OPTIONS ON FUTURES SPREADS: HEDGING, SPECULATION, AND VALUATION

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INTRODUCTION

When a new futures contract is proposed, the following two questions are often asked: (1) who can benefit by trading in this contract and (2) how should this contract be priced? The purpose of this study is to examine an option on a futures spread (OFS) and to address these two questions. OFS contracts have been considered by at least three futures exchanges: The New York Mercantile Exchange (NYMEX) proposed trading an option on a crack (fuel) futures spread, the Cotton Exchange (NYCE) proposed an option on a cotton calendar spread, and the Chicago Board of Trade (CBOT) has considered trading options on the crush (soybean complex) spread. The potential expansion of the OFS contracts to other futures markets is also plausible; some widely traded spreads include the FRAC spread (propane–natural gas), the NOB spread (T-Notes–T-Bonds), and the

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TED spread (T-Bills–Eurodollars). Options on spreads also trade in over-the-counter markets.

A futures spread is defined as a strategy that calls for a simultaneous long position and short position in distinct futures contracts.[1] Spreads include intercommodity (cross-commodity), intracommodity (intermarket or time spreads), and quality spreads (e.g., sweet crude minus sour crude oil). Informally, the spread also refers to the numerical difference between the prices of two futures contracts. A call option on a futures spread is similar to a call option on a futures contract, except that the underlying "security" is a spread instead of a single contract. When the option is exercised, the exercisor takes a spread position in the futures market. The profit on exercise is the immediate mark-to-market between the exercise price and the contemporaneous futures spread.

The valuation of options on spreads is fundamentally different from the valuation of other types of options. First, the behavior of the spread is affected by the behavior of two traded contracts; a spread cannot be modeled as if it is a single asset. Second, the spread value may become negative; this possibility is disallowed in most option-pricing frameworks. Prices and indexes cannot become negative; most options are written against asset prices or indices. Third, the value of an option on a spread (like other contracts that depend on two asset values) depends critically on the covariance between the price changes of the two assets.

Therefore, options on futures spreads satisfy one aspect of speculative demand that is not satisfied by any other futures contract. A long (short) OFS position corresponds to a belief that correlation will drop (rise). Unlike single asset derivatives, the option on a spread depends on the correlation between the price changes of the long and short positions; options on futures contracts and the individual futures contracts themselves are insensitive to changes in the correlation between the contracts per se. It is also reasonable to expect that this correlation changes through time. The correlation, in general, is sensitive to the relative storability of the commodities, the seasonality of the production schedule, and other microeconomic factors. Speculators with superior forecasts of future correlations can take advantage of their information by trading OFS contracts.[2]

[1]The term *basis* is often used to describe the difference between futures and spot prices, or between two futures prices. Hence, the results might be rephrased in terms of options on a basis.

[2]This article allows time-varying correlation, but does not allow correlation to vary stochastically. As such, "trading correlation" in the model could be compared to "trading volatility" in the Black–Scholes model.

An option on a spread also performs a hedging role for a producer: his profit is the spread between output and input prices minus the cost of production, if he produces. Firms tend to produce when the OFS is in the money. Those committed to short-run output plans will hedge with futures; those with output flexibility will hedge by selling call options on futures spreads. The calls are exercised against the firm when the spread is sufficiently high, precisely when the firm produces. The options expire worthless when the firm produces nothing. In other words, the option hedge stabilizes the revenues of the firm. Since the firm owns a collection of real options on spreads, it hedges best by selling sequences of financial options on spreads.

From a valuation perspective, the option on a spread is similar to an exchange option [see Margrabe (1978) and Carr (1988)]. The option to exchange one asset for another is like an option on a spread with an exercise price of zero. The option on a spread can also be thought of as an option on a portfolio that is long one asset and short another. A simplified valuation formula has been advanced for OFS contracts by Wilcox [Goldman, Sachs, and Co. (1990)].[3] Rubinstein (1991) values the spread option using a double-integral solution in the case where both underlying futures contracts follow geometric Brownian motion.

This article develops simplified option-pricing formulae that overcome both the economic limitations of the Goldman model and the computational complexity of the Rubinstein model. The results can also be viewed as an extension of Rubinstein's in the sense that the underlying commodities are allowed to earn mean-reverting stochastic convenience yields; Rubinstein's model applied directly here would require that convenience yield be assumed constant.

It is important from a modeling perspective to allow convenience yield to vary. This generalization helps explain pricing results that appear anomalous in a constant convenience yield model. For example, stochastic convenience yields may give rise to futures pricing curves that are either monotonic or nonmonotonic in the time to maturity. Also, the convenience yield behavior may cause Black (1976) implied volatility for long-term futures options to fall above or below that of shorter-term

[3]The Goldman model uses Bachelier's (1900) formula for an option value when the underlying follows arithmetic Brownian motion. Since the spread may become negative, this approximation is appealing. Wilcox credits Bob Litterman and Scott Richard with the development of the pricing model.

futures options.[4] Brennan (1991) developed a model where convenience yield tends to a long-run positive mean level, and is inventory-dependent. Also, Gibson and Schwartz (1990) measured the convenience yield of crude oil, and found that it is both stochastic and mean-reverting. This study solves the Gibson–Schwartz commodity derivative model in closed form for futures and options on futures, and then extends the results to OFS contracts.

The organization of this article is as follows. It first answers the question, "Who uses spread options?", discussing the economics of the crack spread option, addressing the corporate hedging motives, variance-minimizing hedging strategy, and speculative considerations associated with these contracts. It then provides arbitrage pricing bounds and performs the valuation of crack spread options under the Gibson–Schwartz four state variable specification. Futures contracts and options on futures are valued first; then the option on a futures spread is valued. The valuation ultimately reduces to a single integral in the exact case, but may be readily determined using an analytic approximation formula. The development and verification of the analytic approximation is found in the appendix. The article shows some sample calculations for fuel derivative securities, and then concludes.

USES OF SPREAD OPTIONS

Spread Options as a Solution to the Firm's Hedging Problem

This section shows, in a simplified example, how OFS contracts perfectly satisfy a firm's hedging needs. General motivations for the introduction of spread option contracts are discussed without developing a comprehensive model of optimal hedging policy.

Consider a firm that converts one commodity to another at a cost. For example, an oil refiner converts crude oil into a portfolio of refined products at a marginal refining cost. Let P_O represent the value of a unit of output, P_I the value of a unit of input, and X the per-unit conversion cost. The firm has a real option to convert Q units

[4] Volatility meaning Black (1976) implied volatility. It is theoretically inappropriate to apply the Black pricing model under these conditions. However, it will be shown that a suitably transformed version of the Black model can be used to price options on futures in an environment with mean-reverting and stochastic convenience yields.

of the input good into Q units of output, subject to incurring the cost of conversion.[5]

For the moment, assume the firm operates in discrete time, and has the ability to purchase inputs, convert the inputs, and sell the outputs simultaneously. If the firm maximizes revenue, then next period's operating revenue (ignoring fixed costs) is given by

$$R = Q[\mathrm{Max}(P_O - P_I - X, 0)] \tag{1}$$

where P_O and P_I are interpreted as the contemporaneous spot prices of the input and output. If the spread exceeds the cost of production, output is produced at capacity; otherwise, nothing is produced. Importantly, the payoff resembles an option because the firm has the option not to produce. Even more important, this payoff cannot be replicated (i.e., hedged) perfectly with any combination of options on either the input or the output good. In this case, the variance-minimizing hedge consists of Q short call options on the cash spread, with a striking price equal to the marginal cost of conversion. In other words, since the company owns a real option on a spread, its best hedge is to sell forward an option on a spread.

The above justifies the existence of options on cash spreads; but why are options on futures spreads valuable? A simple answer is that futures are less costly to trade (and in particular, easier to short) than cash. Futures are, therefore, substitutes for cash. A second answer is illustrated by the following: Suppose production does not take place instantaneously, but according to the following time line:

t_0	t_1	t_2	t_3
Hedge decision	Commitment to production	Purchase of inputs	Sale of outputs

At t_1, the firm commits to a production run of Q units, as long as the firm can lock in a profit. At t_2, the input is purchased at the prevailing spot price. At t_3, the output is sold at the prevailing spot price. The minimum variance hedge at time t_1 (if the firm commits to production) is a simple futures hedge; long Q units of the input futures with maturity t_2, and short Q units of the output futures with maturity t_3. This strategy locks in the input and output prices, and therefore locks in the spread if it is positive.

[5]While the example is couched in terms of commodity firms, financial firms sometimes fall into this category as well. For example, banks obtain deposits at borrowing rates, originate loans at lending rates, and incur various costs to make this transformation. Banks may wish to hedge fluctuations in this spread, or control their exposure to the variability of the spread.

Now consider the minimum variance hedge ratio at t_0. At this time, the firm has the option not to produce; it hedges best by selling the call option on production; i.e., the firm is short Q call options on the *futures* spread, with option maturity t_1. The long leg of the spread is a futures contract on the input with maturity t_2, and the short leg of the spread is a futures contract on the output with maturity t_3. If the call is exercised at t_1, the firm produces, and pays its profit to the exercisor. If the call is not exercised, no production takes place. In either case, however, the firm retains the call option premium. The option hedge therefore eliminates all risk associated with production choice. Note that the exposure of the firm cannot be fully hedged with any other combination of cash, futures, cash options or futures options positions.[6] In this sense, the OFS contract makes the market for hedging instruments more complete; in this idealized case, it provides the only hedge with zero variance.

Spread Options as a Speculative Tool

This section shows why traders might use OFS contracts in speculative strategies without determining optimal speculative behavior or positions. To the extent that the firm or any speculator has advantaged information relative to the market, it may be in a position to speculate profitably. In any individual trade, the expected gains to trade are weighed against the losses due to increased risk. With imperfect but superior information, speculators will adopt finite speculative positions to take advantage of their information.

There are several economic variables about which speculators might have superior information. For example, if a refining firm has information about the price or supply of any of the underlying fuels, it may choose to optimally speculate with futures contracts on those fuels. If the information pertains to changes in the convenience yield or storage costs, it might be profitable to trade futures calendar spreads, since futures prices of different maturities are affected differently by changes in convenience yield and storage costs. If a speculator has information about the volatility of the spot contracts, he might use options on the individual futures contracts to speculate. In every case, the speculator seeks to trade the simplest contract that is sensitive to the variable about which he is informed. Other risks in his speculative strategy may be hedged with the more primitive securities.

Why, then, would an informed trader choose to speculate with complex options on futures spreads? Essentially, the trader must have

[6]This statement excludes dynamic trading strategies in zero transaction cost environments.

advantaged information about the *distribution* of the difference between the future prices of the two contracts. For example, a speculator may want to profit from a perceived increase in the variability in the futures spread; he must purchase options on the spread to profit if he is correct. The variance of the difference between the two contracts depends on the individual contract volatilities and on the correlation between the contracts. However, to speculate on volatility of a single contract, a trader prefers options on futures to options on spreads. Therefore, traders will use the futures spread option to speculate on the correlation between two futures prices. No other traded contract value is sensitive to the value of the correlation coefficient, and therefore no other contract provides an appropriate speculative investment vehicle for correlation traders. Beyond the correlation coefficient, trading strategies may be generated for other aspects of the spread distribution. Traders informed of the skewness or kurtosis of the spread distribution will use options on futures spreads to design optimal speculative strategies.

VALUATION

The preceding section established situations where both hedgers and speculators might use OFS contracts. Given that these contracts trade over-the-counter, and that they may trade on futures exchanges, it is desirable to know how to price these contracts. Two approaches to valuation are considered. In the first, no assumptions are made about the economics of the underlying economy; this provides irrefutable (but not necessarily very restrictive) bounds on option prices. The validity of these restrictions depends only on the absence of arbitrage. In the second approach, the underlying characteristics of two commodity markets are specified exactly, and precise pricing relationships are determined for futures, options on futures, and OFS contracts. In the second approach, it is assumed that both commodity prices follow geometric Brownian motion with mean-reverting stochastic convenience yields: this is the crude oil price process specification of Gibson and Schwartz (1990).

Arbitrage Relationships

The classical arbitrage bounds and put–call parity results can be readily applied to options on futures spreads; these results are shown below. Also, four properties unique to options on futures spreads are derived. The following notation is adopted:

C, P:	the values of European calls and puts on futures spreads
C^*, P^*:	the values of American calls and puts on futures spreads
C_L, P_L:	the values of European calls and puts on futures contract L
C_S, P_S:	the values of European calls and puts on futures contract S
τ:	the time to maturity of an option contract
r:	the constant risk-free rate of interest
$b(\tau)$:	$\exp(-r\tau)$; a risk-free discount factor
F_L:	futures price on commodity L (arbitrary maturity $\geq \tau$)
F_S:	futures price of commodity S (same maturity as L)
F:	futures spread (possibly negative, $F = F_L - F_S$)
X_L:	exercise price of option on futures L
X_S:	exercise price of option on futures S
X:	exercise price of option on futures spread (possibly negative for spread options)

Contract L refers to the long contract obtained on exercise, and S to the short contract. The call payoff is simply $\text{Max}(F_L - F_S - X, 0)$ at maturity, and the put payoff is $\text{Max}(X - (F_L - F_S), 0)$.

Classical Pricing Bounds

The basic arbitrage relationships for options on futures spreads are directly comparable to the well-known results for options on futures. An application of the results in Brenner, Courtadon, and Subrahmanyam (1985) shows that the calls and puts satisfy the following bounds conditions:

$$b\ \text{Max}(F - X, 0) \leq C \tag{2}$$

$$\text{Max}(F - X, 0) \leq C^* \tag{3}$$

$$b\ \text{Max}(X - F, 0) \leq P \tag{4}$$

$$\text{Max}(X - F, 0) \leq P^* \tag{5}$$

Equations (2)–(5) can be proved with simple dominance arguments. The put–call parity results for futures can be taken directly from Ramaswamy and Sundaresan (1985):

$$C = P + bF - bX \tag{6}$$

$$C^* \leq P^* + F - bX \tag{7}$$

Put-call parity for European options (6) can be shown by the exact equivalence of the final payoffs on both sides of the equation; eq. (7) can be verified by dominance.

Normally, the call option can be bounded above by the underlying price, and the put option by the exercise price. These usual properties do not hold for options on spreads: call (put) option prices are not bounded from above by the spread (exercise price) because the spread (exercise price) may be negative, and option values must be positive.

Results Unique to Options on Spreads

The general properties of options on futures are well known, and therefore not reviewed here. In this section, four special results for options on futures spreads are presented. Properties 1–3 can be used to place tighter bounds on spread option prices than the bounds used in the Classical Pricing Bounds section above. Property 4 highlights the distinction between volatility of a spread and the volatility of individual futures contracts, and the impact of this distinction on spread option pricing.

Property 1 The value of a call option on a futures spread is less (greater) than the value of a simple exchange option if the exercise price is positive (negative).

Comment: This statement is obvious since call options decrease in the exercise price, and an exchange option is an option on a spread with a zero exercise price. See Margrabe (1978) and Carr (1988) for the pricing of exchange options.

Property 2 $C(X) \leq \min\{C_L(X_L) + P_S(X_S)\}$ if $X_L - X_S = X$, where the minimum is taken over all $X_L \geq 0$.
The value of an option on a spread with exercise price X is less than or equal to the value of any combination of a call on the long position and a put on the short position at respective exercise prices X_L and X_S, provided $X_L - X_S = X$.

Comment: The argument follows by pure dominance. Consider any $X_L \geq 0$, with $X_S = X_L - X$. In every state where the spread option is exercised, either or both of the call and put would have been exercised. The converse is false. The option on a spread takes away a "disentanglement option" from the combination of the two options; they must be exercised simultaneously. Since the value of the disentanglement option is positive, the value of the option on a spread is less than the value of the separate options. Finally, since this is true for all X_L, it is true for the X_L that minimizes the sum of the call and put values.

Property 3 Let $\Pr(z)$ represent the certainty-equivalent probability associated with event z. If certainty-equivalent distributions exist for futures prices F_L and F_S at the European option maturity date, and the distribution has compact support with no mass points, then:

$$C(X) \leq C_L(X_L^*) + P_S(X_S^*) \tag{8}$$

where X_L^* and X_S^* are determined implicitly by

$$\Pr(F_L \geq X_L^*) = \Pr(F_S \leq X_S^*)$$

Comment: This follows from Breeden and Litzenberger (1978). The differential of a European call under the conditions above is $\partial C/\partial X = -b$ Pr(Call expires in the money). Using put-call parity, and the exercise price constraint, calculate the interior minimum. The second order conditions are satisfied trivially since the certainty-equivalent densities (proportional to the second derivative of an option price with respect to its exercise price) are positive. If in addition the certainty equivalent distributions are lognormal, a simple implicit equation can be determined for the exercise prices X_L and X_S.

Property 4 If the option value is increasing in the volatility of the spread,[7] it is decreasing in the correlation between the underlying securities.

Comment: The variance of the spread is related inversely to the covariance of the components: $\mathrm{Var}(X - Y) = \mathrm{Var}\, X + \mathrm{Var}\, Y - 2\,\mathrm{cov}(X, Y)$. Therefore, if correlation increases with individual variance unaltered, the covariance increases, the variance of the spread drops, and the option value drops.

[7]Option values do generally increase with volatility of the underlying. However, to be precise, it is possible that increases in the volatility may be accompanied by changes in other moments of the distribution of future underlying values, causing some option values to drop.

Property 1 distinguishes spread options from exchange options. Taken as a set, properties 2–4 distinguish options on futures spreads from spreads of options on futures. As a practical matter, properties 2–4 take advantage of information in futures option prices to bound OFS prices; the classical upper bounds are impossible to obtain under the assumptions used in the Classical Pricing Bounds section above.

Distribution-Dependent Pricing

In this section, more precise OFS pricing results are obtained under a more restrictive set of assumptions about the underlying commodities. A model of derivative product prices is developed here under the usual perfect market assumptions that take the behavior of the spot prices and convenience yields as exogenous. The stochastic processes used to model the spot price and convenience yield are chosen based on the theoretical arguments of Brennan (1991) and empirical findings of Gibson and Schwartz (1990). Their models admit mean-reverting spot prices in equilibrium. The Gibson–Schwartz futures pricing model is solved in closed-form, and the futures prices are used to develop values for the other contingent claims.

First, futures and options on futures for a single commodity are valued. Then, options on futures are valued so that in empirical applications, market option data might be used to recover implied parameter estimates, particularly implied volatility. After completing these single-contract valuations, the option on a futures spread is derived from its component parts.

Valuing Commodity Futures Contracts

In this section, futures contracts on a commodity that follows the Gibson–Schwartz specification are valued. This approach is appropriate when convenience yields are stochastic and mean-reverting. Gibson–Schwartz did not solve the model in closed form. The valuation problem is converted here to one that can be solved in closed form as done by Vasicek. The solution allows the valuation of options on futures contracts as well.

The price of the spot commodity (S) evolves according to a geometric Brownian motion process with drift α and volatility σ. However, the total return on the commodity position depends on price appreciation (α) plus convenience yield (δ), which is stochastic and mean-reverting.

The risk-free interest rate is assumed constant and equal to r. The long-run mean level of the convenience yield is given by μ, the speed of reversion to long-run mean is given by κ, and the instantaneous volatility of the convenience yield is given by v. The convenience yield process is correlated to the price process by a constant parameter ρ. A priori one expects that $\rho > 0$; higher convenience yield is generally associated with higher spot prices.

Formally, the model is represented as follows:

$$\begin{aligned} dS &= \alpha S\, dt + \sigma S\, dW \\ d\delta &= \kappa(\mu - \delta)\, dt + v\, dZ \\ dW dZ &= \rho dt \end{aligned} \tag{9}$$

where dW and dZ are standard Wiener processes. The convenience yield follows an Ornstein–Uhlenbeck process which allows negative short-term convenience yields. The possibility of short-term negative convenience yields is not excluded a priori, but the long-run mean convenience yield is restricted to be positive.

The pricing of futures contracts can now be determined. Let τ represent the time to maturity of futures contract, λ the market price per unit convenience yield risk (assumed constant), and $F = F(S, \delta, \tau)$ the price of a futures contract. The assumption of a constant interest rate obviates the need to distinguish between futures and forward contracts in an otherwise perfect market [Cox, Ingersoll, and Ross (1981)]. Therefore, one can derive the forward price, and take advantage of the property that the futures price equals the forward price. The forward contract satisfies the usual terminal boundary condition

$$F(S, \delta, 0) = S \tag{10}$$

In addition, standard arbitrage arguments lead to the determination of the partial differential equation for the forward price: [(cf., Gibson and Schwartz (1990)]:

$$\begin{aligned} 0 =& \tfrac{1}{2} F_{SS}\sigma^2 S^2 + F_{S\delta}\rho\sigma v S + \tfrac{1}{2} F_{\delta\delta} v^2 + F_\delta[\kappa(\mu - \delta) - \lambda v] \\ &+ (r - \delta) S F_S - F_\tau \end{aligned} \tag{11}$$

The pricing of the forward contract includes a risk premium for spot price risk and for convenience yield risk, but no compensation for the time value of money, since no capital is required to establish a futures position. In addition, the equation reflects the rate of return shortfall in the commodity price due to the convenience yield.

To solve this equation, apply the following substitution:

$$F(S, \delta, \tau) = \mathrm{S}e^{r\tau} H(\delta, \tau) \tag{12}$$

and then verify that $H(\delta, \tau)$ satisfies

$$0 = \tfrac{1}{2} v^2 H_{\delta\delta} + H_\delta[\kappa(\mu - \delta) + (\rho\sigma - \lambda)v] - \delta H - H_\tau \tag{13}$$

with the terminal boundary condition $H(\delta, 0) = 1$. This equation is similar to the discount bond pricing equation of Vasicek (1977). The difference is that the convenience yield parameters take the place of the interest rate parameters. Therefore, H is a convenience yield discount factor—the value of the long futures position is discounted because it does not enjoy the benefits of holding the physical commodity.

The solution to eq. (13) is given in closed form in Vasicek (1977):

$$H(\delta, \tau) = \exp\left[-\Delta\tau + (\Delta - \delta)U(\tau) - \frac{v^2}{4\kappa} U^2(\tau) \right]$$

$$\Delta = \mu + \frac{v(\rho\sigma - \lambda)}{\kappa} - \frac{v^2}{2\kappa^2}; \qquad U(\tau) = (1 - e^{-\kappa\tau})/\kappa \tag{14}$$

These equations, together with (12), give an analytical expression for futures prices.

As a simple check, note that the futures pricing formula reduces to the constant convenience yield case if $\delta = \mu$, $\kappa \rightarrow \infty$, and $v = \rho = 0$. (The constant convenience yield futures price is $F = \mathrm{S}\exp[(r - \delta)\tau]$.) In the constant convenience yield model, the futures price is monotonic in τ. In the stochastic convenience yield model, the futures pricing function may be increasing, decreasing, convex or concave in τ.[8] Also, futures prices depend on volatility parameters. Increasing spot volatility causes futures prices to rise (fall) if $\rho > (<) 0$. Increasing convenience yield volatility has an ambiguous effect on futures prices.

There are also some useful regularities observed by futures hedge ratios. For example, the elasticity ("beta") of futures with respect to the spot price is unity. The hedge ratio with respect to δ is $F_\delta = -\mathrm{UF} < 0$. The futures price is linear in the spot price and convex in the

[8] An extreme point (or two), if it exists is given by $\tau^* = [2\ln(v/\kappa) - \ln(-b \pm \sqrt{(b^2 + 2(\Delta - r)v^2/\kappa^2))}]/\kappa$ where $b = \Delta - \delta - v^2/[2\kappa^2]$.

convenience yield. The cross-partial ($F_{S\delta}$) is negative; the futures-spot hedge ratio drops as convenience yield rises.

Since the futures price depends on two stochastic factors, it is impossible to fully hedge one futures contract (even dynamically) with another. However, since the factors are correlated, it is possible to hedge some of the convenience yield risk via its correlation with the underlying spot price. Allowing for this correlation, the overall futures-spot elasticity is $(1 - Uv\rho/\sigma)$; the futures–futures elasticity can be calculated by taking the ratio of the two futures–spot elasticities. This calculation allows one to maximally hedge a single futures contract with another.

The next section continues the analysis of equilibrium futures prices in order to determine the pricing of options on futures contracts. Options on futures are shown to provide upper bounds for OFS prices. In addition, it will be shown that the Black (1976) implied volatility of these options plays an important role in OFS pricing.

Valuing Options on Commodity Futures

The futures price behavior derived from part (i) may be used to price options on futures. The futures price is shown here to follow a process with time-dependent variance and stochastic drift. The stochastic drift has no explicit effect on option prices, however, because the futures price incorporates the effect of the stochastic drift. The variance can simply be integrated over time to value options on futures using the Black (1976) commodity option pricing model.

The evolution of the price of the futures contract is determined below from Itô's Lemma and algebraic simplification using (14):

$$dF = \alpha_F F\,dt + \sigma_F F\,dY \tag{15}$$

where

$$\begin{aligned}\alpha_F &\equiv \alpha + \delta - \lambda v U(\tau) - r\\ \sigma_F^2 &\equiv \sigma^2 - 2\rho\sigma v U(\tau) + v^2 U^2(\tau)\\ dY &\equiv [\sigma\,dW - vU(\tau)\,dZ]/\sigma_F\end{aligned} \tag{16}$$

The drift term (α_F) can be interpreted as the drift on the spot price, plus the convenience yield, plus a time-dependent risk premium for convenience yield risk ($\lambda < 0$), minus the risk-free interest rate. The drift depends on the level of the convenience yield, but (importantly) the proportional volatility does not. The stochastic process for the futures price, therefore, reduces to a geometric Brownian motion process with time-varying volatility and stochastic drift.[9] For the purposes of option pricing, the futures drift is replaced with its certainty-equivalent growth rate of zero [Cox and Ross (1976)]. The volatility adjustment appears below.

The results follow the classic option pricing derivation of Black and Scholes (1973); the methodology for deriving the partial differential equation is not repeated here. Let $W = W(F, \tau_W)$ be the price of the call option on futures, and let τ_w be the maturity of option, with $\tau_w \leq \tau$, and $\tau_G = \tau - \tau_W$, the maturity differential between futures and options. The differential equation corresponds to the Black (1976) model for pricing European options on futures. The option price satisfies

$$\tfrac{1}{2}\sigma_F^2 F^2 W_{FF} - rW - \partial W/\partial \tau_W = 0 \tag{17}$$

This is equivalent to the Black equation except that volatility is time-varying. Using Merton (1973) one can integrate the time-varying volatility over the life of the option to define:

$$\sigma_I^2 \tau_W \equiv \int_{\tau_G}^{\tau} \sigma_F^2(s)\, ds \tag{18}$$

$$\begin{aligned}\sigma_I^2 \tau_W = {} & (\sigma^2 - 2\rho\sigma v/\kappa + v^2/\kappa^2)(\tau - \tau_G) + 2(v^2/\kappa - \rho\sigma v) \\ & \times \{\exp[-\kappa\tau] - \exp[-\kappa\tau_G]\}/\kappa^2 - v^2\{\exp[-2\kappa\tau] \\ & - \exp[-2\kappa\tau_G]\}/\{2\kappa^3\}\end{aligned} \tag{19}$$

The term σ_I is also equal to the Black implied volatility. The pricing of the option on a futures contract is the same as the Black futures option model with integrated volatility substituted for constant volatility:

$$\begin{aligned}W &= b(\tau_w)[FN(d_1) - XN(d_2)] \\ d_1 &= [\ln(F/X) + \tfrac{1}{2}\sigma_I^2 \tau_W]/[\sigma_I \sqrt{\tau_W}\,] \\ d_2 &= d_1 - \sigma_I \sqrt{\tau_w}\end{aligned} \tag{20}$$

The usual properties of this option pricing formula are well known. In the constant convenience yield model, implied volatility does not vary with option or futures maturity. In the stochastic model, Black implied volatility (σ_L) is not necessarily constant; nor is it simply

[9]This result is sensitive to the choice of stochastic process for δ. In particular, if the variance of the convenience yield depends on the level of the convenience yield, then the volatility of the futures contract will also depend on the level of the convenience yield.

a monotonic function of time. Intuitively, implied volatility includes both spot volatility and convenience yield volatility; the strength of the second factor depends on the time to maturity of the futures contract. As time to maturity increases, the integrated spot variance increases linearly, but the integrated convenience yield variance approaches a constant. At first, for small τ, the convenience yield volatility may increase enough to cause implied volatility to increase, but eventually, implied volatility must decrease with time to maturity. Therefore, the implied volatility structure may be decreasing or humped. Furthermore, the relationship between implied volatility and ρ, the correlation between the spot price and convenience yield, is nonmonotonic for some parameter choices.

In general, it is not appropriate to calculate implied volatility from one model (i.e., the Black model) when pricing from another model (i.e., one with stochastic convenience yield). However, the implied volatility has an important interpretation here. The next section shows how the implied volatility can be used as an input to price OFS contracts. The dependence of implied volatility on time to maturity admits a more robust model for valuing OFS than would be possible with a constant convenience yield model.

Valuing Options on Futures Spreads

In this section, the European option on a futures spread is evaluated.[10] The futures pricing system may be rewritten using the equations below. L designates the long position obtained on exercise, and S designates the short position. From this point forward, parameter values are subscripted with L or S to designate whether they apply to the long or the short position.

The futures pricing system satisfies:

$$
\begin{aligned}
dF_L/F_L &= \alpha_{FL}\,dt + \sigma_{FL}\,dY_L; \qquad F_L(0) = f_L \\
dF_S/F_S &= \alpha_{FS}\,dt + \sigma_{FS}\,dY_S; \qquad F_S(0) = f_S \\
dW_L dW_S &= \rho(F_L, F_S)\,dt
\end{aligned}
\tag{21}
$$

[10]The value of the early exercise feature for American options is not explicitly examined in this study. In the case of options on futures, early exercise may take place because of the rate of return shortfall in holding the futures contract. The rate of return shortfall equals the risk-free rate, since no capital is required to establish the position. Therefore, futures options are often exercised early. For the purposes of this article, it is assumed that the early exercise feature is not important for pricing. However, for the sake of completeness, the value of this feature should be examined.

The drift and variance terms are defined in eq. (16). The correlation coefficient is determined from the primitive correlation relationships. Invoking (16) again, one obtains:

$$\begin{aligned}\sigma_{FL}\,dY_L &= \sigma_L dW_L - v_L U_L(\tau)dZ_L \\ \sigma_{FS}\,dY_S &= \sigma_S dW_S - v_S U_S(\tau)dZ_S\end{aligned} \tag{22}$$

The product of the two terms is the instantaneous covariance (since dY is normalized to unit variance):

$$\begin{aligned}\rho(F_L,F_S)\sigma_{FL}\sigma_{FS} &= \sigma_L\sigma_S\rho(S_L,S_S) - \sigma_L v_S U_S(\tau)\rho(S_L,\delta_S) \\ &\quad - \sigma_S v_L U_L(\tau)\rho(S_S,\delta_L) + v_L v_S U_L(\tau)U_S(\tau)\rho(\delta_L,\delta_S)\end{aligned} \tag{23}$$

Equation (23) can be used to identify the instantaneous futures correlation, $\rho(F_L,F_S)$, since σ_{FL} and σ_{FS} are known.

Let $C = C(F_L, F_S, \tau_C; X)$ represent the value of the call option on a futures spread with exercise price X. Following Margrabe (1978), under the usual perfect market assumptions, one can replicate the option on a spread (like the exchange option) by continuously adjusting positions in F_L and F_S. It is straightforward to derive the partial differential equation for the value of the option on a spread:

$$\begin{aligned}rC &= \tfrac{1}{2}\,\sigma_{FL}^2 F_L^2 C_{LL} + \rho\sigma_{FL}\sigma_{FS}F_L F_S C_{LS} \\ &\quad + \tfrac{1}{2}\,\sigma_{FS}^2 F_S^2 C_{SS} - \partial C/\partial\tau_C \\ C(F_L,F_S,0;X) &= \mathrm{Max}(F_L - F_S - X, 0)\end{aligned} \tag{24}$$

This equation is identical to that of Margrabe, except that (1) $X \neq 0$ and (2) the variance and covariance coefficients are time-varying.

To simplify (24), one can again invoke Merton (1973). The volatility and covariance are time-varying, but not state-varying, implying that variance and covariance can be integrated with respect to time. Recognizing this, the solution to eq. (24) is the same as the constant coefficient solution, but with integrated second moments substituted for constant second moments.

The integrated variance coefficient is used to calculate futures option prices; the integrated variance is simply the implied variance multiplied by the option maturity. The integrated covariance term is shown below:

$$\begin{aligned}\text{cov} = [&\sigma_L\sigma_S\rho(S_L,S_S) - \sigma_L v_S\rho(S_L,\delta_S)/\kappa_S - \sigma_S v_L\rho(S_S,\delta_L)/\kappa_L \\ &+ v_L v_S\rho(\delta_L,\delta_S)/(\kappa_L\kappa_S)]\tau_W \\ &- v_S\phi_S(\tau)/\kappa_S[\sigma_L\rho(S_L,\delta_S) - v_L\rho(\delta_L,\delta_S)/\kappa_L] \\ &- v_L\phi_L(\tau)/\kappa_L[\sigma_S\rho(S_S,\delta_L) - v_S\rho(\delta_S,\delta_L)/\kappa_S] \\ &- \phi_{S+L}(\tau)[v_L v_S\rho(\delta_L,\delta_S)]/[\kappa_L\kappa_S]\end{aligned}$$

where

$$\phi(\tau) = [\exp(-\kappa\tau) - \exp(-\kappa\tau_G)]/\kappa$$

and the κ terms are κ_S, κ_L, and $\kappa_S + \kappa_L$ for ϕ_S, ϕ_L, and ϕ_{S+L} respectively. The integrated correlation, ρ_I can then be simply determined from the integrated second moments:

$$\rho_I = \text{cov}/[\sigma_{IL}\sigma_{IS}\tau_W] \tag{26}$$

In the constant convenience yield model, the correlation coefficient is constant. In the stochastic model, the integrated correlation coefficient may be a nonmonotonic function of time. This gives rise to a "correlation structure" that is used to price OFS contracts. Ultimately, shifts in the correlation structure are critical determinants of changes in OFS contract prices.

The spread option pricing problem now reduces to the solution of the following p.d.e., which is equivalent to (24) but has coefficients that are treated as constant:

$$\begin{aligned} rC &= \tfrac{1}{2}\sigma_{IL}^2F_L^2C_{LL} + \rho_I\sigma_{IL}\sigma_{IS}F_LF_SC_{LS} \\ &\quad + \tfrac{1}{2}\sigma_{IS}^2F_S^2C_{SS} - \partial C/\partial\tau_C \\ C(F_L,F_S,0;X) &= \text{Max}(F_L - F_S - X, 0) \end{aligned} \tag{27}$$

One solution to this valuation problem is offered in Rubinstein (1991). Recognizing that the p.d.e. solution is equivalent to the expected terminal value of the option under the risk-neutral joint density of the futures prices, he states that:

$$C = b(\tau_C)\int_{F_L=0}^{\infty}\int_{F_S=0}^{\infty}\text{Max}(F_L - F_S - X, 0)P(F_L,F_S)\,dF_L dF_S \tag{28}$$

$P(F_L, F_S)$ is the joint lognormal density under the certainty-equivalent futures growth rates of zero. The terms F_L and F_S are abbreviated forms of $F_L(\tau)$ and $F_S(\tau)$. This double-integral solution can be simplified into a single integral solution. Using Bayes' rule and reversing the order of integration, one can also express the integral as the following:

$$C = \int_{F_S=0}^{\infty} \left[b(\tau_C) \int_{F_L=0}^{\infty} \mathrm{Max}(F_L - (F_S + X), 0) P(F_L \mid F_S)\, dF_L \right] \times P(F_S)\, dF_S \tag{29}$$

The term in brackets corresponds to the Black formula for options on futures. However, the distribution of F_L is conditioned on the realization of F_S, and the exercise price is taken to be $(F_S + X)$. To evaluate the term in brackets, the conditional density of F_L given F_S must be derived. The conditional density is lognormal, since the conditional density of the logarithm is normal. Therefore, the Black–Scholes functional can be applied to simplify the expression in brackets. The distributions of $\ln(F_L)$ and $\ln(F_S)$ along with the projection of $\ln(F_L)$ on $\ln(F_S)$ is described in eq. (30). $n = n(\mu, \sigma)$ is the normal density for a random variable with mean μ and variance σ^2. The symbol "$\sim^*$" is read "is distributed according to the certainty-equivalent distribution".

$$\begin{aligned}
\ln(F_L) &\sim^* n(\ln(f_L) - \tfrac{1}{2}\sigma_{FL}^2 \tau_C, \sigma_{FL}\sqrt{\tau_C}) \\
\ln(F_S) &\sim^* n(\ln(f_S) - \tfrac{1}{2}\sigma_{FS}^2 \tau_C, \sigma_{FS}\sqrt{\tau_C}) \equiv n(\mu_S, g_S) \\
\ln(F_L) \mid \ln(F_S) &\sim^* n(\mu_C, \sigma_C\sqrt{\tau_C}) \\
\mu_C &= \ln(f_C) - \tfrac{1}{2}(\sigma_{FL}^2 - \rho\sigma_{FL}\sigma_{FS})\tau_C \\
\delta_C &= r - \tfrac{1}{2}\rho\sigma_{FL}\sigma_{FS} + \tfrac{1}{2}\rho^2\sigma_{FL}^2 \\
f_C &= f_L(F_S/f_S)^{\beta} \\
\beta &= \rho\sigma_{FL}/\sigma_{FS} \\
\sigma_C^2 &= \sigma_{FL}^2(1 - \rho^2)
\end{aligned} \tag{30}$$

The distributions apply to the expiration date of the spread option. f_L and f_S are the time 0 values of F_L and F_S. The term in brackets can now be written analytically using well-known Black–Scholes operator:

$$\begin{aligned}
[..] &\equiv G(F_S) \equiv f_C \ \exp(-\delta_C\tau_C)N(s_1) - (F_S + X) \ \exp(-r\tau_C)N(s_2) \\
s_1 &= [\ln(f_C/(F_S + X)) + (r - \delta_C + \tfrac{1}{2}\sigma_C^2)\tau_C]/[\sigma_C\sqrt{\tau_C}] \\
s_2 &= s_1 - \sigma_C\sqrt{\tau_C}
\end{aligned} \tag{31}$$

This is the Black–Scholes formula as applied to the conditional density of F_L given F_S. The marginal density of F_S is also given by a lognormal density

$$P(F_S) = \exp[-\tfrac{1}{2}((\ln(F_S) - \mu_S)/g_S)^2]/[F_S g_S\sqrt{2\pi}] \qquad (32)$$

The combination of (29) with (31) and (32) gives a single-integral solution for the value of the spread option in the base valuation case:

$$C = \int_{F_S=0}^{\infty} G(F_S)P(F_S)\,dF_S \qquad (33)$$

This single-integral expression may be used to compute spread option prices more efficiently than possible with a double integral calculation.

This solution for the pricing of OFS contracts may be computed more quickly than the Rubinstein solution; the approximation in the next section is even faster. However, the contribution of the model extends beyond the speed of its application. The OFS model presented here allows convenience yield to be stochastic and mean-reverting and that this stochastic problem can be converted into one with constant convenience yield. Values from the implied volatility structure and the implied correlation structure are incorporated as inputs in the OFS valuation formula.

An Analytic Approximation

When speed of evaluation is important, it is worthwhile to consider analytic approximations to option prices. Following the techniques of Jarrow and Rudd (1982) one can approximate (33) with the following formula:

$$\begin{aligned} C(F) &= \exp(-r\tau_C)[\sqrt{\mu_2}\,d\,N(d) + n(d)\{\sqrt{\mu_2} - \kappa_3 d/[6\mu_2] \\ &\quad + \kappa_4(d^2/\mu_2 - 1)/[24\mu_2^{3/2}]\}] \\ d &= (F_L - F_S - X)/\sqrt{\mu_2} \qquad (34) \end{aligned}$$

The terms μ_2, κ_3, and κ_4 are defined in the appendix. The approximation uses the normal distribution as the approximating distribution, since the normal distribution accounts for possibly negative spread values. The first two terms are the same as the Bachelier/Goldman option pricing terms, except that they are tied to the underlying economic assumptions of the model presented here. The third term is an adjustment for the skewness of the spread distribution relative to the normal, and the fourth term is an adjustment for the kurtosis of the spread distribution. To apply this model, use the Black implied volatility (σ_{IL} and σ_{IS}) for σ_L and σ_S, and ρ_I for ρ in (A4). Calculate the moments μ_2 to μ_4, κ_3 and κ_4, and calculate the call price from (A9). Sample calculations from this formula are shown in the next section. Approximation errors are less than 1/32 if $\sqrt{\mu_2}$ does not exceed 3.74.

NUMERICAL ILLUSTRATIONS

Empirical testing of the model is left to future research due to the inaccessibility of exchange-traded OFS prices. Nevertheless, it would be possible to examine the behavior of the underlying commodities. For example, Gibson–Schwartz used a time series approach to test the commodity specification for crude oil. The pricing formulae for futures and options developed here could be used to find implied parameter values from cross-sectional data. If the model is perfectly correct, then all implied values would remain constant across time periods. Unfortunately, without exchange-traded OFS prices, it would be impossible to understand the behavior of implied correlation over time. Therefore, using either of these two methods, one could test the pricing specification in principle without testing the OFS pricing formula per se.

In this section, a single set of calculations is shown for parameters chosen to roughly match model and market prices on July 12, 1992 (an arbitrarily chosen date), for fuel futures and options contracts. This simulation is included for three reasons: (1) to give potential users of this complex model an opportunity to check their price and implied value calculations, (2) to illustrate the departure of the pricing results from the constant convenience yield model, and (3) to demonstrate numerically the impact and significance of implied volatility and correlation changes in OFS pricing.

As an example, consider the option on the heating oil crack spread. Upon exercise of this call option on this spread, the holder of the option undertakes a long position in heating oil futures and a short position in crude futures. The base-case assumptions used are shown in Table I. A futures contract with 4 months remaining is shown. Options on futures expire approximately 2 weeks prior to the futures, which expire near the end of the month preceding the contract month. Options on crack spreads proposed by the NYMEX expire the same time as the options on futures. Because of different reporting conventions for heating oil (price per gallon) and crude oil (price per barrel), it is necessary to define the spread as 42 times the quoted heating oil price minus the quoted crude price. The standard reporting conventions are maintained throughout the example to facilitate comparison to market prices.

TABLE I

Options on Futures Spreads: Assumptions

Futures maturity (years)	τ			0.3726
Option maturity gap (years)	τ_G			0.0274
Interest rate (risk-free)	r			5.00%
Parameter		*Heating Oil*		*Crude Oil*
Multiplier		42		1
Spot price	S	\$0.6190		\$21.5000
Volatility of spot price	σ	8.00%		10.00%
Convenience yield	δ	12.00%		8.50%
Long-run mean	μ	2.00%		4.00%
Speed of mean-reversion	κ	7.00		10.00
Volatility of convenience	v	6.00%		5.00%
Price of convenience risk	λ	−1.00		−1.50
Correlations				
Own Spot-Own Conv	$\rho(S, \delta)$	0.90		0.90
Own Spot-Other Conv		0.30		0.30
Conv-conv	$\rho(\delta_L, \delta_S)$		0.20	
Spot-spot	$\rho(S_L, S_S)$		0.70	

Table II shows futures, options, and OFS calculations for the base case example. Futures and futures option prices are computed from eqs. (12) and (20). For the sake of comparison, the implied convenience yields for the constant model are shown. Both values fall between the short-term and long-run convenience yield. The integrated volatility is equivalent to the (implied) volatility used in pricing the options on futures according to the Black model.

The OFS calculation shows an implied correlation figure of 0.7395. For these parameter values, fluctuations in the implied correlation are driven mostly by changes in the spot–spot correlation. The raw standard deviation measure is calculated as $\sqrt{\mu_2}$, the standard deviation of the terminal certainty-equivalent distribution. Relative to a normal distribution with the same mean and variance, the skewness of the CEQ distribution is slightly negative (−0.0207), and its kurtosis is slightly positive (0.0162). Because of the small departure from normality, the Jarrow–Rudd approximations to the OFS prices shown are all accurate within \$0.0007 of the exact integral price.

For some values of the convenience yield, the futures prices are non-monotonic in time to maturity (this is shown in Fig. 1 for heating oil). In models with constant convenience yield, futures prices increase

TABLE II

Futures Prices and Implied Volatilities

		Heating Oil	Crude Oil
Futures prices		\$0.6164	\$21.4388
Implied convenience yield		6.14%	5.76%
Implied volatilities (Black)	σ_I	7.48	9.65%

Futures Option Prices

Heating Oil Strike	Calls	Puts	Crude Oil Strike	Calls	Puts
0.52	0.0948	0.0000	18	3.3803	0.0003
0.54	0.0751	0.0000	19	2.4038	0.0067
0.56	0.0556	0.0001	20	1.4756	0.0614

Options on the Futures Spread

		Heating Oil–Crude Oil
Implied correlation	ρ_I	0.7395
Implied raw standard deviation	$\sqrt{\mu_2}$	0.8531
Skewness relative to normal	κ_3	−0.0207
Kurtosis relative to normal	κ_4	0.0162

Strike	Calls	Puts
0	4.3746	0.0060
1	3.3917	0.0047
2	2.4095	0.0040
3	1.4423	0.0183
4	0.6023	0.1599
5	0.1305	0.6696
6	0.0111	1.5318
7	0.0003	2.5025
8	0.0000	3.4838

(decrease) in maturity if $r > (<)\delta$. Figure 2 shows that implied volatilities decline in option maturity as the convenience yield distribution approaches its long-run stationary distribution. Note that increases in the convenience yield volatility cause the implied volatility to *decline*. This counter-intuitive result occurs because the futures price itself also adjusts to the change in the volatility of the convenience yield; implied volatility is a function of the futures price.

While the results are not shown here, different levels of the spot–spot correlation can give rise to implied correlation structures that are both increasing (for high correlation levels) and decreasing (for low correlation levels).

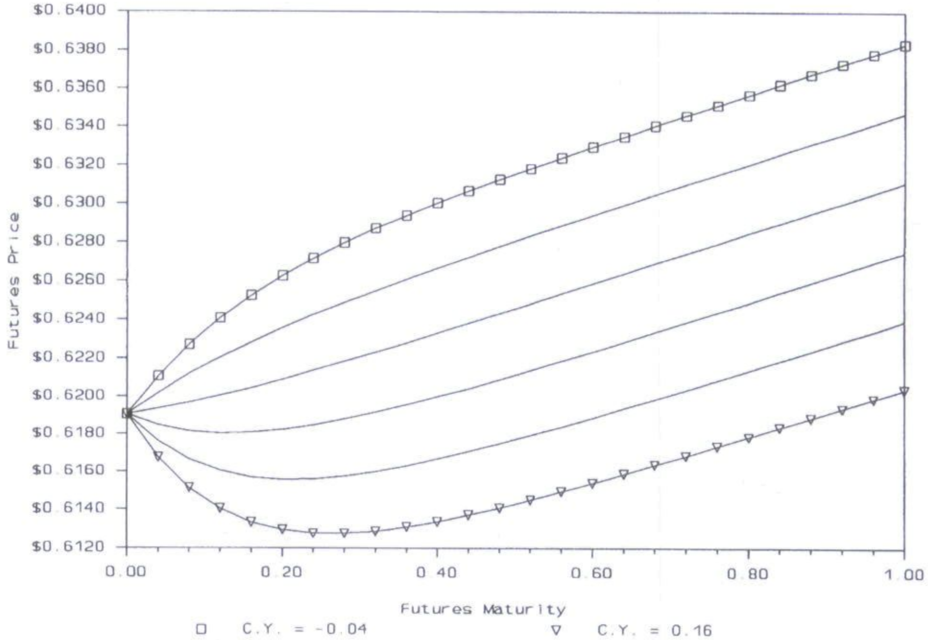

FIGURE 1
Futures pricing with random convenience (other parameters in text).

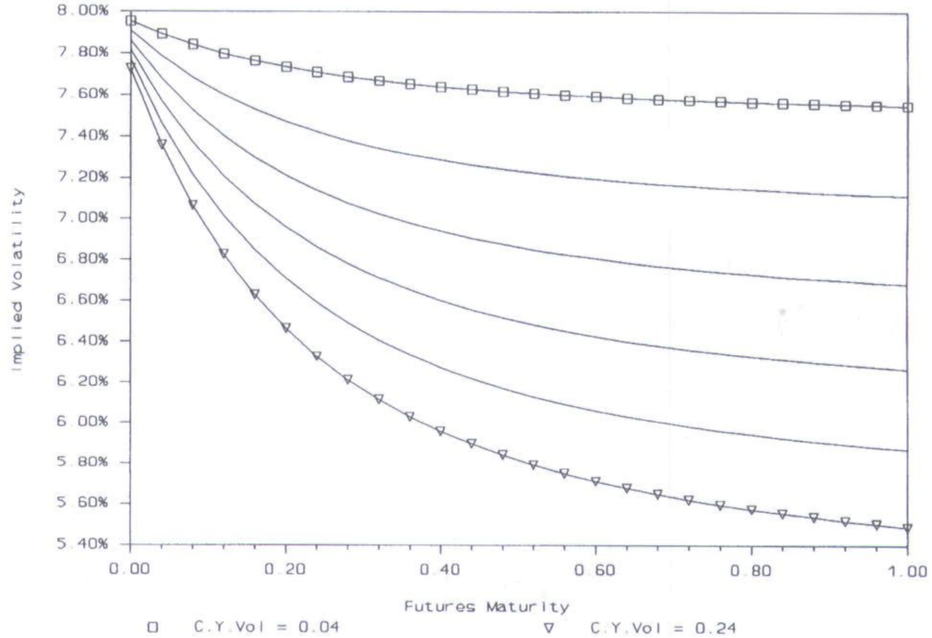

FIGURE 2
Implied volatility with random convenience (other parameters in text).

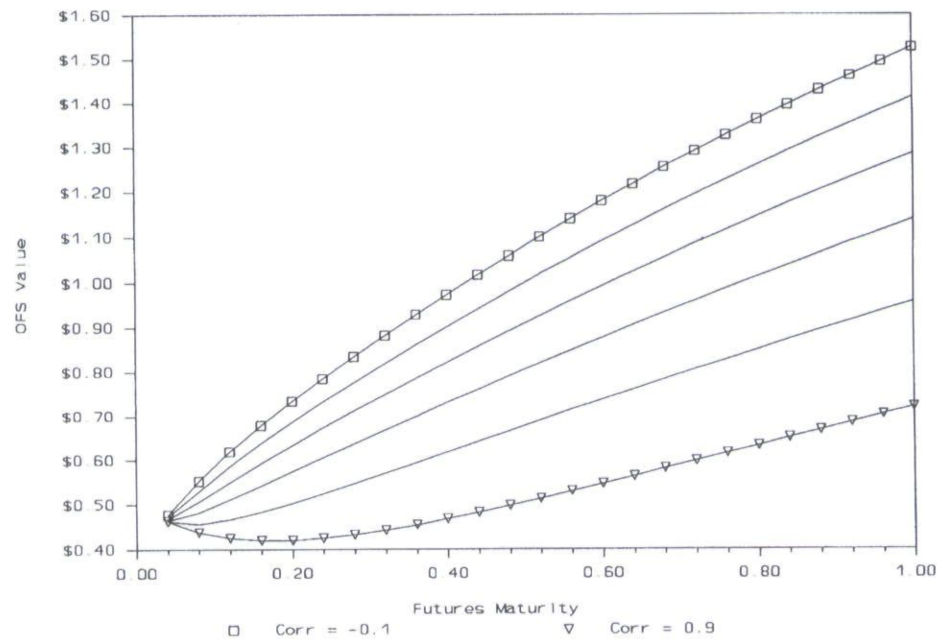

FIGURE 3
Effect of correlation on OFS prices (near-the-money strike; $X = \$4.00$).

Correlation has a dramatic effect on OFS prices, as Figure 3 shows. An OFS contract with a $4.00 striking price (close to the money) is used to show how its value varies with different levels of the spot–spot correlation. The price variation in some cases, due to misspecified correlation, can exceed 50%. This is good news for speculators who think they have superior knowledge of the correlation coefficients. The option on a spread represents an effective means to "trade correlation."

CONCLUSION

This article studies options on futures spreads, examines their uses, and derives pricing relationships. These options can be used as hedging instruments for firms that convert one commodity to another at a cost, and have some output flexibility. The options can also be used to speculate on knowledge of the distribution of the difference between two futures prices at some future date. In particular, one could speculate on the variance of the spread, which is sensitive to the correlation between the two futures contracts. This speculative strategy cannot be replicated with any (currently) publicly traded securities.

In terms of pricing, a technique is implemented that is more comprehensive from an economic perspective than the Goldman pricing model, and that can be transformed for computational purposes to the Rubinstein (1991) model. The computational complexity of the Rubinstein model is reduced by deriving a single integral solution and an analytic approximation for the OFS price. As an extension to Rubinstein's results, the model takes into account the frequently observed property of commodities that convenience yield is stochastic and mean-reverting. Therefore, a four-state variable system (two stochastic commodity prices and two stochastic convenience yields) for the complete panel of futures, futures options, and spread options prices is solved.

Future research may entail applications to different markets, theoretical improvements to the model and empirical validation of the pricing results. While the application of the OFS model to stochastic convenience yields is appropriate for application to commodity markets in particular, it may also be used in markets where dividends are stochastic (options on stock index futures spreads) and where interest rates are stochastic (options on currency futures spreads). Theoretical extensions to this work might examine the American option component explicitly, and allow the volatility of the spot prices to vary stochastically. Also, the effect of stochastic interest rates, since they are correlated with fuel prices, might have an impact on futures and option pricing. Empirical extensions to this work might include the recovery and study of implied parameter values from historical data and model prices. Finally, when options on futures spreads start to trade, it will be interesting to document the behavior of the implied correlation between the futures contracts through time.

APPENDIX

Analytic Approximation to the Value of an Option on a Spread

In this appendix, the analytic approximation to eq. (33) is calculated. Note that the value of the drift (α) is not restricted below. If the underlying contract is a futures contract, then $\alpha = 0$, while, in general, α is the certainty-equivalent growth rate. This allows one to generalize these results to any option on a spread where the components follow geometric Brownian motion processes.

The Approximating Distribution

First, an option is valued assuming a normal distribution for the terminal value of the underlying contract. The normal distribution is then used as an approximating distribution for the true distribution of the spread values. This result was first discovered by Bachelier (1900), and has been applied to options on spreads by Wilcox et al. (1990).

Let L follow an arithmetic Brownian motion process with certainty-equivalent drift α and volatility σ ($dL = \alpha dt + \sigma dW$). A call option on index L has exercise price X and time to maturity τ. Assume the risk-free interest rate is a constant (r). The call can be valued in equilibrium (not arbitrage since L is a non-traded asset) as:

$$C = \sigma\sqrt{\tau}\ \exp(-r\tau)[d\,N(d) + n(d)] \tag{A1}$$

where $d = [L_0 + \alpha\tau - X]/[\sigma\sqrt{\tau}]$, $N(.)$ is the cumulative Gaussian (normal) density, and $n(.)$ is the marginal Gaussian density.

The mean of the certainty-equivalent distribution for L is $(L_0 + \alpha\tau)$. The variance is $\sigma^2\tau$. The skewness is zero, and the adjusted kurtosis is zero. The skewness is defined as the third central moment, and the adjusted kurtosis is defined as the difference between the fourth central moment and thrice the square of the second central moment. See Jarrow and Rudd [1982, eq. (3)].

The True Distribution

Define two correlated geometric Brownian motion processes:

$$\begin{aligned} dL &= \alpha_L L\,dt + \sigma_L L\,dW_L \\ dS &= \alpha_S S\,dt + \sigma_S S\,dW_S \\ dW_L dW_S &= \rho\,dt \end{aligned} \tag{A2}$$

The drift rates should be interpreted as the certainty-equivalent drift rates. It is useful to calculate (and define) the following expectation:

$$\begin{aligned} m_{kj} \equiv E[L^k S^j] = [L_0^k S_0^j]\ \exp[\{k(\alpha_L + \tfrac{1}{2}(k-1)\sigma_L^2) \\ + j(\alpha_S + \tfrac{1}{2}(j-1)\sigma_S^2 + jk\rho\sigma_L\sigma_S\}\tau] \end{aligned} \tag{A3}$$

To verify this result, it is sufficient to show that $L^k S^j$ follows a lognormal process with constant coefficients; the expectation is simply the initial value times the exponential of the drift term.

The function m follows a recursive relationship in both k and j:

$$
\begin{aligned}
m_{00} &= 1 \\
m_{kj} &= m_{k-1,j} L_0 \exp[\{\alpha_L + (k-1)\sigma_L^2 + j\rho\sigma_L\sigma_S\}\tau] \\
m_{kj} &= m_{k,j-1} S_0 \exp[\{\alpha_S + (j-1)\sigma_S^2 + k\rho\sigma_L\sigma_S\}\tau]
\end{aligned} \tag{A4}
$$

The moments of the future difference between L and S determine the departure of the spread distribution from normality, and hence the accuracy of the analytic approximation. Let the difference between L and S be designated as Z. Then the mean of Z is $m = m_{10} - m_{01}$. The central moments up to order four are shown below:

$$
\begin{aligned}
\mu_2 &= (m_{20} - 2m_{11} + m_{02}) - m^2 \\
\mu_3 &= (m_{30} - 3m_{21} + 3m_{12} - m_{03}) - 3m(m_{20} - 2m_{11} + m_{02}) + 2m^3 \\
\mu_4 &= (m_{40} - 4m_{31} + 6m_{22} - 4m_{13} + m_{04}) - 4m(m_{30} - 3m_{21} \\
&\quad + 3m_{12} - m_{03}) + 6m^2(m_{20} - 2m_{11} + m_{02}) - 3m^4
\end{aligned} \tag{A5}
$$

The skewness and adjusted kurtosis are determined to be

$$
\begin{aligned}
\kappa_3 &= \mu_3 \\
\kappa_4 &= \mu_4 - 3\mu_2^2
\end{aligned} \tag{A6}
$$

Application of the Jarrow–Rudd (1982) Technique

From eqs. (6) and (11) in Jarrow and Rudd (1982), the value of the call option is approximately given by:

$$
\begin{aligned}
C(F) \approx C(A) &- \exp(-r\tau)[\kappa_3(F) - \kappa_3(A)]/3!\, dA(X)/dS_t \\
&+ \exp(-r\tau)[\kappa_4(F) - \kappa_4(A)]/4!\, d^2A(X)/dS_t^2
\end{aligned} \tag{A7}
$$

where F is the true distribution, A is the approximating distribution, κ_3 is the third central moment, κ_4 the adjusted kurtosis, X the exercise price, and S_t the terminal index value. In other words, the value approximation is the call value under the normal distribution, $C(A)$, plus adjustments for the departure of the true distribution from the skewness (κ_3) and kurtosis (κ_4) of the normal distribution.

The first two moments of the true and approximating distributions are matched. To accomplish this, define the initial value of Z, $Z_0 =$

$L_0 - S_0$. Also require that

$$\mu_2 = \sigma^2 \tau \quad \text{and}$$
$$\alpha = (m - Z_0)/\tau \qquad \text{(A8)}$$

Then, the call option formula in eq. (A1) can be used as the approximating formula. The general formula is specialized as follows:

$$C(F) = \exp(-r\tau)[\sqrt{\mu_2}\, d\, N(d) + n(d)\{\sqrt{\mu_2} - \kappa_3 d/[6\mu_2]$$
$$+ \kappa_4(d^2/\mu_2 - 1)/[24\mu_2^{3/2}]\}]$$
$$d = (L_0 - S_0 - X)/\sqrt{\mu_2} \qquad \text{(A9)}$$

where the F subscript on κ_3 and κ_4 has been suppressed, since the higher adjusted normal moments are zero.

To apply this model, use the Black implied volatility for σ_L and σ_S, and ρ_I for ρ in (A4). Calculate the moments μ_2 to μ_4, κ_3 and κ_4, and calculate the call price from (A9). L corresponds to the long position, and S to the short position taken upon exercise.

The instantaneous hedge ratio approximations are analytic but difficult, since the κ terms depend on the initial values of the futures prices. Also, while the hedge ratio with relation to the spread size is informative, the hedge ratio with respect to each of the individual futures contracts is not, since the contract price movements are correlated.

Accuracy of the Analytic Approximation

In this section, the accuracy of eq. (A9) in approximating the OFS valuation eq. (33) is examined. The development of (33) shows that the stochastic convenience yield model can be transformed into a jointly lognormal model. This observation is important here because it limits the sources of approximating error to five: the parameters of the joint lognormal distribution for the two underlying futures values.

The sources of approximating error may be reduced to two by studying the moment functions in eqs. (A5). For example, increases in the long variance ($\sigma_L^2 \tau$) have a symmetric effect on the moments of the spread distribution as increases in the short variance ($\sigma_S^2 \tau$). This translates into symmetric impacts on the valuation formula, and, therefore, on the approximating error. Also, increases in the covariance term ($\rho\sigma_L\sigma_S$) have exactly the opposite effect as the variance increases.

Therefore, the approximating error uncertainty with respect to the second moments can be spanned by a single factor. The standard deviation of the spread distribution, $\sqrt{\mu_2}$ is chosen as the single summary factor for the second-order moments.

For the first-order moments, the effect of changes in the short futures and long futures have symmetric (and opposite) effects. Therefore, the concept of moneyness, i.e., $F_L - F_S - Xe^{-r\tau}$, is used as a summary measure for approximation error due to the first order moments. Plotting the error (Integral–Approximation) over suitable ranges of moneyness and spread volatility, Figure A1 shows the pricing errors that obtain using the approximation formula. The outermost curve corresponds to a value of $\sqrt{\mu_2} = 3.74$. The curves show both increasing positive and negative error as $\sqrt{\mu_2}$ grows. Figure A1 shows that as long as $\sqrt{\mu_2}$ is below 3.74, one can expect the pricing error to be bounded by \$0.025, less than 1/32. The effect of time can be incorporated by noting that μ_2 increases linearly with time. For example, the approximation would be equally accurate for a 1-year OFS with an annual spread standard deviation of 3.74 and a 3-month OFS with an annual spread standard deviation of 7.48.

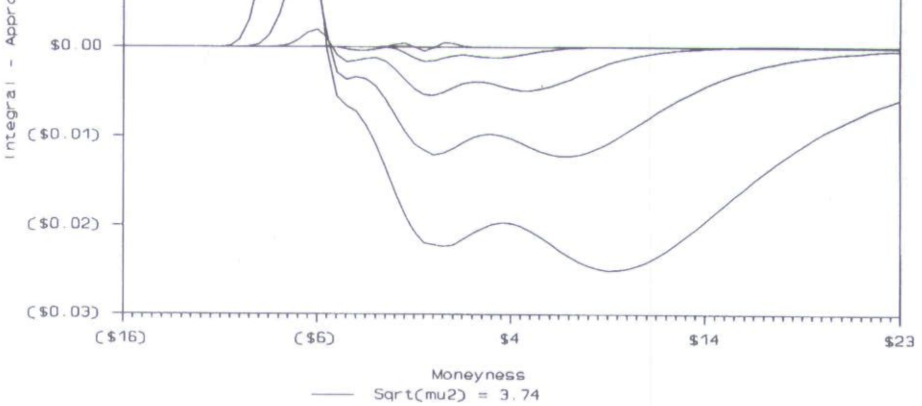

FIGURE A1
Accuracy of Jarrow–Rudd approximation (option on Crack Futures Spread).

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