An Alternative Formulation on the Pricing of Foreign Currency Options

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INTRODUCTION

Traditionally, the pricing of foreign currency options has been based upon the Biger and Hull (1983), Garman and Kohlhagen (1983) and Grabbe (1983) models. Each model is based upon the assumption that a riskless hedge can be created by investing in foreign bonds, the option and domestic bonds. In the Biger and Hull and Garman and Kohlhagen models it is assumed that foreign and domestic interest rates are constant. The Grabbe model allows for the possibility of stochastic interest rates which is similar to the approach adopted by Merton (1973). Grabbe assumes that the bond price dynamics of foreign and domestic bonds follows a geometric Brownian motion. Both models can be criticized. Namely, interest rates are not likely to be constant[1] in the Garman and Kohlhagen model and the assumption in the Grabbe model that bond returns follow a geometric Brownian motion may imply negative interest rates (that is, the price of the bond may be greater than the face value) and also the variance of the bond price at maturity will not be zero.[2]

This study proposes an alternative model to value foreign currency options by modeling the foreign and domestic bond processes as Brownian bridges. This process ensures that the bond price is known with certainty at the time the bond is initiated and at maturity. That is, the variance at $t = 0$ and $t = T$ (time of maturity) is zero. This type of process was proposed by Ball and Torous (1983) for the valuation of options on bonds. Other types of valuation models to price options on bonds have been proposed by Brennan and Schwartz (1983) and Courtadon (1982). However, each of these models is dependent upon investor's risk preferences and also requires the use of numerical methods to obtain solutions.

By assuming the bond price dynamics is generated by a Brownian bridge, there is still a possibility that the price of the bond may be greater than the face value. However, the bridge process better describes the characteristics of a bond because the price of the bond at maturity is known with certainty and the resulting variance of the bond price is

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[1]If both the domestic and the foreign interest rates were constant, the interest rate parity theorem would imply that there is no uncertainty in the exchange rate and that it is constant.

[2]If that were the case, the bond price at maturity would not be certain.

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zero. If one were to assume that the bond price dynamics is generated by a geometric Brownian motion, the variance at maturity would not be zero.

THE OPTION PRICING MODEL

In the spirit of Ball and Torous (1983), assume that foreign and domestic bond prices are generated by a Brownian bridge process. The bond price for the foreign bond is given by:

$$F(t) = \exp[-\mu_F(T - t) + \alpha(t)] \tag{1}$$

where

- $F(t)$ is the price of a pure discount foreign bond at time t,
- μ_F is the yield to maturity,
- T is the time that the bond matures,
- $\alpha(t)$ is a Brownian bridge process as defined by,

$$d\alpha(t) = -\frac{\alpha(t)}{T - t}dt + dW(t) \tag{2}$$

and

- $dW(t)$ is a white noise process which is normally distributed with a mean of zero and variance σ_F^2.

Assume also that the exchange rate follows a geometric Brownian motion. This type of process is assumed in both the Garman and Kohlagen (1983) and Grabbe (1983) models. However, there is some evidence to indicate that the variance of the returns of foreign currency are nonstationary [see Ang and Peterson (1984) and Tucker and Scott (1987)]. Imposing the condition of nonstationarity in the variance term means that numerical methods need to be used to determine the option value. To obtain closed form solutions the lognormality assumption is adopted.

$$\frac{dS(t)}{S(t)} = \mu_s\, dt + dX(t) \tag{3}$$

where

- $S(t)$ represents the price of a unit of foreign currency in the domestic denomination.
- μ_s is the expected futures return on the exchange rate,
- and $dX(t)$ is a white noise process with mean zero and variance σ_s^2.

Further assume that the domestic bond price is given by:

$$D(t) = \exp[-\mu_D(T - t) + \gamma(t)] \tag{4}$$

where

- $D(t)$ is the price of a pure discount domestic bond at time t,
- μ_D is the yield to maturity,
- $\gamma(t)$ is a Brownian bridge as defined by

$$d\gamma(t) = \frac{-\gamma(t)\, dt}{T - t} + dZ(t) \tag{5}$$

and

$dZ(t)$ is a white noise process with mean zero and variance σ_D^2.

By applying Ito's lemma to eq. (1) the following return on the foreign bond equation is obtained:

$$\frac{dF(t)}{F(t)} = \left(\frac{\sigma_F^2}{2} - \frac{\log F(t)}{T - t}\right)dt + dW(t) \tag{6}$$

Similarly the return on the domestic bond is given by

$$\frac{dD(t)}{D(t)} = \left(\frac{\sigma_D^2}{2} - \frac{\log D(t)}{T - t}\right)dt + dZ(t) \tag{7}$$

To evaluate the domestic and foreign bonds jointly, it is easier to have them denominated in one currency. For convenience, the foreign investment is modeled in the domestic currency. The value of the foreign bond domestic currency is given by $Q = FS$. By applying Ito's lemma to Q, the return on the foreign investment is given by:

$$\frac{dQ(t)}{Q(t)} = \left[\frac{\sigma_F^2}{2} - \frac{\log F(t)}{T - t} + \mu_s + \rho_{FS}\sigma_F\sigma_S\right]dt + dW(t) + dX(t) \tag{8}$$

Equation (8) can be rewritten as:

$$\frac{dQ(t)}{Q(t)} = \mu_Q\, dt + dY(t) \tag{9}$$

where

$$\mu_Q = \left[\frac{\sigma_F^2}{2} - \frac{\log F(t)}{T - t} + \mu_S + \rho_{FS}\sigma_F\sigma_S\right]$$

and

$dY(t)$ is a white noise process with mean zero and variance

$$\sigma_Q^2 = \sigma_F^2 + 2p_{FS}\sigma_F\sigma_S + \sigma_S^2$$

Assuming that the call is a function of $C = C(Q, D, \tau)$, (where τ is the time to maturity) one can create a riskless hedge by investing in Q, C, and D.[3] Using standard techniques as outlined in Merton (1973), the following partial differential equation results as a necessary condition to the hedge.

$$\frac{1}{2}Q^2\sigma_Q^2\frac{\partial^2 C}{\partial Q^2} + \rho_{QD}QD\sigma_Q\sigma_D\frac{\partial^2 C}{\partial Q\partial D} + \frac{1}{2}D^2\sigma_D^2\frac{\partial^2 C}{\partial D^2} = \frac{\partial C}{\partial \tau} \tag{10}$$

Equation (10) is the same PDE as found in Merton (1973). The boundary condition (at maturity $Q = S$) is

$$C(S, 1, 0) = \begin{cases} S - E & \text{if } S > E \\ 0 & S \leq E \end{cases} \tag{11}$$

where E is the exercise price. Thus, the resulting call price can be shown to be[4]

$$C(Q, D, \tau) = FSN(d_1) - DEN(d_2) \tag{12}$$

[3]Among Q, C, and D, there is one redundant asset which can be spanned by the other two.

[4]Since the bond dynamics of the bridge process differs from the geometric process only in terms of their means and that the assets are traded securities, the risk neutral valuation method can be used. (See, Hull, pp. 95–96.)

An alternative mathematic derivation of eq. (10) can be obtained from the authors upon request.

where

$N(-)$ is the cumulative unit normal distribution,

$$d_1 = \frac{\log \frac{FS}{DE} + \frac{1}{2}(\sigma_Q^2 - 2\rho_{QD}\sigma_Q\sigma_D + \sigma_D^2)\tau}{\sqrt{(\sigma_Q^2 - 2\rho_{QD}\sigma_Q\sigma_D + \sigma_D^2)\tau}}$$

and

$$d_2 = d_1 - \sqrt{(\sigma_Q^2 - 2\rho_{QD}\sigma_Q\sigma_D + \sigma_D^2)\tau}$$

Note the similarity in structure between eq. (12) and the Grabbe (1983) formulation.[5] It is important to note, however, that the variance and covariance terms of the bridge process are different from the geometric process because the means of the two processes are different. In estimating the variance and covariance terms it is simpler to expand the variance and covariance terms of Q into its constituents parts. Recall from eq. (9) that $\sigma_Q^2 = \sigma_F^2 + 2\rho_{FS}\sigma_F\sigma_S + \sigma_S^2$ and noting that $\rho_{QD}\sigma_Q\sigma_D = \rho_{FD}\sigma_F\sigma_D + \rho_{SD}\sigma_S\sigma_D$, the variance term in d_1 and d_2 can be restated as:

$$\sigma_F^2 + \sigma_S^2 + \sigma_D^2 + 2\rho_{FS}\sigma_F\sigma_S - 2\rho_{SD}\sigma_S\sigma_D - 2\rho_{FD}\sigma_F\sigma_D \tag{13}$$

This relationship corresponds to the variance term as found by Rumsey (1991). Rumsey examines the question of cross-currency options assuming the geometric Brownian motion for the bond price dynamics. A cross-currency option is an option in which the currency of the exercise price is different from the currency of the underlying asset. In that case, both the asset price and the strike price are stochastic. Rumsey finds that if the asset price is measured in terms of the currency of the exercise price, then the cross-currency option can be valued by eq. (12).

ESTIMATION OF THE PARAMETERS

Formulas for estimating the variance and covariance terms are provided to show that these parameters are different from the geometric Brownian techniques because the means of the processes are different. The variance of the foreign bond is given by:

$$\hat{\sigma}_F^2 = \frac{\frac{1}{2}\sum_{j=1}^{N}\left[\log \frac{P_F(t_j + \Delta t)}{P_F(t_j)} + \frac{\log P_F(t_j)}{T - t_j}\right]^2 (T - t_j)}{(T - t_j - \Delta t)\Delta t} \tag{14}$$

A similar formulation applies to the variance of the domestic bond. The estimation of the return on the exchange rate can be estimated using the same procedure as in the Black–Scholes (1973) model. The estimate of the covariance between the foreign and domestic bonds is given by:

$$\rho_{FD}\hat{\sigma}_F\sigma_D = \frac{\frac{1}{N}\sum_{j=1}^{N}\left[\log \frac{F(t_j + \Delta t)}{F(t_j)} \log \frac{D(t_j + \Delta t)}{D(t_j)} - \frac{\log F(t_j) \log D(t_j) (\Delta t)^2}{(T - t_j)^2}\right](T - t_j)}{(T - t_j - \Delta t)\Delta t} \tag{15}$$

[5]See Grabbe (1983), p. 246.

The estimate of the covariance between the foreign bond and the exchange rate is given by:

$$\rho_{FS}\hat{\sigma}_F\sigma_S = \frac{\frac{1}{N}\sum_{j=1}^{N}\left[\log\frac{F(t_j+\Delta t)}{F(t_j)}\log\frac{S(t_j+\Delta t)}{S(t_j)} - \frac{\log(F(t_j))\left(\mu_S - \frac{\sigma_S^2}{2}\right)\Delta t^2}{(T-t_j)}\right]}{(T-t_j-\Delta t)\log\left(\frac{T-t_j}{T-t_j-\Delta t}\right)} \tag{16}$$

A similar procedure can be adopted to measure the covariance between the domestic bond and the foreign currency as well.

SUMMARY

This article formulates an alternative option pricing model to value foreign currency. This model differs from Grabbe (1983) and Garmen and Kahlhagen (1983) by assuming that the bond price dynamics are generated by a Brownian bridge. Theoretically, this type of process is superior to assuming that the bond returns are geometric or constant. This theoretical superiority needs to be tested empirically. While the pricing equation resembles that of Grabbe, the variance and covariance terms are estimated quite differently. A procedure to estimate the variance and covariance terms is provided.

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