageurs will ensure that it does. In such a situation, the futures market is referred to as being a "full carry" market.

The expected spot price at delivery is

$$E(\tilde{P}_{s,t}) = P_s e^{\overline{r}_s t} \tag{A2}$$

where

$$\tilde{P}_{s,t} = \text{spot price at delivery; and}$$

$$\overline{r}_s = \text{expected spot return}$$

Now using eq. (A2), P_f can be represented as

$$P_f = \frac{E(\tilde{P}_{s,d})}{e^{\overline{r}_f t}} = \frac{P_s(e^{\overline{r}_s t})}{e^{\overline{r}_f t}} \quad \text{or}$$

$$\frac{P_f}{P_s} = e^{(\overline{r}_s - \overline{r}_f)t} \tag{A3}$$

Equation (A3) is purely definitional and has nothing to do with arbitrage. Combining eq. (A1), which does depend on arbitrage, with eq. (A3)

$$e^{it} = e^{(\bar{r}_s - \bar{r}_f)t}$$

$$i = \bar{r}_s - \bar{r}_f$$

$$\bar{r}_f = \bar{r}_s - i = RP \tag{A4}$$

Equation (A4) [which is also eq. (8)] reveals that the only component making up the expected futures return is the spot risk premium. Thus, spot/futures arbitrage leads to a situation in which the spot risk premium is "passed" to the futures market.

Relax the assumption regarding zero storage cost and convenience yield and designate the storage costs as SC which includes the actual rather than the opportunity costs of storage. Further, assume that SC and CY are constant. The arbitrage free equation now becomes

$$\frac{P_f}{P_s} = e^{(i+CY-SC)t} \tag{A5}$$

Equation (A3) must now be rewritten as

$$\frac{P_f}{P_s} = e^{(\overline{\Delta} P_s - \bar{r}_f)t} \tag{A6}$$

where $\overline{\Delta} P_s$ = expected spot price change.

The holder of the spot commodity now has three sources of return: the spot price change, storage costs, and the convenience yield. Thus the expected spot return, $\bar{r}_s$, also has three components and takes the form

$$\bar{r}_s = \overline{\Delta} P_s - SC + CY = i + RP \tag{A7}$$

The latter part of this equation represents the usual market equilibrium result where the expected return on an asset is equal to the risk-free rate plus the risk premium. Rearranging eq. (A7),

$$\overline{\Delta} P_s = i + RP + SC - CY \tag{A8}$$

Of particular interest is the negative term, CY, on the right-hand side of eq. (A8). As convenience yield increases, the expected spot price change decreases. Thus, as inventories grow tight, convenience yield increases, the spot price increases relative to the futures price, and, therefore, the ratio of P_f/P_s decreases [see eq. (A6)]. This is the less than full carry situation so often observed in the futures market which is carefully analyzed by Working (1948).

Now substituting eq. (A8) into eq. (A6), and equating it to eq. (A5),

$$\bar{r}_f = RP \tag{A9}$$

Equation (A9) is identical to eq. (A4) [and eq. (8)]. That is, even in the presence of storage costs and convenience yield, the expected futures return continues to be equal to the spot price risk premium. In summary, the expected return on the futures contract is equal to RP whether or not one is operating in a "full carry" market or a storage cost and convenience yield market.

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