A Theoretical Comparison of Composite Index Futures Contracts

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INTRODUCTION

In February, 1982 the Kansas City Board of Trade (KCBT) began trading the first stock index futures contract. The contract's underlying asset is the Value Line Index (VLI), a geometric average composed of 90% NYSE and 10% OTC stocks. Since then more stock index futures contracts have been introduced such as the S&P 500 Index futures traded at Chicago Mercantile Exchange (CME) and the Major Market Index (MMI) futures traded at the Chicago Board of Trade (CBOT). The newer indices are all based on different stock compositions; however, they all apply arithmetic averaging methods.[1] In essence, they compete against each other based upon which composition better reflects the overall interests and activities of traders, generating trading volume accordingly.[2]

The KCBT, realizing that the uniqueness of its averaging scheme and, ceteris paribus, its underperformance relative to the arithmetic average, enlisted a subsidiary corporation to calculate an arithmetic average VLI. It began trading VLI futures based on this new average in March, 1988. Since then all stock index futures have employed arithmetic averaging. However, in spite of its underperformance, geometric averaging remains useful for some financial futures trading. Examples are the U.S. Dollar Index futures of New York Cotton Exchange and the Commodity Research Bureau (CRB) Index futures of the New York Futures Exchange.

This article utilizes a theoretical framework to analyze the difference between the geometric average index and the arithmetic average index. It is a well-known

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[1]While both S&P 500 and MMI futures contracts adopt the arithmetic averaging procedure, the bases to which the averaging method applies are different. The former contract considers the total market capitalization of the firm's outstanding shares for each stock; whereas, the latter evaluates the price of each stock. Stoll and Whalley (1992, p. 105) provide an example to illustrate the difference between these two methods and the geometric average method.

[2]It is assumed that the cash settlement index determines the market activity; therefore, the averaging method is of concern because it directly affects the index. For a general cash settlement contract, the choice of settlement index is discussed in Jones (1982), Lien (1989b), and Cita and Lien (1992).

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mathematical property that the geometric average underperforms the arithmetic average (referred to as the Holder inequality). However, this property by itself does not imply that the futures contract based on the former is less desirable than the futures contract based on the latter. For example, if a futures contract is evaluated by its hedging effectiveness [Ederington (1979); Lien (1990b)], then only the second moments of the futures and spot prices are relevant. Underperformance has no bearing in this case. Nonetheless, within a representative hedger framework, Lien (1989b, 1990a) demonstrates that a futures contract based on some sort of arithmetic average index (with weights reflecting the hedger's cash positions) dominates all possible index specifications in terms of both hedger's utility and trading volume. However, this conclusion is dubious when information asymmetry prevails in the markets.

While the analysis here is directly useful for understanding the current Dollar index and CRB Index futures contracts, it is primarily directed at determining whether, in general, stock index futures contracts should use the geometric or arithmetic average index. The results are derived from a theoretical model of stock index futures which explicitly takes into account information asymmetry. Specifically, the model resembles that of Subrahmanyam's (1991), which in turn follows Kyle (1985) and Admati and Pfleiderer (1988). Within this framework three types of agents exist in any speculative market: informed traders (or insiders), liquidity traders, and market makers. Insiders trade on the basis of their superior private information and gain profits at the expense of fully rational but less informed agents (liquidity traders) who trade for liquidity needs. Liquidity needs include the desire for: immediate consumption, idiosyncratic wealth shocks, tax planning, risk exposure adjustment, and portfolio insurance. In the case of competing futures markets, liquidity traders are further partitioned into non-discretionary liquidity traders, who trade certain amount of orders in various markets, and discretionary liquidity traders who choose between security and index futures markets to trade given orders. Finally, market makers are the agents who clear the markets.

This study first considers the case of a uniquely traded stock index futures contract (i.e., one not competing against any other stock index futures) and the exchange's problem of choosing between arithmetic and geometric averaging. The conclusions are conditioned by three performance criteria: trading volume, expected profits for insiders, and expected losses for liquidity traders. It turns out that the geometric average index futures outperforms the arithmetic average index version in terms of trading volume and expected losses for liquidity traders. Also, when the weights applied in the averaging methods are rather heterogeneous, the geometric average index futures is more desirable in terms of expected profits for insiders. Therefore, exchanges should adopt geometric average index futures in the absence of competition. Also, the KCBT correctly adopted the geometric Value Line Index when initiating the first stock index futures contract. This result also has direct implications for commodity futures contracts that rely upon cash settlement methods, such as feeder cattle and live cattle futures as suggested in Kahl, Hudson, and Ward (1989). Specifically, if each local market is treated as an isolated security market, then the commodity futures market with cash settlement is similar to the stock index futures market under consideration. This study finds that a better contract specification for the commodity futures is using the geometric mean of cash prices in various locations as compared to the conventional arithmetic mean, provided each cash price is lognormally distributed.[3]

[3]Kahl, Hudson, and Ward (1989) compared seven different indices, all applying the arithmetic averaging method. Using geometric mean to represent the average of geographical variables is popular among geologists. For example, see Jenson (1991).

This study also considers the case of competitive futures contracts. In particular, comparisons are made between an equally weighted geometric average index futures and a value-weighted arithmetic average index futures. This situation is analogous to the one holding prior to March, 1988 vis-a-vis the VLI and the S&P 500 Index futures. Herein the strategic behavior of discretionary liquidity traders needs to be taken into account. A representative discretionary liquidity trader allocates trade orders in security markets and the two futures markets to minimize the aggregate expected losses while satisfying his liquidity needs. Unfortunately the results are too complicated to produce immediate analytical implications. When the number of the underlying securities is large, it is easily shown that if the aggregate liquidity needs from discretionary liquidity traders are proportional to one of the two index compositions, then the corresponding index futures contract will gain a larger market share. Furthermore, the other index futures may be driven out of the market. This result is consistent with the conventional wisdom regarding the difficulties faced by the original VLI futures in competition with the S&P 500 Index futures.

A numerical example with two securities is constructed to further explore the implications of the results. While theoretically it is possible for the two contracts to complement each other, the example indicates they are most likely competing against each other. The geometric index futures market is shown to be more liquid than the corresponding arithmetic average index futures market. This means that the discretionary liquidity trader prefers to trade in the geometric index futures market unless his liquidity needs are unbalanced. As long as the proportion of each security in the liquidity demand schedule exceeds 0.3, the geometric index futures contract obtains more trading volume even if the competing contract is fully consistent with the liquidity demand schedule. From this example, it seems that the KCBT should have retained the geometric VLI futures contract upon reconstructing the weights and (perhaps) withdrawing some minor securities from the index specification, instead of adopting a new and inferior arithmetic average version when facing competition from other index futures contracts. Overall, the results strongly support the use of geometric average index futures contracts. The conclusion is established within the asymmetric information framework with rational economic agents; also, security price is assumed to be independently lognormally distributed. When some of the assumptions fail, the robustness of the above policy recommendation is unknown and left for future research.

The remainder of this article is organized as follows. The next section formalizes the model. The third section discusses the case of a single index futures contract while the fourth discusses the case of two index futures contracts competing against each other. The final section provides some concluding remarks.

THE BASIC MODEL

Suppose there are N individual securities simultaneously traded in the market. Trading takes place at time 0 and the securities are liquidated at time 1. Let μ_i, $i = 1, \ldots, N$, denote the value of security i at time 0 (which is public knowledge), and let S_i denote security i's liquidation value at time 1. At time 0, S_i is a log-normally distributed random variable independent of each other:

$$\log S_i = \theta_i + e_i \tag{1}$$

where $\theta_i = E(\log S_i)$ is the expectation of $\log S_i$; and e_i is a normal random variable with $E(e_i) = 0$, $E(e_i^2) = \sigma_i^2$ and $E(e_i e_j) = 0$ whenever $i \neq j$. The lognormality assumption is well accepted in the literature [see Eytan and Harpaz (1986); Eytan, Harpaz, and Krull (1988); Erhardt and Tucker (1990)]. From (1), the mean and variance of S_i are as follows:

$$E(S_i) = \exp(\theta_i + \sigma_i^2/2) \tag{2}$$

$$\mathrm{var}(S_i) = \exp(2\theta_i + \sigma_i^2)\left[\exp(\sigma_i^2) - 1\right] \tag{3}$$

Moreover, S_i may be expressed as:

$$S_i = E(S_i) + \varepsilon_i \tag{4}$$

where $\varepsilon_i = \exp(\theta_i)\left[\exp(e_i) - \exp(\sigma_i^2/2)\right]$. A martingale equilibrium is assumed so that $E(S_i) = \mu_i$.

Three categories of agents trade in the markets: liquidity traders, informed traders (or insiders), and market makers. Liquidity traders are assumed to be risk neutral and trade for reasons not directly related to the future payoffs of financial assets.[4] Their demand schedules are treated as exogenously determined. Let Z_i denote the net trade demand for security i by liquidity traders. All Z_i's are assumed to be mutually independent normal random variables with zero mean and variance var(Z_i). Also, Z_i and e_j are independent for every $i, j = 1, \ldots, N$. Informed traders have access to information concerning the future value of securities and trade in the markets expecting to earn profits for superior information. It is assumed that there are k_i informed traders in the ith security market who observe the value of ε_i (or e_i) perfectly.[5] Each informed trader, assumed to be risk neutral, trades in the ith security market to maximize expected profit, knowing only the underlying distribution of the liquidity traders' net demand. All orders from either liquidity traders or insiders are simultaneously submitted to the market makers. Upon observing total orders, v_i, market makers set a market price $P_i = P_i(v_i)$ to absorb the net demand and thus clear the market. Market makers are assumed to be risk neutral and competitive and therefore expect to earn zero profit. Note that market makers cannot separate insiders' orders from liquidity traders' orders.

Following Kyle (1985), the market equilibrium for each security is defined by the following two conditions: (1) Each insider chooses the optimal trading strategy to maximize his expected profit knowing how market makers set the market price; (2) Market makers choose a price setting rule to earn zero expected profit knowing how insiders behave. For analytical tractability, the analysis is restricted to linear, symmetric Nash equilibria.[6] Proposition 1 characterizes the unique equilibrium outcome.

Proposition 1. *For every security market, there is a unique linear, symmetric Nash equilibrium. At the equilibrium, each informed trader* j *in security market* i *submits an order* x_{ji}, *where*

[4]The assumption of risk neutrality simplifies the analysis and leads to a unique linear symmetric equilibrium. Spiegel and Subrahmanyam (1992) assume risk neutral informed traders and risk averse uninformed traders. Under certain conditions, the existence of a linear equilibrium is maintained. In the case of risk averse informed traders, it seems unlikely to have a linear equilibrium.

[5]This assumption follows Kyle (1985) and Subrahmanyam (1991). Essentially, the informed trader is assumed to have access to an observation of the ex post liquidation value of the security. An alternative and more realistic assumption allows for imperfect observations. That is, the informed trader observes ε_i with the contamination of a noise, say u_i. The assumption, however, leaves the main conclusions unchanged.

[6]Although mixed strategies are feasible for insiders, they are dominated by pure strategies when dealing with a linear equilibrium [Kyle (1985)].

$$x_{ji} = \varepsilon_i/((k_i + 1)\lambda_i) \tag{5}$$

$$\lambda_i = (E(S_i)/(k_i + 1)) \cdot \left[k_i(\exp(\sigma_i^2) - 1)/\mathrm{var}(Z_i)\right]^{1/2} \tag{6}$$

The market price for security i *is given by*

$$P_i = \mu_i + \lambda_i v_i \tag{7}$$

The proof of the proposition is provided in the Appendix.

In general, one may define the liquidity of a market as the order flow necessary to induce the market price to change one unit. Given the linear pricing rule, the liquidity of the ith security market is $1/\lambda_i$. From (6), it can be shown that λ_i decreases with increasing k_i. That is, when there are more insiders, the competition is intensified which leads to greater market liquidity. Suppose the liquidity traders' total order is z_i, then their expected loss is

$$E[(P_i - \mu_i)Z_i \,|\, Z_i = z_i] = \lambda_i z_i^2 \tag{8}$$

Thus liquidity traders incur smaller losses in a more liquid market.

TRADING A SINGLE INDEX FUTURES CONTRACT

A single composite index futures contract is introduced now into the model. Two possible index aggregation procedures are considered: arithmetic and geometric averaging; both employing the same weighting scheme. They are referred to as contract A and G, respectively. The future value of contract A is given by

$$S_A = \sum_{i=1}^{N} w_i S_i = \sum_{i=1}^{N} w_i \mu_i + \sum_{i=1}^{N} w_i \varepsilon_i \tag{9}$$

where w_i is the weight of security i in the index construction: $w_i \geq 0$, $\sum_{i=1}^{N} w_i = 1$. Assuming an identical weighting scheme for the two contracts, the future value of contract G is then

$$S_G = \exp\left(\sum_{i=1}^{N} w_i \log(S_i)\right) = \exp\left(\sum_{i=1}^{N} w_i \theta_i + \sum_{i=1}^{N} w_i e_i\right) \tag{10}$$

Similar to the security market, participants in the futures market are either liquidity traders, or market makers, or insiders. Since the index futures contract is priced off its underlying securities, security-specific information is useful for predicting the value of the contract. Thus an insider in security markets is also an insider in the index futures market. Market makers determine the price of the index futures upon absorbing the net trade order from insiders and liquidity traders. Again, market makers are assumed to be competitive and expecting zero profit. The orders of liquidity traders are exogenously given and denoted by Z_A (or Z_G) in the case of contract A (or contract G). Insiders have no knowledge of Z_A (or Z_G) except that it is normally distributed with zero mean and variance $\mathrm{var}(Z_A)$ [or $\mathrm{var}(Z_G)$]. To simplify the analysis,[7] assume $E(S_i) = 1$, for any $i = 1, \ldots, N$. The same equilibrium concept is then applied to the index futures contract

[7]Since $E(S_i) > 0$, the units of each security may be adjusted to assure that $E(S_i) = 1$.

and the linear symmetric Nash equilibrium is characterized in the presence of contract A and contract G, respectively.

Proposition 2. *Assume contract* A *is introduced into the market, there exists a unique linear symmetric Nash equilibrium in the index futures market. At the equilibrium, an informed trader who observes* ε_i *or* e_i *in the* i*th security market submits an order* x_{Ai} *to market makers:*

$$x_{Ai} = w_i \varepsilon_i / \{(k_i + 1)\lambda_A\} \tag{11}$$

$$\lambda_A = (\mathrm{var}(Z_A))^{-1/2} \left[\sum_{i=1}^{N} k_i w_i^2 (\exp(\sigma_i^2) - 1)/(k_i + 1)^2 \right]^{1/2} \tag{12}$$

The price of the contract A *is given by:*

$$P_A = E(S_A) + \lambda_A v_A = 1 + \lambda_A v_A \tag{13}$$

where v_A *is the total net trade order from the insiders and the liquidity traders.*

Proposition 3. *Suppose contract* G *is introduced into the market. There exists a unique linear, symmetric Nash equilibrium. At equilibrium, an insider who observes* ε_i *or* e_i *submits an order* x_{Gi} *to the market makers:*

$$x_{Gi} = \varepsilon_{iG} / \{(k_i + 1)\lambda_G\} \tag{14}$$

$$\varepsilon_{iG} = \exp\left\{ \sum_{j=1}^{N} \left(w_j^2 - w_j\right)\sigma_j^2/2 \right\} \cdot \left(\exp\left(w_i e_i - \left(w_i^2 \sigma_i^2/2\right)\right) - 1\right) \tag{15}$$

$$\lambda_G = \exp\left\{ \sum_{i=1}^{N} \left(w_j^2 - w_j\right)\sigma_j^2/2 \right\} (\mathrm{var}(Z_G))^{-1/2} \cdot \left\{ \sum_{i=1}^{N} k_i \left(\exp\left(w_i^2 \sigma_i^2\right) - 1\right)/(k_i + 1)^2 \right\}^{1/2} \tag{16}$$

The price of the contract G *is given by:*

$$P_G = E(S_G) + \lambda_G v_G \tag{17}$$

where v_G *is the total net trade order from the insiders and the liquidity traders.*

The above two theorems are, in essence, straightforward applications of Proposition 1. Proof and additional details are provided in the Appendix. Again, $1/\lambda_A$ and $1/\lambda_G$ measure the liquidity of the market for contract A and contract G, respectively. An increasing number of insiders in any security market promotes competition among insiders in the index futures market, leading to greater liquidity. Due to different weights for securities in the index composition, the influence of increasing the number of insiders upon futures markets differ across securities. For securities carrying larger weights the influence is greater.

In choosing between the two contract specifications, it is assumed that the exchange considers the following three factors: trading volume, expected profits of insiders, and expected losses of liquidity traders. Market makers prefer a heavily traded contract because the revenue from commission-fee is proportional to the trading volume. Of course, insiders would like to exploit superior information in a market which generates more profits, while liquidity traders opt to minimize the cost of satisfying liquidity needs. In this model, trading volume is determined by the net trade orders of the liquidity traders and insiders. While insiders choose their trade orders optimally, the demand schedule of liquidity traders is exogenously given. Different realizations of Z_A and Z_G will affect the comparison of contract A's and contract G's trading volume. To proceed, assume $Z_A = Z_G$ so that exogenous conditions are always constant, also $\mathrm{var}(Z_A) = \mathrm{var}(Z_G)$. Alternatively, it may be assumed that $\mathrm{var}(Z_A) = \mathrm{var}(Z_G)$. In that case this study analyzes only where the two random variables have the same realization: $z_A = z_G$.

Proposition 4. *Consider an insider with information on* ε_i, *a larger expected trade order will be submitted in the presence of contract* G. *That is,* $E|x_{Gi}| > E|x_{Ai}|$ *where* x_{Gi} *and* x_{Ai} *are given in eq.* (14) *and* (1) *respectively. As a result, contract* G *generates more expected trading volume than contract* A.

Although insiders are more active in contract G, the comparison between the expected profits for either contract is ambiguous. More precisely, when contract A prevails, the insider's expected profit is

$$\Pi_A = E((S_A - P_A)x_{Ai} \mid \varepsilon_i) = w_i^2 \varepsilon_i^2 / ((k_i + 1)/\lambda_A) \tag{18}$$

In the case of contract G, the expected profit is:

$$\Pi_G = E((S_G - P_G)x_{Gi} \mid \varepsilon_i) = \varepsilon_{iG}^2 / ((k_i + 1)\lambda_G) \tag{19}$$

Upon substituting ε_{iG}^2, λ_A, and λ_G into the above equations, it can be seen that the comparison is generally ambiguous. For a very special case in which a specific security dominants the index composition, the following proposition results.

Proposition 5. *If security* j *dominates all other securities such that* w_j *is close to* 1, *then all insiders, regardless of their sources of security-specific information, expect more profits from contract* G *than from contract* A, *i.e.,* $E(\Pi_G) > E(\Pi_A)$.

Lastly, consider liquidity traders. They prefer to resolve their liquidity needs in the market which requires lower expected losses. Let Z_o denote the trade order from a particular liquidity trader such that Z_o is independent of other liquidity traders' orders. For any realization of Z_o, say z_o, the liquidity traders' expected losses from trading contract A and contract G are:

$$L_A = E((P_A - S_A)Z_o \mid z_o) = \lambda_A z_o^2 \tag{20}$$

and

$$L_G = E((P_G - S_G)Z_o \mid z_o) = \lambda_G z_o^2 \tag{21}$$

respectively.

Proposition 6. *Suppose a liquidity trader attempts to resolve the liquidity needs by trading in the index futures market. Then contract* G *is preferred to contract* A *in the sense that less expected loss will be incurred.*

The above result is due to the fact that, given the assumptions, contract G is more liquid than contract A (i.e., $\lambda_G < \lambda_A$). To this point, Propositions 4 to 6 collectively indicate that exchanges have an incentive to choose the geometric averaging method over the arithmetic averaging method when constructing the composite indices. This suggests that the initial specification of the VLI was correct because it is more able to take into account the preferences of insiders, liquidity traders, and market makers.

It is yet necessary to justify the presence of liquidity traders in index futures markets. Specifically, a trader with liquidity needs may trade in all of the security markets in fixed proportions rather than the market for contract A. Subrahmanyam (1991) argued that, since contract A aggregates independent trade orders of insiders with various security specific information, the transaction cost is smaller and, thus, a discretionary liquidity trader would choose the index futures market over security markets. Moreover, Eytan and Harpaz (1986) establish the equivalence of contracts A and G for portfolio insurance purposes.[8] These arguments serve to validate the existence of liquidity traders who prefer exchanging contract G.

COMPETING INDEX FUTURES MARKETS

Suppose now that markets for both contract A and contract G prevail. To analyze the competition between the original VLI futures and the S&P 500 index futures, assume the VLI contract (being contract G) applies the same weight $(1/N)$ to every security, whereas the S&P 500 contract (being contract A) uses weights (w_i) that reflect the individual security's total market capitalization, i.e., the stock price multiplied by the number of outstanding shares.[9] Using this assumption, consider the optimal trading strategy of a discretionary liquidity trader attempting to trade T_i shares of security i due to liquidity needs. Since trading in all security markets and the two index futures markets is available, the optimal strategy is to allocate trade orders among the markets such that the expected loss is minimal. Specifically, let t_i $(i = 1, \ldots, N)$, t_A, and t_G denote the trading order in the ith security market, the market for contract A, and the market for contract G, respectively. The discretionary liquidity trader faces the following problem:

$$\text{Min} \sum_{i=1}^{N} E[(P_i - S_i)t_i \mid t_i] + E[(P_A - S_A)t_A \mid t_A] + E[(P_G - S_G)t_G \mid t_G]$$

subject to the constraint: $(t_1, \ldots, t_N) + t_A(w_1, \ldots, w_N) + t_G(1/N, \ldots, 1/N) = (T_1, \ldots, T_N)$. Further, assume that each individual liquidity trader's decision has no effect on the variance of aggregate trading orders from liquidity traders.[10] Consequently, the optimal trading strategy is characterized as follows.

Proposition 7. *Consider a discretionary liquidity trader's attempt to trade* $\mathrm{T_i}$ *shares of* i*th security for every* $\mathrm{i} = 1, \ldots, \mathrm{N}$. *The optimal trading strategy is to submit* $\mathrm{t_i^*}$ *trade order to the* i*th security market,* $\mathrm{t_A^*}$ *trade orders to the market for contract* A, *and* $\mathrm{t_G^*}$ *trade orders to the market for contract* G, *where*

[8]Due to this equivalence result, in terms of satisfying liquidity needs discretionary liquidity traders are indifferent between contract A and contract G when both adopt the same weighting scheme. However, insiders behave differently in the two markets, leading to different equilibrium outcomes. The resulting differences in market liquidity generates a bias in favor of contract G.

[9]This analysis also pertains to the case where the two index futures employ the same weighting scheme despite the fact that arithmetic averaging is adopted in one contract and geometric averaging in the other. In reality, it seems unlikely the CFTC would approve a new index contract displaying the same weight allocation as an existing contract, thus placing liquidity traders at a severe disadvantage especially if the trade volume becomes smaller in either market.

[10]The relaxation of this assumption actually reinforces the result. This is because market liquidity is positively related to the variance of liquidity traders' total trading orders. For example, when contract A is introduced to compete against contract G, discretionary liquidity traders switch some of their orders to the market for contract A. The market becomes more liquid and attracts more orders, meaning the liquidity traders' losses are further reduced.

$$t_A^* = \left[\left(N\lambda_G + \sum_{i=1}^{N}(\lambda_i/N)\right)\left(\sum_{i=1}^{N}\lambda_i w_i T_i\right) - \left(\sum_{i=1}^{N}\lambda_i T_i\right)\left(\sum_{i=1}^{N}\lambda_i w_i/N\right)\right]\Big/ D \quad (22)$$

$$t_G^* = \left[\left(\lambda_A + \sum_{i=1}^{N}\lambda_i w_i^2\right)\left(\sum_{i=1}^{N}\lambda_i w_i T_i\right) - \left(\sum_{i=1}^{N}\lambda_i w_i T_i\right)\left(\sum_{i=1}^{N}\lambda_i w_i\right)\right]\Big/ D \quad (23)$$

with

$$D = \left(N\lambda_G + \sum_{i=1}^{N}(\lambda_i/N)\right)\left(\lambda_A + \sum_{i=1}^{N}\lambda_i w_i^2\right) - (1/N)\left(\sum_{i=1}^{N}\lambda_i w_i\right)^2$$

Also,

$$t_i^* = T_i - t_A^* w_i - t_G^*/N, \qquad i = 1,\ldots,N \quad (24)$$

The number of discretionary liquidity traders are normalized now to one. Let Z_{Ni}, Z_{NA}, and Z_{NG} denote the trade order of non-discretionary liquidity traders in the ith security market, the market for contract A, and the market for contract G, respectively. Then, $\mathrm{var}(Z_i) = \mathrm{var}(t_i) + \mathrm{var}(Z_{Ni})$, $\mathrm{var}(Z_A) = \mathrm{var}(t_A) + \mathrm{var}(Z_{NA})$, and $\mathrm{var}(Z_G) = \mathrm{var}(t_G) + \mathrm{var}(Z_{NG})$. From eqs. (22)–(24), $\mathrm{var}(t_i)$, $\mathrm{var}(t_A)$, and $\mathrm{var}(t_G)$ are all functions of λ_i, λ_A, and λ_G, which, in turn, are functions of $\mathrm{var}(Z_i)$, $\mathrm{var}(Z_A)$, and $\mathrm{var}(Z_G)$. The latter, upon solving this equation system, can be written as functions of $\mathrm{var}(Z_{Ni})$, $\mathrm{var}(Z_{NA})$, and $\mathrm{var}(Z_{NG})$ which are exogenously given. To proceed further, assume $k_i = k$, $\sigma_i^2 = 1$; $i = 1,\ldots,N$. Also, assume $\mathrm{var}(Z_{Ni})$, $\mathrm{var}(Z_{NA})$, and $\mathrm{var}(Z_{NG})$ are specifically determined such that $\mathrm{var}(Z_A) = \mathrm{var}(Z_G) = 1$ and $\mathrm{var}(Z_i) = \mathrm{var}(w_i Z_A) = w_i^2$. The first condition assures both contracts are treated equally. The second condition follows Subrahmanyam (1991); it says that securities with larger weights in the arithmetic average index tend to have more liquidity trading volume. An alternative assumption, with $\mathrm{var}(Z_i) = \mathrm{var}(Z_G/N) = 1/N^2$, is also investigated. The conclusions remain essentially unchanged.

Given the above assumptions, the liquidity level of the security and index futures markets may be simplified to:

$$\lambda_i = \theta k^*/w_i \quad (6')$$

$$\lambda_A = \theta k^*\left(\sum_{i=1}^{N} w_i^2\right)^{1/2} \quad (12')$$

$$\lambda_G = k^* N^{-1/2}\exp((1 - N)/2N) \quad (16')$$

where $\theta^2 = \exp(1) - 1$ and $k^* = k^{1/2}/(k + 1)$. To derive (16′) apply the Taylor expansion (to the first order) to approximate the exponential term[11] in eq. (16). For these approximates, again, $\lambda_A > \lambda_G$.[12] Further substitution of the above expressions into

[11]This approximation is well justified when w_i is small. Since contract G assigns equal weight, $1/N$, to every security, this approach is quite acceptable if the number of underlying securities is large.

[12]From eq. (12′), λ_A achieves the minimum at $\theta k^* N^{-1/2}$ when $w_i = 1/N$ for all i. Thus, regardless of the weighting scheme, $\lambda_A > \lambda_G$ if $\exp((1 - N)/2N) < \theta = (\exp(1) - 1)^{1/2}$. The latter condition holds because $1 - N < 0$ and $e > 2$. This result equals the approximation method as the exact value exhibits the same property (see the proof of Proposition 6).

eqs. (22)–(23) leads to:

$$t_G^* = \left\{\left(\sum_{i=1}^{N} T_i/w_i\right)\left(1 + \left(\sum_{i=1}^{N} w_i^2\right)^{1/2}\right) - N\sum_{i=1}^{N} T_i\right\}H^{-1} \tag{22$'$}$$

$$t_A^* = \left[\left(N^{1/2}\theta^{-1}\exp((1-N)/2N) + \sum_{i=1}^{N}(1/(Nw_i))\right)\left(\sum_{i=1}^{N} T_i\right) - \sum_{i=1}^{N}(T_i/w_i)\right]H^{-1} \tag{23$'$}$$

where

$$H = \left(N^{1/2}\theta^{-1}\exp((1-N)/2N) + \sum_{i=1}^{N}(1/(Nw_i))\right)\left(1 + \left(\sum_{i=1}^{N} w_i^2\right)^{1/2}\right) - N$$

In general, t_G^* and t_A^* may be of the same or opposite signs. When they have the same signs, the two futures contracts compete against each other. When signs are opposite, they complement each other. A similar conclusion regarding wheat futures markets is derived in Lien (1989a). However, a comparison of t_G^* and t_A^* is usually ambiguous, it depends on factors such as T_i and w_i. In any case, it is possible for some liquidity traders to be more active in the market for contract A while others are more active in the market for contract G (see the following numerical example). Suppose the number of the underlying securities is large with one security having a very small weight (e.g., $w_j = 1/N^3$ for some j), then two extreme cases can be easily characterized. For one case, if liquidity need is consistent with the weighting scheme of contract A (i.e., $T_i = w_i$), then the discretionary liquidity trader will not participate in the market for contract G despite the fact that the latter is more liquid. The result can be easily found from eqs. (12$'$) and (16$'$). For the other extreme, when $T_i = 1/N$, the trader will not trade contract A. In all other cases, trading in both markets is certainly desirable.[13] Yet, the contract employing a weighting scheme closer to the representative liquidity needs will acquire greater popularity among discretionary liquidity traders. The result is consistent with the conventional wisdom.

When the number of securities is small or moderate, the implications of eqs. (22$'$) and (23$'$) are unclear. A numerical example is given for further study. Assume there are two securities with only one insider in the market for each security. Liquidity traders' total demand is 200, i.e., $T_1 + T_2 = 200$. Table I shows the discretionary liquidity traders' optimal trade orders in contract A, contract G, and each individual security. For example, suppose contract A calculates the index by assigning 20% to security 1 and 80% to security 2. Also, assume the discretionary liquidity trader attempts to establish a short position of 40 units in security 1 and 160 units in security 2. That is, his demand allocation is matched perfectly with contract A. As indicated in the fourth row of Table I, his optimal trade orders are as follows: short 75.1 units in contract A, short 63.0 units in contract G, short 68.4 units in security 2 while long 65 units in security 1. On the other hand, when the liquidity trader's demand allocation is consistent with contract G, the third row of Table I describes optimal trade orders as follows: short 32.1, 141.4, 22.9, and 36 in contract A, contract G, security 1, and security 2, respectively. Recall that contract G

[13]Since a liquidity trader's loss in a market is proportional to the square of his trading order, a diversification effect results when participating in both markets. In fact, an index market remains active in absence of non-discretionary liquidity traders.

allocates equal weights to both securities. Thus, when contract G matches the liquidity trader's demand schedule, the optimal trading volume for contract A is small, about 23% of the trading volume for contract G. In case that contract A is consistent with the demand schedule, contract G remains attractive with a trading volume about 84% of contract A's trading volume.

Also, from the table it can be seen that composite index futures contracts are always more liquid than an individual security market, whereas the market for contract G is more liquid than that for contract A. For example, suppose the weighting scheme of contact A is (0.4, 0.6), the liquidity level are 1/0.47, 1/0.25, 1/1.63, and 1/1.09 for the markets of contract G, contract A, security 1, and security 2, respectively. For an alternative weighting scheme, say (0.3, 0.7), the above numbers become 1/0.5, 1/0.28, 1/2.18, 1/0.94. Note that the market liquidity level for contract G remains constant as its weighting scheme is assumed constant. Thus, liquidity traders tend to trade more in the geometric index futures market. For the extreme case where liquidity needs are concentrated on one security, contract A dominates contract G if it appropriately reflects the needs. However, as long as the proportion of each security in the liquidity demand schedule exceeds 0.3, contract G always dominates contract A in terms of trading volume regardless of the weighting scheme of the latter contract.

From this example, one may argue that the KCBT should have retained its geometric VLI futures contract upon changing the weight allocations. The decision of reverting to an arithmetic version only promoted the competitive advantage of other contracts. One might conclude also that both Dollar and the CRB Index futures maintain their current specification, provided they are not too far off the prevailing commercial interests, even if competitive contracts appear.

CONCLUDING REMARKS

This article employs a theoretical model to investigate the differences between arithmetic and geometric average index futures contracts. For the general case, the latter is superior vis-a-vis trading volume, insiders' profits, and liquidity traders' losses. Thus, it appears that when the exchange decides to initiate a new and unique index futures contract, the geometric averaging method should be adopted. However, in cases where both types of contracts coexist and compete in the marketplace, the arithmetic average contract will possibly gain a larger market share only if the geometric average contract is far off the general commercial interests. This result indicates that the VLI futures lost its market share because its sharp divergence from liquidity demand schedule promoted the trading of competing futures contracts. To remedy this problem, KCBT should have reconstructed the weighting scheme while retaining the geometric averaging method, instead of adopting a new but inferior arithmetic average contract as it actually did in 1988.

In sum, this study finds that geometric index futures contracts are superior to arithmetic average contracts in stock index markets. In the absence of competition, the contract performs better than the corresponding arithmetic version. The existence of a competitive contract only implies that the exchange should investigate the possibility of reconstructing the weights for each security. The geometric averaging method should always be maintained. The conclusion is derived within the asymmetric information framework with addition of several assumptions. For example, the price of each security is assumed to be lognormally distributed. The assumption, however, is widely agreed upon in the literature. There is also an implicit assumption that traders accurately calculate the expectation of the prices in either geometric or arithmetic index markets and act accordingly. Cakici, Harpaz, and Yagil (1990) examined the pricing efficiency of VLI futures from 1982

Table I
DISCRETIONARY LIQUIDITY TRADER'S OPTIMAL TRADE ORDERS IN VARIOUS MARKETS AND CORRESPONDING MARKETS LIQUIDITY LEVELS[a]

(w_1, w_2)	(T_1, T_2)	t_A^*	t_G^*	t_1^*	t_2^*	λ_A	λ_G	λ_1	λ_2
10^{-4}, 0.9999	100, 100	0.017	199.7	0.017	8.4×10^{-7}	0.66	0.28	6554.2	0.66
10^{-4}, 0.9999	0.02, 199.98	99.99	0.04	–0.01	99.99	0.66	0.28	6554.2	0.66
0.2, 0.8	100, 100	32.1	141.4	22.9	3.6	0.54	0.28	3.28	0.82
0.2, 0.8	40, 160	75.1	63.0	–6.5	68.4	0.54	0.28	3.28	0.82
0.3, 0.7	100, 100	45.7	119.4	26.6	8.27	0.50	0.28	2.18	0.94
0.3, 0.7	60, 140	66.5	82.9	–1.39	52.0	0.50	0.28	2.18	0.94
0.4, 0.6	100, 100	55.4	104.6	25.5	14.4	0.47	0.28	1.63	1.09
0.4, 0.6	80, 120	60.9	95.1	8.5	35.8	0.47	0.28	1.63	1.09
0.5, 0.5	100, 100	59.0	99.3	20.9	20.9	0.46	0.26	1.31	1.31

[a]Assume $k = 1$, $\sigma_i^2 = 1$, $\mathrm{var}(Z_A) = \mathrm{var}(Z_G) = 1$, $\mathrm{var}(Z_i) = w_i^2$, $N = 2$, $T_1 + T_2 = 200$.

to 1987. They concluded the market failed to recognize the underperformance and overvalued the contract from 1982 to 1986 with the exception of 1987. As cash settlement methods gradually become popular, the conclusions of this study are directly useful for index construction in commodity and financial futures markets. The validity of each underlying assumption and the impacts of relaxation of any assumption clearly warrant future research.

Appendix

Proof of Proposition 1: The market makers' expected profit in market i is $E[(S_i - P_i(v_i))v_i \mid v_i]$, which is zero the equilibrium. Hence, $P_i(v_i) = E(S_i \mid v_i) = E(S_i) + \lambda_i v_i$, where λ_i is the regression coefficient of ε_i on the total order v_i, that is,

$$\lambda_i = \mathrm{cov}(\varepsilon_i, v_i)/\mathrm{var}(v_i) \tag{A1}$$

The insider's expected profit, when he submits an order x_{ji}, is

$$\begin{aligned} E[(S_i - P_i)x_{ji} \mid \varepsilon_i] &= E[(\varepsilon_i - \lambda_i(x_{ji} + (k_i - 1)x_{si} + z_i))x_{ji} \mid \varepsilon_i] \\ &= \varepsilon_i x_{ji} - \lambda_i x_{ji}^2 - (k_i - 1)x_{si}x_{ji}\lambda_i \end{aligned} \tag{A2}$$

where x_{si} denotes another insider's order, also based on the observation ε_i. The insider chooses x_{ji} to maximize (A2). The first order condition is

$$\varepsilon_i - 2\lambda_i x_{ji} - (k_i - 1)x_{si}\lambda_i = 0$$

At the symmetric equilibrium, we know $x_{si} = x_{ji}$. Therefore

$$x_{ji} = \varepsilon_i/((k_i + 1)\lambda_i)$$

Upon substituting x_{ji} into (A1), one derives a quadratic function in λ_i. The positive root (as required by the second order condition) yields eq. (6). The uniqueness of the equilibrium follows directly.

Proof of Proposition 2: The proof is similar to that of Proposition 1. It is known that λ_A (resp. λ_G) is the regression coefficient of $\sum_{i=1}^n w_i\varepsilon_i$ (resp. where $\varepsilon_{iG} = S_G - E(S_G) = \exp\{\sum_{i=1}^n w_i \ln(S_i)\} - \exp\{\sum_{i=1}^n w_i\theta_i + (1/2)\sum_{i=1}^n w_i^2\sigma_i^2\}$) on v_A (resp. v_G). Thus,

$$\lambda_A = \mathrm{cov}\left(\sum_{i=1}^n w_i\varepsilon_i, v_A\right)\Big/ \mathrm{var}(v_A) \tag{A3}$$

$$\lambda_G = \mathrm{cov}(\varepsilon_{iG}, v_G)/\mathrm{var}(v_G) \tag{A4}$$

After contract A is introduced, insider j with information on ε_i chooses x_{ji} and x_{Ai} to maximizes his expected profit: $E[(S_A - P_A)x_{Ai} \mid \varepsilon_i] + E[(S_i - P_i)x_{ji} \mid \varepsilon_i]$. Let x_{Asi} denote the order of any other insider who also observes ε_i. Then

$$\begin{aligned} &E[(S_A - P_A)x_{Ai} \mid \varepsilon_i] \\ &\quad = E\left[\left(\sum_{i=1}^n w_iS_i - \sum_{i=1}^n w_iE(S_i)\right.\right. \\ &\qquad \left.\left. -\lambda_A\left(x_{Ai} + (k_i - 1)x_{Asi} + \sum_{j\neq i} k_jx_{Aj} + z_A\right)\right)x_{Ai} \mid \varepsilon_i\right] \end{aligned}$$

Assume $E\left(\sum_{j\neq i} k_j x_{Aj} \mid \varepsilon_j\right) = 0$. The first order condition for x_{Aj} is given by

$$w_i\varepsilon_i - 2\lambda_A x_{Ai} - (k_i - 1)\lambda_A x_{Asi} = 0$$

Again, at the symmetric equilibrium, $x_{Asi} = x_{Ai}$, implying

$$x_{Ai} = w_i\varepsilon_i/(\lambda_A(k_i + 1)) \tag{A5}$$

Note that $E(x_{Aj} \mid \varepsilon_i) = 0$ for $j \neq i$. By substituting (A5) into (A3), the positive root of the resulting quadratic equation of λ_A yields eq. (12). The uniqueness of equilibrium follows directly.

Proof of Proposition 3: The proof is similar to that of Proposition 2.

Proof of Proposition 4: Upon applying the Taylor expansion to approximate $\exp(x)$ by $1 + x$, one derives the following:

$$x_{Gi} = [\exp(w_ie_i - (1/2)w_i^2\sigma_i^2) - 1]\Bigg/\left\{(k_i + 1)\left[\sum_{i=1}^{N} w_i^2\sigma_i^2 k_i(k_i + 1)^{-2}\operatorname{var}(Z_G)^{-1}\right]^{-1/2}\right\}$$

$$x_{Ai} = w_i[\exp(e_i - (1/2)\sigma_i^2) - 1)]\Bigg/\left\{(k_i + 1)\left[\sum_{i=1}^{N} w_i^2\sigma_i^2 k_i(k_i + 1)^{-2}\operatorname{var}(Z_A)^{-1}\right]^{-1/2}\right\}$$

To show $E|x_{Gi}| > E|x_{Ai}|$, it is necessary only to establish

$$E\left|\exp(w_ie_i - (1/2)w_i^2\sigma_i^2) - 1\right| > E\left|w_i[\exp(e_i - (1/2)\sigma_i^2) - 1)]\right|$$

First, apply integration by parts to derive

$$E\left|\exp(w_ie_i - (1/2)w_i^2\sigma_i^2) - 1\right| = 4^{(1/2)}\int_0^{w_i\sigma_i^2}(2\Pi)^{-1/2}\exp(-x^2/2)\,dx$$

$$E\left|w_i[\exp(e_i - (1/2)\sigma_i^2) - 1)]\right| = 4\int_0^{\sigma_i^2/2}(2\Pi)^{-1/2}\exp(-x^2/2)\,dx$$

Define $g(x) = \int_0^x(2\Pi)^{-1/2}\exp(-x^2/2)\,dx$. Then, $E|x_{Gi}| > E|x_{Ai}|$ if, and only if, $g(w_ix) > w_ig(x)$. Moreover, noting that $g(0) = 0$, and $g''(x) < 0$ when $x > 0$. One can conclude $g(x)$ is a concave function when $x > 0$. Consequently $g(w_ix + (1 - w_i)0) > w_ig(x) + (1 - w_i)g(0)$ or, equivalently, $g(w_ix) > w_ig(x)$.

Proof of Proposition 5: Compare $E(w_i^2\varepsilon_i^2/\lambda_A)$ with $E(\varepsilon_{iG}^2/\lambda_G)$:

$$E(w_i^2\varepsilon_i^2/\lambda_A) = \lambda_A(k_i + 1)^2E^2|x_{Ai}|$$

$$E(\varepsilon_{iG}^2/\lambda_G) = \lambda_A(k_i + 1)^2E^2|x_{Gi}|\exp\left(\sum_{i=1}^{N}(w_i^2 - w_i)\sigma_i^2\right)$$

From Proposition 4, $E|x_{Gi}| > E|x_{Ai}|$. When security j dominates others (i.e., w_j is close to 1), the last term in the above equation is near 1. Hence, $E(\varepsilon_{iG}^2/\lambda_G)$ tends to exceed $E(w_i^2\varepsilon_i^2/\lambda_A)$.

Proof of Proposition 6: From Proposition 2, when $\mathrm{var}(Z_A) = \mathrm{var}(Z_G)$, $\lambda_G = \lambda_A \exp\left(\sum_{i=1}^{N} (w_i^2 - w_i)\sigma_i^2\right)$ which implies $\lambda_G < \lambda_A$. Thus, contract G is more liquid than contract A, and subsequently, the discretionary liquidity trader incurs less expected losses in the market for contract G than in the contract A market.

Proof of Proposition 7: Note that a discretionary liquidity trader's problem is to choose $t_1^*, t_2^*, \ldots, t_N^*, t_A^*$, and t_G^* to maximize $\sum_{i=1}^{N} \lambda_i t_i^2 + \lambda_A t_A^2 + \lambda_G t_G^2$ subject to $(t_1, \ldots, t_N) + t_A(w_1, \ldots, w_N) + t_G(1/N, \ldots, 1/N) = (T_1, \ldots, T_N)$. The first order conditions are given by

$$2\lambda_i t_i^* - q_i = 0, \qquad i = 1, \ldots, N \tag{A6}$$

$$2\lambda_G t_G^* - (1/N)\sum_{i=1}^{N} q_i = 0 \tag{A7}$$

$$2\lambda_A t_A^* - \sum_{i=1}^{N} q_i w_i = 0 \tag{A8}$$

where q_i's, $i = 1, \ldots, N$, are multipliers. Aggregating eq. (A6) over i and then substituting into (A7) and (A8), one derives

$$\sum_{i=1}^{N} \lambda_i t_i^* = N\lambda_G t_G^* \tag{A9}$$

$$\sum_{i=1}^{N} \lambda_i t_i^* w_i = \lambda_A t_A^* \tag{A10}$$

Combining the constraint and the above two equations, one establishes eqs. (22) and (23).

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