ANALYTIC APPROXIMATION OF THE OPTIMAL EXERCISE BOUNDARIES FOR AMERICAN FUTURES OPTIONS

IN JOON KIM

A call (put) option on a futures contract gives its owner the right to assume a long (short) position in the futures contract at the exercise price at or before expiration. Since their introduction in 1982, markets for futures options have experienced significant growth, both in terms of trading volume and available contracts. Options are now traded on the futures contracts on stock market indexes; on the futures contracts on Treasury instruments; on the futures contracts on foreign currencies; and on the futures contracts on some commodities. The valuation of these options, aside from its theoretical appeal to academician, is of considerable interest to investors and traders who wish to hedge their positions in the underlying futures contracts.

In a seminal article, Black (1976) provided a complete description of forward and futures contracts, as well as a framework for the valuation of European options on forward contracts and on spot commodities. He derived a closed-form valuation formula for European calls on forward contracts. Although he focused on European options

I am grateful to Marti Subrahmanyam, Stephen Figlewski and Stephen Brown for helpful comments. Any remaining errors are my own.

■ *In Joon Kim is an Assistant Professor in the Department of Management & Policy at Korea Advanced Institute of Science and Technology.*

The Journal of Futures Markets, Vol. 14, No. 1, 1–24 (1994)

CCC 0270-7314/94/010001-24

on forward contracts, his results apply to European options on futures contracts as well, if the interest rate is constant. American options on futures contracts are more difficult to value analytically than European options. Ramaswamy and Sundaresan (1985); Brenner, Courtadon, and Subrahmanyam (1985); and Ball and Torous (1986) established that American options on futures contracts are subject to early exercise even in the absence of dividend payments from the spot underlying the futures contracts as long as the interest rate is positive.[1] For this reason, Black's formula cannot be used, and no analytic solution has been available for American options written on futures contracts.

Ramaswamy and Sundaresan (1985) and Brenner, Courtadon, and Subrahmanyam (1985) pointed out that the determination of the optimal exercise boundary is an important element in the valuation of American options on futures contracts. A correct understanding of the optimal exercise boundary is crucial for the valuation of American options. The optimal exercise boundary is defined by the critical futures price, above which it is optimal to exercise the American calls on futures contracts. Ramaswamy and Sundaresan (1985) and Brenner, Courtadon, and Subrahmanyam (1985) employed the implicit-finite difference method in Brennan and Schwartz (1977, 1978) to determine simultaneously the optimal exercise boundary and the value of American options on futures contracts. Barone-Adesi and Whaley (1987) and Whaley (1986) employed the quadratic approximation of MacMillan (1986) to value early-exercise premiums associated with American options on futures contracts. This list is only partial, but it illustrates techniques available in the literature to deal with the early-exercise feature of American options.

Recently Kim (1990) derived analytic valuation formulas for American options written on assets paying continuous proportional dividends by considering an option exercisable at a finite number of points in time and evaluating its continuous limit as the time interval shrinks to zero. The derivation of valuation formulas was possible by formulating the valuation problem in economically and mathematically meaningful ways. Valuation formulas for American options on futures contracts can be obtained as a special case of Kim's formulas. Although the analytic solutions provide better intuition for the economic meaning of American options than numerical methods, the implementation technique suggested by Kim is not necessarily more efficient than finite-

[1] If the interest rate is zero, American call options on futures contracts would not be exercised prematurely.

difference methods. The purpose of this article is to present analytic valuation formulas for American options on futures contracts and to develop a computationally efficient technique to implement them.

The article is organized as follows. In the next section, the results in Kim (1990) are adapted to the valuation problem of American calls on futures contracts. Based on the properties of the optimal exercise boundary, the precise specification of the valuation problem is presented and a valuation formula is derived by utilizing the results in Kolodner (1956) and McKean (1965). In the second section, American puts on futures contracts are examined. Numerical methods to implement the valuation formulas are then discussed and a technique is developed that is shown to be as accurate as the implicit finite-difference method, but is computationally simple and efficient. The precise understanding of the optimal exercise boundary helps in the development of a useful approximation technique. The conclusions appear in the last section.

AMERICAN CALLS ON FUTURES CONTRACTS

Consider an American call on a futures contract that has an exercise price of K and that expires at time T. The underlying futures contract expires at or after time T.[2] The usual conditions are assumed that the markets are perfect with continuous trading. The futures price F is assumed to follow a lognormal diffusion process with volatility parameter σ, and the interest rate r is a positive constant. In this simplified setting, futures prices would be identical to forward prices as shown by Cox, Ingersoll, and Ross (1981). Black (1976); Ramaswamy and Sundaresan (1985); and Brenner, Courtadon, and Subrahmanyam (1985) examined the valuation of options on futures contracts in this setting.

Denote the value of a *live* American call on a futures contract at time t for the optimal exercise boundary $G(\cdot)$ by $C(F, \tau; G(\cdot))$, where $\tau = T - t$ is time to expiration. The American call function $C(F, \tau; G(\cdot))$ is differentiable with respect to τ and twice differentiable with respect to F, defined on domain $\{(F, \tau); 0 < F < G(\cdot), 0 < \tau \leq T\}$. The optimal exercise boundary $G(\cdot)$ is a continuous function of time to expiration and represents the futures price *above* which American calls are exercised optimally.[3] The function of $C(S, \tau; G(\cdot))$ represents the value of the American call conditional on the futures price being below $G(\cdot)$. If the

[2]If options and the underlying futures contract expires on the same date, a European option on a futures contract is equivalent to a European option on spot.

[3]See Van Morebeke (1976) for a proof of continuity of the optimal exercise boundary.

futures price is above the optimal exercise boundary, the American call is dead and its value is defined to be $(F - K)$. Black (1976) has derived the partial differential equation that governs the value of European calls on futures contracts. The value of American futures options satisfies the same partial differential equation:

$$\frac{1}{2}\sigma^2 F^2 C_{FF} - rC = C_\tau \tag{1}$$

This equation defines the discounted conditional expectation under the risk-neutral futures price process which is governed by a lognormal diffusion process with a zero drift and volatility parameter σ. By comparing eq. (1) and the proportional dividend model of Merton (1973), one can see that options on futures contracts are equivalent to options on assets with continuous dividend yield equal to the risk-free interest rate. Ball and Torous (1986) have shown that options on a futures contract are equivalent to options on a futures bond which would be viewed as an asset with continuous dividend yield equal to the risk-free interest rate in this setting. Equation (1) holds even if the commodity or the instrument underlying the futures contract pays dividends or a service stream as long as the futures price follows a lognormal diffusion process with volatility parameter σ.

An American call will be exercised at expiration if it has not been exercised early and if the futures price is above the exercise price at expiration. Kim (1990) has shown that $G(0) = K$ for American futures calls, where $G(0)$ represents the limit of $G(\tau)$ as τ tends to zero.[4] Note that it is impossible for the futures price at expiration to be above K without crossing the optimal exercise boundary earlier. This means that the *live* American calls on the futures contract will have a value of zero at expiration.

$$\lim_{\tau \to 0} C(F, \tau : G(\cdot)) = 0 \tag{1a}$$

In addition to the expiration condition, the complete characterization of the valuation problem requires the following boundary conditions:

$$\lim_{F \to G(\tau)} C(F, \tau; G(\cdot)) = G(\tau) - K \tag{1b}$$

$$\lim_{F \to 0} C(F, \tau; G(\cdot)) = 0 \tag{1c}$$

$$\lim_{F \to G(\tau)} C_F = 1 \tag{1d}$$

[4]See the Appendix for a less rigorous but intuitive proof in a binomial setting.

The value of American calls at the time of exercise is equal to the immediate exercise value by definition, as specified in the boundary condition (1b). The boundary condition (1c) is a regularity condition that results from the fact that zero is an absorbing barrier for a lognormal diffusion process. The boundary condition (1d) is imposed to ensure the optimality of the exercise boundary.[5]

The free-boundary problem stated in terms of the partial differential eq. (1) subject to (1a)–(1d) has been examined by Kolodner (1956) and McKean (1965). A solution is obtained by applying their results to this problem.[6] The value of an American call is expressed as the sum of the European call value and the early-exercise premium.[7]

$$C(F,\tau;G(\cdot)) = c(F,\tau) + \int_0^{\tau} re^{-r(\tau-s)}[F\aleph(d_1(F,\tau-s;G(s))) - K\aleph(d_2(F,\tau-s;G(s)))]\,ds \tag{2}$$

Here $c(F,\tau)$ represents Black's European call pricing formula given by

$$c(F,\tau) = Fe^{-r\tau}\aleph(d_1(F,\tau;K)) - Ke^{-r\tau}\aleph(d_2(F,\tau;K))$$

where $\aleph(\cdot)$ is the unit normal distribution function and

$$d_1(F,\tau-s;K) = \frac{\ln(F/K) + \frac{1}{2}\sigma^2(\tau-s)}{\sigma\sqrt{\tau-s}}$$

$$d_2(F,\tau-s;K) = d_1(F,\tau-s;K) - \sigma\sqrt{\tau-s}$$

The second term of the right-hand side of eq. (2) represents the exact functional form for the early-exercise premium, which is approximated by a power function in the quadratic approximation of MacMillan (1986) and Barone-Adesi and Whaley (1987). The early-exercise premium is equivalent to the interest on a portfolio of European calls, each expiring in $\tau - s$ and exercisable at K if the futures price is above $G(s)$. These European calls are not ordinary options. Although their payoff is $(F - K)$ upon exercise, early exercise is allowed only when the futures price is above $G(s)$. It is easy to see that as the interest rate tends to zero, the early-exercise premium vanishes and American calls become worth more alive than dead for any futures price.

[5]This condition is known as the "high-contact condition" [Merton (1973)] or the "smooth-pasting condition".

[6]The function $C(F,\tau)$ is analytic in F and τ.

[7]See the Appendix for the derivation.

It is interesting to observe that the value of a futures option appears to be unaffected by any payout from the asset underlying the futures contract. However, note that the greater the payout from the asset, the lower the futures price. In other words, although the relationship between the value of a futures option and the futures price is independent of the payout from the asset, the value of the futures option is lower, the greater the payout from the asset.

If the futures price reaches the optimal exercise boundary for the first time, it is optimal to exercise the American call. By imposing the boundary condition (1b) on eq. (2), the following equation is obtained that implicitly defines the optimal exercise boundary.[8]

$$G(s) - K = c(G(s), s) + \int_0^s re^{-r(s-\xi)}[G(s)\aleph(d_1(G(s), s-\xi; G(\xi))) - K\aleph(d_2(G(s), s-\xi; G(\xi)))]\,d\xi \tag{3}$$

When $r = 0$, a solution to eq. (3) is $G(s) = \infty$, which implies that premature exercise would not occur. This is consistent with the implication of eq. (2). To compute the value of calls using eq. (2), the optimal exercise boundary $G(s)$ must be determined first by solving eq. (3). Note that any payout from the asset underlying the futures contract does not affect the optimal exercise boundary, although the probability of exercise decreases as the payout increases.

AMERICAN PUTS ON FUTURES CONTRACTS

American put options can be valued in the same way as American calls. Consider an American put that has an exercise price of K and that expires at time T. Denote the value of a *live* American put for the optimal exercise boundary $B(\cdot)$ by $P(F, \tau; B(\cdot))$ that is differentiable with respect to τ and twice differentiable with respect to F, defined on domain $\{(F, \tau); B(\cdot) < S \leq \infty, 0 < \tau \leq T\}$. The optimal exercise boundary $B(\cdot)$ is defined as the time dependent futures price *below* which it is optimal to exercise the American puts. The analytic solution can be found by formulating the American put valuation problem as a free-boundary problem. The mathematical formulation in the previous section allows expression of the value of a *live* American put as the sum of an early-

[8]See the Appendix for the derivation.

exercise premium and the value of an equivalent European put:

$$P(F,\tau;B(\cdot)) = p(F,\tau) + \int_0^{\tau} re^{-r(\tau-s)}[K\aleph(-d_2(F,\tau-s;B(s))) - F\aleph(-d_1(F,\tau-s;B(s)))]\,ds \tag{4}$$

where $p(F,\tau)$ represents Black's European put pricing formula given by

$$p(F,\tau) = Ke^{-r\tau}\aleph(-d_2(F,\tau;K)) - Fe^{-r\tau}\aleph(-d_1(F,\tau;K))$$

The optimal exercise boundary $B(\cdot)$ is implicitly defined by

$$K - B(s) = p(B(s),s) + \int_0^{s} re^{-r(s-\xi)}[K\aleph(-d_2(B(s),s-\xi;B(\xi))) - B(s)\aleph(-d_1(B(s),s-\xi;B(\xi)))]\,d\xi \tag{5}$$

Equation (4) indicates that when the interest is zero, the early-exercise privilege is worthless and the American put is equivalent to a comparable European put. In this case, eq. (5) also implies that the optimal exercise boundary is zero for any s so that premature exercise cannot be optimal. This is consistent with the intuition of this study.

A NUMERICAL METHOD

Once the optimal exercise boundary is determined, the value of American options on futures contracts can be computed by numerical integration of eq. (2) or (4), which is a relatively simple task. The difficulty associated with implementing the valuation formulas stems from the fact that it is not possible to obtain the optimal exercise boundary in explicit form for finitely lived American futures options from the integral eq. (3) or (5), although it can be determined numerically. The focus of this section is on numerical techniques to solve eq. (3) or (5).

Note that s appears in both the integrand and the upper integral limit in eq. (3). This means that eq. (3) for each s represents a separate integral equation. Kim (1990) suggested solving integral equations recursively. Define

$$\phi(G(s),s-\xi;G(\xi)) = re^{-r(s-\xi)} \times [G(s)\aleph(d_1(G(s),s-\xi;G(\xi))) - K\aleph(d_2(G(s),s-\xi;G(\xi)))]$$

Now divide τ into n subintervals s_i for $i = 1, 2, \ldots, n$ so that $\tau = s_n$ and $s_0 = 0$. Assume $s_k - s_{k-1} = \Delta t$ for any k. Integrals are approximated using the trapezoidal rule. $G(s_1)$ is determined by

$$G(s_1) - K = c(G(s_1), s_1) + \frac{\Delta t}{2}[\phi(G(s_1), s_1; G(0)) + \phi(G(s_1), 0; G(s_1))]$$

Once $G(s_1)$ is found, $G(s_2)$ is determined by

$$\begin{aligned} G(s_2) - K = c(G(s_2), s_2) + \frac{\Delta t}{2}[&\phi(G(s_2), s_2; G(0)) \\ &+ 2\phi(G(s_2), s_2 - s_1; G(s_1)) + \phi(G(s_2), 0; G(s_2))] \end{aligned}$$

Work backward recursively to solve for $G(s_n)$.

$$\begin{aligned} G(s_n) - K = c(G(s_n), s_n) + \frac{\Delta t}{2}\Bigg[&\phi(G(s_n), s_n; G(0)) \\ &+ \sum_{i=1}^{n-1} 2\phi(G(s_n), s_n - s_i; G(s_i)) + \phi(G(s_n), 0; G(s_n))\Bigg] \end{aligned}$$

This recursive procedure in which the determination of $G(s_k)$ requires knowing $G(s_i)$ for $i = 1, 2, \ldots, k - 1$ is similar to that used to compute the critical stock price for Geske and Johnson's (1984) formula.

This method is employed to solve eq. (3) and obtain a numerical solution for the optimal exercise boundary $G(s_i)$. Then, using that numerical solution, eq. (2) is numerically integrated to compute the theoretical value of American calls on futures contracts. The results are presented in Table I. Values of 6-month and 3-year calls on futures contracts obtained by varying parameter values for the volatility and the interest rate are shown. Three-year calls on futures contracts are presented to examine the behavior of eqs. (2) and (3) for a wide range of cases. Longer term options are traded in organized exchanges as well as OTC markets and their importance is drawing some attention. For comparisons, the values of the futures calls computed using the implicit finite-difference method are included in the table. As expected, both methods yield identical results for the short-term and the long-term calls on futures contracts. The implicit finite-difference method relies on grid spacing in two-dimensional space (futures price and time to maturity) while numerical solution to eqs. (2) and (3) needs discretization in only one dimension (time to expiration). No definite statement can be made

TABLE I
Comparisons of the Numerical Methods for the Valuation of American Calls

		$\tau = 0.5$		$\tau = 3$	
Parameters	*Futures Price*	*Finite Diff.*[a]	*Eqs. (2) & (3)*[b]	*Finite Diff.*[a]	*Eqs. (2) & (3)*[b]
$r = 8\%$, $\sigma = 0.20$	90	1.71	1.71	7.25	7.25
	95	3.24	3.24	9.33	9.33
	100	5.46	5.46	11.70	11.70
	105	8.37	8.37	14.37	14.37
	110	11.90	11.90	17.32	17.32
$r = 8\%$, $\sigma = 0.30$	90	3.86	3.86	12.52	12.52
	95	5.78	5.78	14.89	14.89
	100	8.18	8.19	17.46	17.46
	105	11.05	11.05	20.22	20.22
	110	14.34	14.34	23.16	23.16
$r = 10\%$, $\sigma = 0.20$	90	1.70	1.70	6.97	6.97
	95	3.22	3.22	8.99	8.99
	100	5.42	5.42	11.31	11.31
	105	8.32	8.32	13.93	13.93
	110	11.83	11.83	16.84	16.84
$r = 10\%$, $\sigma = 0.30$	90	3.82	3.83	12.05	12.05
	95	5.74	5.74	14.36	14.36
	100	8.12	8.13	16.87	16.87
	105	10.97	10.97	19.58	19.58
	110	14.25	14.25	22.47	22.47
$r = 12\%$, $\sigma = 0.20$	90	1.68	1.68	6.71	6.71
	95	3.19	3.19	8.68	8.68
	100	5.38	5.38	10.95	10.95
	105	8.26	8.26	13.52	12.52
	110	11.77	11.77	16.40	16.40
$r = 12\%$, $\sigma = 0.30$	90	3.79	3.79	11.62	11.63
	95	5.69	5.69	13.88	13.88
	100	8.07	8.07	16.34	16.34
	105	10.90	10.90	18.99	18.99
	110	14.16	14.16	21.85	21.85

[a]The value of American calls with the exercise price of $100 computed using the implicit finite-difference method.
[b]The numerical integration of eq. (2) given the numerical solution for the optimal exercise boundary obtained by numerically solving eq. (3).

on the relative accuracy of the two numerical methods because it is not yet possible to evaluate the effect of discretization errors in determining $G(s)$ on the value of American options. However, simulation results indicate that the implicit finite-difference method requires finer grid spacing and, hence, more computer time than this numerical method for the same degree of accuracy.

AN APPROXIMATION METHOD

As indicated by Kim (1990), the numerical method presented above requires evaluating the $(i + 1)$th integral equation over the entire range up to s_{i+1} to determine the optimal exercise boundary at s_{i+1}. This is why it is computationally expensive to solve n integral equations by a recursive procedure. The numerical procedure is simplified by developing an approximate function for the optimal exercise boundary. As an approximation for $G(s)$, the following functions are employed based on the observation that the optimal exercise boundary is nondecreasing in time to expiration and that it starts at K and approaches the optimal exercise boundary for perpetual calls on assets paying continuous dividends at the risk-free interest rate as time to expiration increases.

$$G(s; a, b) = G(\infty) + (K - G(\infty)) \exp(-as^b) \tag{6}$$

where a and b are constants, and $G(\infty)$ represents the critical futures price for perpetual American calls defined by[9]

$$\begin{aligned} G(\infty) &= \frac{\beta K}{\beta - 1} \\ \beta &= \frac{\frac{1}{2}\sigma^2 + \sqrt{\frac{1}{4}\sigma^4 + 2\sigma^2 r}}{\sigma^2} \end{aligned} \tag{6a}$$

In eq. (6), a and b are constants that depend only on the parameters of the model, i.e., r and σ. The approximate exercise policies given by eq. (6) satisfy the properties described in Kim (1990).[10] Note that eq. (6) represents the exact optimal exercise boundary for perpetual American calls since $G(s)$ approaches $G(\infty)$ as s tends to infinity. Equation (6) is also consistent with the fact that the optimal exercise boundaries for finitely lived calls on futures contracts are lower than for perpetual American calls with the same exercise price. The parameters a and b are

[9]Note that the critical futures price for perpetual American calls is a time invariant constant. The critical futures price for perpetual American calls can be obtained by solving eq. (3). [See Kim (1990) for the derivation.] It might not make economic sense to consider futures contracts that never mature because the futures price would not be defined for such contracts. However, it is theoretically possible to consider perpetual futures contracts written on spot instruments that pay dividends at the same rate as the interest rate. In this case, the futures price would be equal to the spot price. Since such futures contracts are equivalent to spot instruments, perpetual American options on futures contracts can be viewed as perpetual American options written on assets paying dividends at the same rate as the interest rate.

[10]As r approaches zero, $G(\infty)$ increases without limit, indicating that early exercise is not optimal for perpetual American calls when $r = 0$.

determined in two-step least-squares regressions. In the first step, eq. (3) is numerically solved for a large number of parameter configurations to determine numerically the function $G(s; r, \sigma)$. Then, the right-hand side of (6) is linearized by rearranging terms and taking logarithms and linear regressions are employed to estimate a and b. This yields the parameters a and b for each combination of r, σ. It is found that R^2 for the first step regressions are greater than 0.997 for all cases. Finally, a and b are regressed against r, r^2, σ and σ^2 to obtain the following equations:[11]

$$a = 0.28577 + 10.961r - 26.521r^2 - 0.73902\sigma + 0.50624\sigma^2$$

$$b = 0.45755 - 0.24552r + 1.3934r^2 + 1.2185\sigma + 0.074965\sigma^2$$

An R^2 of 0.996 and 0.986 respectively is obtained for the second stage of regressions. These equations together with eqs. (2) and (6) represent an approximation technique for the valuation of American calls on futures contracts.

Equation (6) is employed in the numerical integration of eq. (2) to compute the theoretical value of American calls. Numerical results are reported in Tables II and III. Table II contains the theoretical values of 3-, 6-, and 9-month American calls on futures contracts obtained from eqs. (2) and (6) as well as those from the implicit finite-difference method. Table II shows that eqs. (2) and (6) provide call values that are as accurate as the implicit finite-difference method.[12] To facilitate comparisons, call values that are computed using the quadratic approximation of MacMillan (1986) and Barone-Adesi and Whaley (1987) are presented also.[13] Their technique is accurate for 3-month calls. As time to expiration increases, however, the call values obtained from their approximation technique are shown to deviate from the accurate values. This is not surprising because the early exercise premium is small for short-term options. For example, the early exercise premium is less than \$0.014 for the 3-month out-of-the-money calls (K = \$90 and \$95) for both $\sigma = 0.2$ and 0.3. Table III presents the theoretical values of 3-year American calls on futures contracts. As Barone-Adesi and Whaley (1987) recognized, the quadratic approximation does not provide accurate values for long-term

[11] This approximate exercise boundary could be useful to improve computational efficiency in the implicit finite-difference method. If the optimal exercise boundary is known in advance, one need not search for the critical stock price in each recursive time step of the implicit finite-difference method.

[12] These values are not sensitive to a small change in coefficients.

[13] See Barone-Adesi and Whaley (1987) for comparisons between the quadratic approximation and Geske and Johnson's (1984) technique.

TABLE II
Comparisons of the Approximation Formulas for the Valuation of American Calls

Parameters	Futures Price	European Option Value	Finite Diff.[a]	Eqs. (2) & (6)[b]	Quad. Approx.[c]
$r = 10\%$, $\sigma =$ 0.20, $\tau = 0.25$	90	0.70	0.70	0.70	0.70
	95	1.84	1.85	1.85	1.85
	100	3.89	3.91	3.91	3.91
	105	6.89	6.94	6.94	6.94
	110	10.68	10.79	10.79	10.78
$r = 10\%$, $\sigma =$ 0.30, $\tau = 0.25$	90	1.97	1.98	1.98	1.98
	95	3.57	3.59	3.59	3.60
	100	5.83	5.86	5.86	5.87
	105	8.73	8.78	8.78	8.79
	110	12.19	12.29	12.29	12.29
$r = 10\%$, $\sigma =$ 0.20, $\tau = 0.50$	90	1.69	1.70	1.70	1.71
	95	3.19	3.22	3.22	3.24
	100	5.36	5.42	5.42	5.44
	105	8.20	8.32	8.32	8.33
	110	11.62	11.83	11.83	11.84
$r = 10\%$, $\sigma =$ 0.30, $\tau = 0.50$	90	3.79	3.82	3.83	3.85
	95	5.68	5.74	5.74	5.77
	100	8.04	8.12	8.12	8.16
	105	10.83	10.97	10.97	11.00
	110	14.03	14.25	14.24	14.27
$r = 10\%$, $\sigma =$ 0.20, $\tau = 0.75$	90	2.53	2.56	2.56	2.60
	95	4.19	4.26	4.26	4.30
	100	6.40	6.52	6.52	6.57
	105	9.14	9.35	9.35	9.39
	110	12.35	12.70	12.69	12.73
$r = 10\%$, $\sigma =$ 0.30, $\tau = 0.75$	90	5.20	5.28	5.28	5.33
	95	7.21	7.33	7.32	7.39
	100	9.59	9.77	9.76	9.83
	105	12.32	12.58	12.58	12.65
	110	15.37	15.75	15.74	15.81

[a] The value of American calls with the exercise price of $100 computed using the implicit finite-difference method.
[b] The numerical integration of eq. (2) under the approximate exercise boundary given by eq. (6).
[c] Solutions using the quadratic Approximation method of MacMillan (1986) and Barone-Adesi and Whaley (1987).

calls. However, even for the long-term calls, the approximation technique used here is shown to be quite accurate for a wide range of parameter values. Errors are at most $0.02 for the cases considered in the table. Compared to the quadratic approximation, eq. (2) together with eq. (6) represents an improvement in valuing American calls on futures contracts in terms of accuracy without the high computational cost associated with the finite-difference methods. The accuracy of the implicit finite-difference method and numerical integration depends

TABLE III
Comparisons of the Approximation Formulas for the Valuation of American Calls

Parameters	*Futures Price*	*European Option Value*	*Finite Diff.*[a]	*Eqs. (2) & (6)*[b]	*Quad. Approx.*[c]
$r = 8\%$, $\sigma =$ 0.20, $\tau = 3$	90	6.81	7.25	7.25	7.54
	95	8.69	9.33	9.33	9.64
	100	10.82	11.70	11.70	12.03
	105	13.16	14.37	14.36	14.70
	110	15.71	17.31	17.31	17.64
$r = 8\%$, $\sigma =$ 0.30, $\tau = 3$	90	11.69	12.52	12.51	12.96
	95	13.83	14.89	14.88	15.35
	100	16.13	17.46	17.45	17.94
	105	18.57	20.22	20.21	20.71
	110	21.14	23.16	23.16	23.66
$r = 10\%$, $\sigma =$ 0.20, $\tau = 3$	90	6.41	6.97	6.96	7.32
	95	8.19	8.99	8.98	9.37
	100	10.19	11.31	11.30	11.71
	105	12.40	13.93	13.92	14.33
	110	14.80	16.84	16.82	17.23
$r = 10\%$, $\sigma =$ 0.30, $\tau = 3$	90	11.01	12.05	12.04	12.59
	95	13.02	14.36	14.35	14.93
	100	15.19	16.87	16.86	17.46
	105	17.48	19.58	19.56	20.18
	110	19.91	22.47	22.46	23.08
$r = 12\%$, $\sigma =$ 0.20, $\tau = 3$	90	6.04	6.71	6.70	7.11
	95	7.71	8.68	8.67	9.11
	100	9.59	10.95	10.94	11.40
	105	11.67	13.52	13.51	13.98
	110	13.94	16.40	16.39	16.85
$r = 12\%$, $\sigma =$ 0.30, $\tau = 3$	90	10.37	11.62	11.62	12.24
	95	12.27	13.88	13.87	14.52
	100	14.30	16.34	16.33	17.01
	105	16.47	18.99	18.98	19.68
	110	18.75	21.85	21.83	22.54

[a]The value of American calls with the exercise price of $100 computed using the implicit finite-difference method.
[b]The numerical integration of eq. (2) under the approximate exercise boundary given by eq. (6).
[c]Solutions using the quadratic Approximation method of MacMillan (1986) and Barone-Adesi and Whaley (1987).

on the number of grid points which determines computer time. The quadratic approximation contains errors that do not decrease with computer time. The numerical integration of eq. (2) is approximately one hundred times faster than the implicit finite-difference method for penny accuracy, while it requires 20% more computer time than the quadratic approximation.

The delta or the hedge ratio of an option is defined as the partial derivative of the option value with respect to the price of the underlying asset. It tells how much the option value will change for a small change in the price of the underlying asset. The hedge ratio is of practical importance for investors who wish to hedge their position in the underlying asset. The derivative of an approximation formula is usually less accurate than the formula itself. This is a limitation of using approximation formulas to compute the hedge ratio. Given the approximate optimal exercise boundary, eq. (2) allows the computation of the accurate hedge ratio efficiently from a formula. The formula is obtained by differentiating eq. (2) with respect to the futures price.[14]

$$C_F = e^{-r\tau}\aleph(d_1(F,\tau;K)) + \int_0^{\tau} re^{-r(\tau-s)} \times \left[\aleph(d_1(F,\tau-s;G(s))) + \frac{(G(s)-K)}{F\sigma\sqrt{\tau-s}}n(d_2(F,\tau-s;G(s)))\right]ds \quad (7)$$

where $n(\cdot)$ is the standard normal density function. Table IV reports the hedge ratios computed from the implicit finite-difference method and compares them with those obtained from eq. (7) and the quadratic approximation. Using numerical integration in eq. (7) with the optimal exercise boundary approximated by eq. (6) is as accurate as the implicit finite-difference method. The quadratic approximation method does not provide the accurate hedge ratios even for the short-term in-the-money calls.

The same approximation method can be employed for American puts where the optimal exercise boundary is approximated by the following function:[15]

$$B(s;c,d) = B(\infty) + (K - B(\infty))\exp(-cs^d) \quad (8)$$

where $B(\infty)$ represents the critical futures price for perpetual American puts defined by[16]

$$B(\infty) = \frac{\theta K}{\theta - 1}$$

$$\theta = \frac{\frac{1}{2}\sigma^2 - \sqrt{\frac{1}{4}\sigma^4 + 2\sigma^2 r}}{\sigma^2} \quad (8a)$$

[14]Other expressions are possible.

[15]R^2 for the first step regressions are greater than 0.999 for all cases.

[16]The critical futures price for perpetual American puts is obtained by solving eq. (5). The critical futures price for perpetual American puts $B(\infty)$ represents the lower bound for the optimal exercise boundary $B(\cdot)$ for finitely lived American puts with otherwise similar terms.

TABLE IV

Comparisons of the Approximation Formulas for the Hedge Ratio (American Calls)

Parameters	*Futures Price*	*Finite Diff.*[a]	*Eqs. (6) & (7)*[b]	*Quad. Approx.*[c]
$r = 10\%$, $\sigma = 0.20$, $\tau = 0.25$	90	0.154	0.154	0.154
	95	0.315	0.315	0.314
	100	0.511	0.511	0.507
	105	0.696	0.695	0.688
	110	0.837	0.836	0.822
$r = 10\%$, $\sigma = 0.30$, $\tau = 0.25$	90	0.260	0.260	0.259
	95	0.387	0.387	0.385
	100	0.521	0.521	0.517
	105	0.646	0.646	0.640
	110	0.753	0.752	0.743
$r = 10\%$, $\sigma = 0.20$, $\tau = 0.50$	90	0.240	0.240	0.238
	95	0.371	0.371	0.367
	100	0.511	0.511	0.503
	105	0.644	0.644	0.630
	110	0.758	0.758	0.736
$r = 10\%$, $\sigma = 0.30$, $\tau = 0.50$	90	0.335	0.335	0.332
	95	0.430	0.430	0.425
	100	0.525	0.525	0.517
	105	0.614	0.613	0.602
	110	0.694	0.693	0.678
$r = 10\%$, $\sigma = 0.20$, $\tau = 0.75$	90	0.284	0.284	0.280
	95	0.395	0.395	0.388
	100	0.510	0.510	0.498
	105	0.620	0.619	0.600
	110	0.718	0.717	0.688
$r = 10\%$, $\sigma = 0.30$, $\tau = 0.75$	90	0.370	0.370	0.365
	95	0.449	0.449	0.441
	100	0.526	0.526	0.514
	105	0.599	0.599	0.583
	110	0.666	0.666	0.644

[a]The hedge ratio for American calls with the exercise price of $100 computed using the implicit finite-difference method.
[b]The numerical integration of eq. (7) under the approximate exercise boundary given by eq. (6).
[c]Solutions using the quadratic Approximation method of MacMillan (1986) and Barone-Adesi and Whaley (1987).

The following estimates are obtained for the parameters c and d with R^2 of 0.999 and 0.998, respectively.

$$c = 0.44634 + 8.1139r - 16.473r^2 + 0.82102\sigma + 0.48361\sigma^2$$

$$d = 0.43023 + 0.23611r - 0.32363r^2 - 0.11062\sigma + 0.063546\sigma^2$$

Tables V and VI compare numerical methods for computing the theoretical value of American puts. Table V presents the theoretical values of 3-, 6-, and 9-month American puts on futures contracts obtained

TABLE V
Comparisons of the Approximation Formulas for the Valuation of American Puts

Parameters	Futures Price	European Option Value	Finite Diff.[a]	Eqs. (4) & (8)[b]	Quad. Approx.[c]
$r = 10\%$, $\sigma =$ 0.20, $\tau = 0.25$	90	10.45	10.56	10.56	10.55
	95	6.72	6.77	6.77	6.77
	100	3.89	3.91	3.91	3.91
	105	2.01	2.02	2.02	2.03
	110	0.93	0.93	0.93	0.94
$r = 10\%$, $\sigma =$ 0.30, $\tau = 0.25$	90	11.73	11.83	11.82	11.82
	95	8.45	8.51	8.50	8.51
	100	5.83	5.86	5.86	5.87
	105	3.85	3.87	3.87	3.87
	110	2.44	2.45	2.45	2.45
$r = 10\%$, $\sigma =$ 0.20, $\tau = 0.50$	90	11.20	11.42	11.42	11.42
	95	7.94	8.07	8.06	8.08
	100	5.36	5.43	5.42	5.44
	105	3.44	3.47	3.47	3.49
	110	2.10	2.12	2.12	2.14
$r = 10\%$, $\sigma =$ 0.30, $\tau = 0.50$	90	13.31	13.53	13.52	13.54
	95	10.44	10.59	10.58	10.61
	100	8.04	8.13	8.13	8.16
	105	6.07	6.14	6.13	6.16
	110	4.51	4.55	4.55	4.58
$r = 10\%$, $\sigma =$ 0.20, $\tau = 0.75$	90	11.81	12.16	12.15	12.18
	95	8.83	9.04	9.04	9.08
	100	6.40	6.52	6.52	6.57
	105	4.50	4.57	4.57	4.61
	110	3.07	3.11	3.11	3.15
$r = 10\%$, $\sigma =$ 0.30, $\tau = 0.75$	90	14.48	14.86	14.84	14.90
	95	11.85	12.11	12.10	12.17
	100	9.59	9.77	9.77	9.83
	105	7.68	7.81	7.80	7.87
	110	6.09	6.19	6.18	6.25

[a]The value of American puts with the exercise price of \$100 computed using the implicit finite-difference method.
[b]The numerical integration of eq. (4) under the approximate exercise boundary given by eq. (8).
[c]Solutions using the quadratic Approximation method of MacMillan (1986) and Barone-Adesi and Whaley (1987).

from eqs. (4) and (8) as well as those from the implicit finite-difference method and the quadratic approximation method. The 3-year American puts are examined in Table VI. Both tables show that eqs. (4) and (8) are accurate for both the short-term and long-term puts while the quadratic approximation is accurate only for options with small early-exercise premiums.

The hedge ratios for American put options can be computed using the following formula, the optimal exercise boundary being defined by

TABLE VI
Comparisons of the Approximation Formulas for the Valuation of American Puts

Parameters	*Futures Price*	*European Option Value*	*Finite Diff.*[a]	*Eqs. (4) & (8)*[b]	*Quad. Approx.*[c]
$r = 8\%$, $\sigma =$ 0.20, $\tau = 3$	90	14.67	16.21	16.21	16.50
	95	12.63	13.79	13.79	14.11
	100	10.82	11.70	11.70	12.03
	105	9.23	9.91	9.91	10.23
	110	7.85	8.37	8.37	8.69
$r = 8\%$, $\sigma =$ 0.30, $\tau = 3$	90	19.56	21.45	21.45	21.91
	95	17.76	19.35	19.35	19.82
	100	16.13	17.46	17.46	17.94
	105	14.63	15.76	15.75	16.25
	110	13.28	14.23	14.23	14.72
$r = 10\%$, $\sigma =$ 0.20, $\tau = 3$	90	13.82	15.77	15.76	16.12
	95	11.89	13.37	13.36	13.75
	100	10.19	11.31	11.31	11.71
	105	8.69	9.55	9.55	9.95
	110	7.39	8.05	8.04	8.44
$r = 10\%$, $\sigma =$ 0.30, $\tau = 3$	90	18.42	20.83	20.82	21.37
	95	16.73	18.74	18.73	19.31
	100	15.19	16.87	16.86	17.46
	105	13.78	16.20	15.19	15.80
	110	12.50	13.70	13.70	14.30
$r = 12\%$, $\sigma =$ 0.20, $\tau = 3$	90	13.01	15.37	15.37	15.77
	95	11.20	12.98	12.98	13.42
	100	9.59	10.95	10.95	11.40
	105	8.19	9.22	9.22	9.68
	110	6.96	7.75	7.75	8.20
$r = 12\%$, $\sigma =$ 0.30, $\tau = 3$	90	17.35	20.26	20.24	20.88
	95	15.75	18.18	18.17	18.84
	100	14.30	16.34	16.32	17.01
	105	12.98	14.69	14.68	15.38
	110	11.77	13.22	13.21	13.91

[a]The value of American puts with the exercise price of $100 computed using the implicit finite-difference method.
[b]The numerical integration of eq. (4) under the approximate exercise boundary given by eq. (8).
[c]Solutions using the quadratic Approximation method of MacMillan (1986) and Barone-Adesi and Whaley (1987).

eqs. (8) and (8a).

$$P_F = -e^{-r\tau}\aleph(-d_1(F,\tau;K)) - \int_0^\tau re^{-r(\tau-s)} \times \left[\aleph(-d_1(F,\tau-s;B(s))) - \frac{(B(s)-K)}{F\sigma\sqrt{\tau-s}}n(d_2(F,\tau-s;B(s)))\right]ds \quad (9)$$

Table VII reports the hedge ratios computed from numerical integration of eq. (9) given the approximate optimal exercise boundary of eq. (8) and compares them to the hedge ratios obtained from the implicit finite-difference method and the quadratic approximation. The results are similar to Table IV.

Numerical integration of eq. (2) or (4) is not as computationally efficient as the quadratic approximation in which a polynomial approximation is used to compute the cumulative normal distribution function.

TABLE VII

Comparisons of the Approximation Formulas for the Hedge Ratio (American Puts)

Parameters	*Futures Price*	*Finite Diff.*[a]	*Eqs. (8) & (9)*[b]	*Quad. Approx.*[c]
$r = 10\%$, $\sigma = 0.20$, $\tau = 0.25$	90	−0.839	−0.839	−0.820
	95	−0.670	−0.670	−0.661
	100	−0.472	−0.472	−0.468
	105	−0.289	−0.289	−0.288
	110	−0.154	−0.155	−0.154
$r = 10\%$, $\sigma = 0.30$, $\tau = 0.25$	90	−0.728	−0.728	−0.716
	95	−0.597	−0.597	−0.590
	100	−0.462	−0.462	−0.458
	105	−0.338	−0.338	−0.336
	110	−0.234	−0.234	−0.233
$r = 10\%$, $\sigma = 0.20$, $\tau = 0.50$	90	−0.741	−0.740	−0.711
	95	−0.600	−0.600	−0.583
	100	−0.457	−0.457	−0.448
	105	−0.326	−0.327	−0.322
	110	−0.219	−0.219	−0.217
$r = 10\%$, $\sigma = 0.30$, $\tau = 0.50$	90	−0.639	−0.639	−0.618
	95	−0.539	−0.539	−0.525
	100	−0.443	−0.443	−0.434
	105	−0.356	−0.356	−0.350
	110	−0.278	−0.278	−0.275
$r = 10\%$, $\sigma = 0.20$, $\tau = 0.75$	90	−0.685	−0.684	−0.644
	95	−0.563	−0.563	−0.538
	100	−0.445	−0.445	−0.430
	105	−0.338	−0.338	−0.330
	110	−0.248	−0.248	−0.242
$r = 10\%$, $\sigma = 0.30$, $\tau = 0.75$	90	−0.591	−0.591	−0.560
	95	−0.507	−0.507	−0.486
	100	−0.429	−0.429	−0.413
	105	−0.357	−0.357	−0.346
	110	−0.293	−0.293	−0.286

[a]The hedge ratio for American puts with the exercise price of $100 computed using the implicit finite-difference method.

[b]The numerical integration of eq. (9) under the approximate exercise boundary given by eq. (8).

[c]Solutions using the quadratic Approximation method of MacMillan (1986) and Barone-Adesi and Whaley (1987).

If a polynomial approximation is obtained for eqs. (2) and (4), the computational efficiency of this approximation method could be much improved. This is left for future research.

SUMMARY

An analytical formulation of free-boundary problems provides a useful means to examine American options for which premature exercise is an important consideration. This article presents the valuation formulas for American options written on futures contracts by utilizing the results in Kim (1990). The focus of this article is on developing a computationally efficient numerical method to implement the valuation formulas for American futures options. A correct understanding of the behavior of the optimal exercise boundary plays an important role in developing a computationally efficient technique to implement the valuation formulas. The implementation technique represents a substantial improvement over existing numerical methods in terms of accuracy and/or computational efficiency. In addition, the results of this study can provide a logical basis to develop computationally more efficient numerical techniques.

APPENDIX

Proof of $G(0) = K$

Consider the evolution of the futures price over a small time interval Δt described by a binomial process.

$$\begin{tikzcd} & uF \quad \text{with probability } q \\ F \arrow[ur] \arrow[dr] & \\ & dF \quad \text{with probability } 1 - q \end{tikzcd}$$

Cox and Rubinstein (1985) have shown that this process converges to a lognormal diffusion process with the volatility parameter σ by choosing $u = e^{\sigma\sqrt{\Delta t}}$ and $d = e^{-\sigma\sqrt{\Delta t}}$. Consider an American call written on this futures contract with the exercise price of K when time to expiration is Δt. Since it is obvious that $G(0) \geq K$, it will be shown that $G(0) \leq K$. Suppose $G(0) = K + \varepsilon$ for $\varepsilon > 0$. One can choose Δt and the futures price strictly above K and strictly below $K + \varepsilon$ such that $uF < K + \varepsilon$ and $dF > K$. In this case, the call will be exercised at expiration regardless of the futures price at that time. Let r be the continuously compounded interest rate and define $R = e^{r\Delta t}$. The current value of the American call would be the discounted expected value of the call at expiration, expectations being taken with respect to the risk-neutral probability p and $1 - p$.

$$C(F, \Delta t) = [p(uF - K) + (1 - p)(dF - K)]/R \tag{A1}$$

Since a newly created futures contract has a zero value, the risk-neutral probability is determined by

$$0 = p(uF - F) + (1 - p)(dF - F)$$

Solving this equation, one obtains $p = (1 - d)/(u - d)$. By using this definition of the risk-neutral probability, eq. (A1) is reduced to

$$C(F, \Delta t) = (F - K)/R$$

Since $R > 1$, $C(F, \Delta t)$ is strictly less than the immediate exercise value $F - K$, implying that it is optimal to exercise the call. It follows that the critical futures price $G(\Delta t)$ should be below the current futures price which is strictly lower than $K + \varepsilon$. This contradicts Proposition 1 in Kim (1990) which states $G(\Delta t) \geq G(0)$. Consequently, the supposition is false. Intuitively, if one knows that it will be optimal to exercise a call next instance regardless of the futures price, then the call on futures contract should be exercised now to earn interest on the futures price. This is greater than the foregone interest on the exercise price. Since the call expires next instance, one need not consider the opportunities to exercise later. If the call is exercised now, it cannot be exercised next instance, which is self-contradictory.

Derivation of Equation (3)

Kolodner (1956) and McKean (1965) solved the free-boundary problem stated in terms of the partial differential eq. (1) subject to (1a)–(1d).

$$C(F, \tau; G(\cdot)) = \int_0^\tau \frac{e^{-[x-H(s)]^2/2\sigma^2(\tau-s)}}{\sigma\sqrt{2\pi(\tau - s)}} e^{-r(\tau-s)} \times \left[\frac{G(s)\sigma^2}{2} + \left[H'(s) - \frac{x - H(s)}{2(\tau - s)}\right](G(s) - K)\right] ds \tag{A2}$$

where

$$H(\sigma) = \ln G(\sigma) - \frac{1}{2}\sigma^2 s, \qquad x = \ln F - \frac{1}{2}\sigma^2 \tau$$

The prime denotes the derivative of a function with respect to its argument. Kim (1990) shows that equation (A2) can be transformed into eq. (2). The optimal exercise boundary $G(s)$ for $0 < \sigma \leq \tau$ is implicitly defined by the following equation.[17]

$$\frac{G(s) - K}{2} = \int_0^s \frac{e^{-[H(s)-H(\xi)]^2/2\sigma^2(s-\xi)}}{\sigma\sqrt{2\pi(s-\xi)}} e^{-r(s-\xi)} \times \left[\frac{G(\xi)\sigma^2}{2} + \left[H'(\xi) - \frac{H(s) - H(\xi)}{2(s-\xi)} \right] (G(\xi) - K) \right] d\xi \tag{A3}$$

The right-hand side of eq. (A3) can be broken into two parts.

$$\int_0^s \frac{e^{-[H(s)-H(\xi)]^2/2(s-\xi)}}{\sqrt{2\pi(s-\xi)}} e^{-r(s-\xi)} G(\xi) \left[\frac{\sigma}{2} + H'(\xi) - \frac{H(s) - H(\xi)}{2(s-\xi)} \right] d\xi - K \int_0^s \frac{e^{-[H(s)-H(\xi)]^2/2(s-\xi)}}{\sqrt{2\pi(s-\xi)}} e^{-r(s-\xi)} \left[H'(\xi) - \frac{H(s) - H(\xi)}{2(s-\xi)} \right] d\xi$$

Denote the two integrals by Λ_1 and Λ_2, respectively. To evaluate Λ_1 the following two identities are used.

$$\frac{e^{-[H(s)-H(\xi)]^2/2(s-\xi)}}{\sqrt{2\pi(s-\xi)}} e^{-r(s-\xi)} G(\xi) = \frac{e^{-[H(s)-H(\xi)+\sigma(s-\xi)]^2/2(s-\xi)}}{\sqrt{2\pi(s-\xi)}} \times G(s) e^{-\alpha(s-\xi)}$$

$$-\frac{\partial}{\partial \xi} \aleph \left[\frac{H(s) - H(\xi) + \sigma(s-\xi)}{\sqrt{s-\xi}} \right] = \frac{e^{-[H(s)-H(\xi)+\sigma(s-\xi)]^2/2(s-\xi)}}{\sqrt{2\pi(s-\xi)}} \left[\frac{\sigma}{2} + H'(\xi) - \frac{H(s) - H(\xi)}{2(s-\xi)} \right]$$

[17]There are many other integral equations that define the unique optimal exercise boundary.

Integrating by parts, the first integral is reduced to

$$\Lambda_1 = -\frac{G(s)}{2} + G(s)e^{-\alpha s}\aleph\left[\frac{H(s) - H(0) + \sigma s}{\sqrt{s}}\right]$$
$$+ \alpha G(s)\int_0^s e^{-\alpha(s-\xi)}\aleph\left[\frac{H(s) - H(\xi) + \sigma(s-\xi)}{\sqrt{s-\xi}}\right]d\xi$$

The second integral is evaluated in a similar way by noting that

$$-\frac{\partial}{\partial\xi}\aleph\left[-\frac{H(s) - H(\xi)}{\sqrt{s-\xi}}\right] = \frac{e^{-[H(s)-H(\xi)]^2/2(s-\xi)}}{\sqrt{2\pi(s-\xi)}} \times\left[H'(\xi) - \frac{H(s) - H(\xi)}{2(s-\xi)}\right]$$

Therefore,

$$\Lambda_2 = \frac{K}{2} - Ke^{-rs}\aleph\left[\frac{H(s) - H(0)}{\sqrt{s}}\right]$$
$$- rK\int_0^s e^{-r(s-\xi)}\aleph\left[\frac{H(s) - H(\xi)}{\sqrt{s-\xi}}\right]d\xi$$

Adding Λ_1 and Λ_2 and rearranging terms, eq. (3) is obtained.

Direct Derivation of Equation (6a)

Let $C(F;G(\infty))$ be the value of the *live* perpetual American calls, defined on $\{F; 0 < F < G(\infty)\}$. The value of the perpetual American calls does not depend on time, and hence would be governed by the ordinary differential equation

$$\frac{1}{2}\sigma^2F^2C_{FF} - rC = 0 \tag{A4}$$

subject to the boundary conditions

$$\lim_{F\to F^*} C(F;G(\infty)) = G(\infty) - K \tag{A4a}$$
$$\lim_{F\to 0} C(F;G(\infty)) = 0 \tag{A4b}$$

The general solution to (A4) is $\kappa_1F^\beta + \kappa_2F^\theta$, where β and θ are defined above. Since $\theta < 0$, the boundary condition (A4b) requires that $\kappa_2 = 0$, and (A4a) requires that $\kappa_1 = (G(\infty) - K)(G(\infty))^{\beta}$. Thus,

$$C(F; G(\infty)) = (G(\infty) - K)\left(\frac{F}{G(\infty)}\right)^{\beta} \tag{A5}$$

The optimal exercise boundary for the perpetual American calls is determined by maximizing $C(F; G(\infty))$ with respect to $G(\infty)$. By differentiating $C(F; G(\infty))$ with respect to $G(\infty)$ and setting it zero, the optimal exercise boundary is obtained as

$$G(\infty) = \frac{\beta K}{\beta - 1} \tag{A6}$$

Finally, the right-hand side of eq. (A6) is substituted for $G(\infty)$ in eq. (A5) to obtain the valuation formula for the *live* perpetual American calls.

$$C(F; G(\infty)) = \frac{K}{1 - \beta}\left(\frac{(\beta - 1)F}{\beta K}\right)^{\beta} \tag{A7}$$

BIBLIOGRAPHY

Ball, C.A., and Torous, W.N. (1986, September): "Futures Options and the Volatility of Futures Prices," *Journal of Finance,* 41:857–870.

Barone-Adesi, G., and Whaley, R. (1987, June): "Efficient Analytic Approximation of American Option Value," *Journal of Finance,* 42:301–320.

Black, F. (1976, January/March): "The Pricing of Commodity Contracts," *Journal of Financial Economics,* 3:167–179.

Black, F., and Scholes, M. (1973, January/March): "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy,* 81:637–659.

Brennan, M.J., and Schwartz, E.S. (1977, May): "The Valuation of American Put Options," *Journal of Finance,* 32:449–462.

Brennan, M.J., and Schwartz, E.S. (1978, September): "Finite Difference Methods and Jump Processes Arising in the Pricing of Contingent Claims: A Synthesis," *Journal of Financial and Quantitative Analysis,* 13:461–474.

Brenner, M.J., Courtadon, G.R., and Subrahmanyam, M.G. (1985, December): "Options on the Spot and Options on Futures," *Journal of Finance,* 40:1303–1318.

Cox, J.C., and Ross, S.A. (1976, January): "The Valuation of Options for Alternative Stochastic Processes," *Journal of Financial Economics,* 3:145–166.

Cox, J.C., Ingersoll, J.E., and Ross, S.A. (1981, December): "The Relationship Between Forward Prices and Futures Prices," *Journal of Financial Economics,* 9:321–346.

Cox, J.C., and Rubinstein, M. (1985): *Options Markets.* Englewood Cliffs, NJ: Prentice-Hall.

Geske, R., and Johnson, H.E. (1984, December): "The American Put Option Valued Analytically," *Journal of Finance,* 39:1511–1524.

Kim, I.J. (1990): "The Analytic Valuation of American Options," *The Review of Financial Studies,* 3:547–572.

Kolodner, I.I. (1956): "Free Boundary Problem for the Heat Equation with Applications to Problems of Change of Phase," *Communications in Pure and Applied Mathematics,* 9:1–31.

MacMillan, L.W. (1986): "An Analytic Approximation for the American Put Price," *Advances in Futures and Options Research,* 1:119–139.

McKean, Jr., H.P. (1965, Spring): "Appendix: A Free Boundary Problem for the Heat Equation Arising from a Problem in Mathematical Economics," *Industrial Management Review,* 6:32–39.

Merton, R.C. (1965, Spring): "Rational Theory of Warrant Pricing," *Industrial Management Review,* 6:13–31.

Merton, R.C. (1973): "Theory of Rational Option Pricing," *Bell Journal of Economics and Management Science,* 4:141–183.

Ramaswamy, K., and Sundaresan, M. (1985, December): "The Valuation of Options on Futures Contracts," *Journal of Finance,* 40:1319–1340.

Van Morebeke, P. (1976): "On Optimal Stopping and Free Boundary Problems," *Archives of Rational Mechanical Analysis,* 60:101–148.

Whaley, R.E. (1986, March): "Valuation of American Futures Options; Theory and Empirical Tests," *Journal of Finance,* 41:127–150.