A Modified Lattice Approach to Option Pricing

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INTRODUCTION

Various numerical procedures for valuing derivative securities have appeared in the option pricing literature, including the Monte Carlo simulation, the lattice approach, and the finite difference approach. These procedures are not only asymptotically equivalent to the Black–Scholes model when pricing European type options, but also capable of handling more complex cases when a closed-form solution to the value of an option does not exist. This article focuses on the lattice approach to option pricing.

Cox, Ross, and Rubinstein (CRR) (1979) pioneered the lattice approach. They developed a discrete-time, binomial approach to option pricing. The essence of their approach is the construction of a binomial lattice of stock prices where the risk neutral valuation rule is maintained. With a particular selection of binomial parameters including probabilities and jumps, they show that the CRR binomial model converges to the Black–Scholes (BS) (1973) model. The CRR methodology has been extended subsequently by various researchers. Hull and White (1988) provided a slightly modified binomial lattice model; Rendleman and Barter (1979) applied the CRR methodology to the pricing of put and call options on debt securities.

Boyle (1986) took the CRR methodology one step further and proposed a trinomial option pricing model, a.k.a. a three-jump model, where the stock price can either move upwards, downwards, or stay unchanged in a given time period. A complete set of trinomial lattice parameters was derived, including three probabilities and three jumps. He also compared the numerical accuracy of his model with that of the CRR binomial model. In a later article Boyle (1988) demonstrated how a five-jump, three-dimensional lattice can be developed to value options on two underlying securities. Omberg (1988) studied a family of discrete-time jump processes in option pricing. The technique of Gauss–Hermite quadrature was applied to the backward recursive integration problem of a compound option pricing model and, as a result, jump processes of any order with known numerical accuracy properties for valuing options were obtained. In particular, he constructed a "sharpened" trinomial process which is nearly identical to the trinomial process developed by Boyle (1986).

This article develops a modified approach to the selection of lattice parameters including probabilities and jumps. It is a general approach that can be applied to any

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multi-dimensional lattice approach. It is recognized that the necessary conditions for a lattice model to converge to the Black–Scholes model do not provide a unique solution to the lattice parameters. Additional restriction(s) on the lattice parameters is (are) required. In the modified approach the selection of lattice parameters is made such that the accuracy of approximation is improved. The approach is then applied to modify both the CRR binomial model and the Boyle trinomial model.

Numerical simulations are conducted to investigate the comparative accuracy of the modified lattice procedures with that of the CRR binomial and Boyle trinomial procedures. The results indicate that the modified binomial procedure is more accurate than the CRR binomial procedure, and the modified trinomial procedures are as accurate as the Boyle trinomial procedure. Furthermore, all trinomial procedures are found to be more accurate than binomial procedures.

The following introduces the modified binomial approximation method. The next section outlines the modified trinomial approximation method. A precise definition of accuracy is outlined and is used to investigate the numerical accuracy of the modified lattice procedures as compared with the CRR and Boyle procedures.

THE MODIFIED BINOMIAL MODEL[1]

Suppose that one wants to model the movements of the price of a non-dividend paying stock during the period from $t = 0$ to $t = T$. The stock price is assumed to follow, in a risk neutral world, the following stochastic process,

$$\frac{dS(t)}{S(t)} = rdt + \sigma dz \tag{1}$$

where r and σ, both constant, are the instantaneous proportional drift and volatility rates, respectively. A logarithmic transformation simplifies the above process to,

$$d \log S(t) = \left(r - \frac{\sigma^2}{2}\right)dt + \sigma dz \tag{2}$$

It follows immediately that the stock price is lognormally distributed. Specifically the holding period return $\log[S(t)/S_0]$ is normally distributed with mean $(r - \sigma^2/2)t$ and variance $\sigma^2 t$. In general the mth non-central moment of the stock price $S(t)$ is given by the following formula,

$$E[S(t)^m \mid S_0] = S_0^m \exp\left[\left(mr + m(m-1)\frac{\sigma^2}{2}\right)t\right] \tag{3}$$

A binomial approximation to the above continuous stochastic process can be developed as follows. The time to maturity (T) of an option on the stock is divided into N equal subintervals of length $h = T/N$. During each time period, say from kh to $(k + 1)h$, with $k = 0, 1, 2, \cdots, N - 1$, stock price movements are assumed to follow a binomial process, which is sometimes called a two-jump process. The stock price jumps from its initial value, $S(kh)$, either upward to $uS(kh)$ or downward to $dS(kh)$, where $0 < d < 1 < u$. This binomial branching process is illustrated in Figure 1, where p and q are the implicit probabilities in the risk neutral world.

This binomial lattice has a total of four parameters, u, d, p, and q. These parameters uniquely determine the evolution of stock prices, which in turn determines a unique

[1]The terms *model, method,* and *procedure* are used interchangeably in this article.

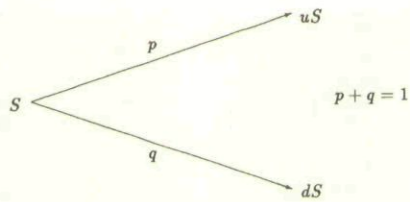

Figure 1

value of an option on the stock. However, these parameters cannot be chosen arbitrarily, because the option price obtained may not converge to the proper limit. It is a well-known fact that the value of an option is the discounted expectation of the terminal payoff under the equivalent martingale measure. If a discrete-time process (such as a binomial process) converges to the continuous-time process followed by the stock price, i.e., a Geometric Brownian Motion, then the expectations from the discrete-time case will converge to the continuous-time one. Thus, the convergence of the option value obtained from a discrete-time model to the one from a continuous-time model is ensured.[2]

Thus, the binomial parameters must be selected such that the discrete-time distribution converges to the lognormal distribution of the stock price in continuous time. According to Lindeberg's Central Limit Theorem,[3] the following conditions are sufficient to ensure this convergence: (a) the probabilities (p and q) are positive in the limit between 0 and 1 but not equal to either 0 or 1; (b) the probabilities sum to 1; (c) jumps (u and d) are independent of the stock price level; (d) the mean of the binomial distribution is equal to the mean of the lognormal distribution; (e) the variance of the binomial distribution is equal to the variance of the lognormal distribution. Mathematically, these restrictions are,

$$p + q = 1, \qquad 0 < p,\ q < 1 \tag{4}$$

$$pu + qd = M \tag{5}$$

$$pu^2 + qd^2 = M^2V \tag{6}$$

where, $M = \exp(rh)$, $V = \exp(\sigma^2 h)$, and $h = T/N$. There are four unknowns, u, d, p, and q in the above three equations. Thus, for a unique solution one more equation is needed. The choices for this additional restriction are theoretically infinite, and there is no obvious criterion to choose among these infinite choices. Ideally it should be selected to achieve the desirable convergence properties of the binomial approximation procedure. Toward this end, the following restriction is proposed,

$$pu^3 + qd^3 = M^3V^3 \tag{7}$$

This condition ensures that the third moment of the discrete-time process is also correct according to the continuous-time process. Since the binomial distribution is a skewed one, this condition might be more sensible and result in a more accurate binomial procedure.

Solving eqs. (4)–(7), a solution to the four binomial parameters is derived, and the resulting binomial model is a modified version of the original CRR model, which isreferred to hereafter as the BIN model:

[2]A general rule for convergence is discussed in He (1990). For details, see his Theorem 2 on p. 536.

[3]For an explanation of Lindeberg's Central Limit Theorem, see Feller (1971), Theorem 3, p. 262.

$$p = \frac{M - d}{u - d}, \qquad q = 1 - p = \frac{u - M}{u - d}$$

$$u = \frac{MV}{2}\left[(V + 1) + \sqrt{V^2 + 2V - 3}\right]$$

$$d = \frac{MV}{2}\left[(V + 1) - \sqrt{V^2 + 2V - 3}\right]$$

It's trivial to show that,

$$ud = (MV)^2 \tag{8}$$

Equation (8) implies that the binomial lattice grows roughly at the rate $(r + \sigma^2)$.

An option on the stock can be valued then by using a backward recursive procedure. Working backwards from the maturity of the option, the option value at time kh can be calculated as the discounted expectation of the option values at time $(k + 1)h$. For a call option on a non-dividend paying stock, such a backward recursion may be carried out by the following recursive valuation equation,

$$C(kh) = [pC_u((k + 1)h) + qC_d((k + 1)h)]/\exp(rh) \tag{9}$$

For other options, especially American options, some minor adjustment to the above recursion equation is needed.

The BIN model differs from the CRR binomial model in the following two aspects. First, in the CRR model the variance of the stock price is only correct in the limit as the time step h approaches zero. In the BIN model, both the mean and variance are correct for any given h. Second, instead of choosing the correct third moment (thus correct skewness), Cox, Ross, and Rubinstein (1979) selected the following condition for modeling simplicity,

$$ud = 1 \tag{10}$$

THE MODIFIED TRINOMIAL MODEL

The trinomial approximation method is a straightforward generalization of its binomial counterpart. Suppose that one wishes to price a call option on a non-dividend paying stock, which matures at time $t = T$ and can be exercised at a striking price of X. Assume that the stock price follows the same stochastic diffusion process as described in the previous section. A trinomial approximation to the continuous-time process of the stock price can be developed as follows.

As in the previous section, the time to maturity T of an option is divided into N equal subintervals of length $h = T/N$. During each time period, say from kh to $(k + 1)h$ (with $k = 0, 1, 2, \cdots, N - 1$), stock price movements are assumed to follow a trinomial process, i.e., a three-jump process. The stock price can either jump from its initial value $S(kh)$, upward to $uS(kh)$, downward to $dS(kh)$, or toward the middle $mS(kh)$. Such a trinomial process of stock prices is illustrated in Figure 2, where $u > m > d > 0$, $u > 1$, and $d < 1$.

This trinomial lattice has six parameters, u, m, d, p_1, p_2, and p_3. They consist of three jump and three probability parameters. These parameters uniquely determine the evolution of stock prices, which in turn determines a unique value for an option on the stock. Again as in the binomial case, they cannot be chosen arbitrarily. They must be restricted such that the trinomial process converges to the lognormal distribution of the stock price in continuous time. The restrictions that are sufficient to guarantee this

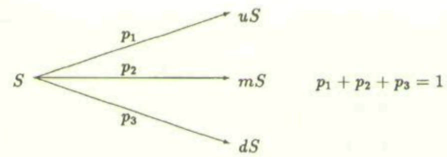

Figure 2

convergence are similar to those in the binomial case. Mathematically, these restrictions are,

$$p_1 + p_2 + p_3 = 1, \qquad 0 < p_1, p_2, p_3 < 1 \tag{11}$$

$$p_1 u + p_2 m + p_3 d = M \tag{12}$$

$$p_1 u^2 + p_2 m^2 + p_3 d^2 = M^2 V \tag{13}$$

For six trinomial parameters, there are only three restriction equations, (11)–(13). Three additional equations are necessary to define a unique solution. One of them is apparent and is stated below,

$$ud = m^2 \tag{14}$$

The motivation of this equation is as follows. Unlike a typical binomial lattice from the previous section, the above trinomial lattice does not usually recombine properly. The above equation ensures that an upward move followed by a downward move is equivalent to a downward move followed by an upward move. A downward move followed by a middle move is the same as a middle move followed by a downward move, and so on. Thus, the number of the nodes in the trinomial lattice increases linearly with the number of time periods used, instead of exponentially. In fact with this restriction, the number of nodes on an N-period trinomial lattice is reduced[4] from $(3^{N+1} - 1)/2$ to $(N + 1)^2$. Such a trinomial lattice is deemed computationally efficient.

For the other two additional restrictions, Boyle (1986) chose $m = 1$ and $u = \exp(\lambda\sigma\sqrt{h})$ and derived a trinomial model, where $\lambda \geq 1$ is a parameter that needs to be determined by its users.[5] However, convergence is likely to be maximized if the two additional restrictions are used instead to give correct values for higher moments of the trinomial distribution, such as the third moment and/or fourth moment. Along this line, two modified trinomial models are developed below.

The First Modified Trinomial Model

Recall that in the Boyle trinomial model it is found that "best results were obtained when the probabilities were roughly equal" (see footnote 5). In the modified *trinomial* lattice where m may differ from 1, it is possible to make these probabilities *exactly*

[4]For example, with $N = 10$ periods, the reduction in total number of nodes on the lattice is from 88,573 nodes to 121 nodes.

[5]The choice of the value of λ is crucial to numerical accuracy of the model. Boyle (1988) demonstrated that p_2 may become negative if λ is very close to or equal to 1, and both p_1 and p_3 may become close to zero if λ is equal to or greater than 2. Negative probability is certainly unacceptable and extremely small probabilities give rise to underflow errors in the computational algorithm. Through a trial-and-error experiment Boyle (1986) found that "best results were obtained when the probabilities were roughly equal". This translates to a λ value of around 1.20.

equal. That is,

$$p_1 = p_2 = p_3 \tag{15}$$

Solving the system of eqs. (11)–(15) one obtains a modified trinomial model, which is referred to here as the TRIN1 model:

$$p_1 = p_2 = p_3 = \frac{1}{3} \tag{16}$$

$$u = K + \sqrt{K^2 - m^2} \tag{17}$$

$$d = K - \sqrt{K^2 - m^2} \tag{18}$$

$$m = \frac{M(3 - V)}{2} \tag{19}$$

where $K = M(V + 3)/4$.

The Second Modified Trinomial Model

To develop the second modified trinomial model, the following two restrictions are imposed,

$$p_1 u^3 + p_2 m^3 + p_3 d^3 = M^3 V^3 \tag{20}$$

$$p_1 u^4 + p_2 m^4 + p_3 d^4 = M^4 V^6 \tag{21}$$

These two equations ensure that the third and fourth moments of the trinomial distribution are correct according to their counterparts of the continuous distribution. By solving the system of eqs. (11)–(14), (20), and (21) a solution is found to be:

$$p_1 = \frac{md - M(m + d) + M^2 V}{(u - d)(u - m)} \tag{22}$$

$$p_2 = \frac{M(u + d) - ud - M^2 V}{(u - m)(m - d)} \tag{23}$$

$$p_3 = \frac{um - M(u + m) + M^2 V}{(u - d)(m - d)} \tag{24}$$

$$u = K + \sqrt{K^2 - m^2} \tag{25}$$

$$d = K - \sqrt{K^2 - m^2} \tag{26}$$

$$m = MV^2, \qquad K = \frac{M}{2}(V^4 + V^3) \tag{27}$$

This modified model is referred to as the TRIN2 model.

Discussion

The modified trinomial models are intuitively more appealing than the Boyle model. Recall that Boyle (1986) found the "optimal" solution through a trial-and-error experiment. A systematic method for solving the problem is presented here. The TRIN1 model in particular allows one to exactly equalize the probabilities. Boyle could do so only "roughly". The reason is that m is allowed to differ from 1 in the modified trinomial lattice. On the other hand only the mean and variance of the discrete-time process are correct in the Boyle model, while the third and fourth moments (skewness and kurtosis) are also correct in the TRIN2 model. Based on this intuition, one may expect that the two modified versions are at least as accurate as the Boyle model.

Furthermore, both modified versions of the trinomial model behave properly in the limit as $\sigma \rightarrow 0$, while the Boyle model does not. Consider the pricing of a call option on a non-dividend paying stock. A little algebra shows that as $\sigma \rightarrow 0$ the Black–Scholes formula is reduced to,

$$C = \max\{S_0 - X\exp(-rT), 0\} \tag{28}$$

Thus, an approximation procedure should produce call prices that converge to this limit as well. However, the Boyle model does not converge at all when $\sigma \rightarrow 0$. It is trivial to show that as $\sigma \rightarrow 0$, the Boyle trinomial lattice is reduced to the one shown in Figure 3a. Obviously the Boyle lattice structure is incorrect in this limiting case. On the other hand, the modified versions do converge to the correct solution when $\sigma \rightarrow 0$, as given by eq. (28), and the lattice structure for the two modified models is reduced to the one shown in Figure 3b and the one shown in Figure 3c, respectively. A numerical example helps to illustrate this point. Consider a current stock price of $S_0 = \$100$. A call option on the stock matures in 7 months with an exercise price of $X = \$90$. Suppose the risk free rate is 10% p.a. Implementing a 100-period trinomial lattice, the Boyle model gives a call price of $C = -\$125.7346$ if $\sigma = 0.81\%$ p.a., and $C = \$11.2044$ if $\sigma = 0.82\%$ p.a., compared to the accurate price of $C = \$14.8672$ from the Black–Scholes formula for both σ values. In contrast, the two modified models give a price of $C = \$14.8672$ for both σ values, which is exactly the same as the Black–Scholes price.[6]

NUMERICAL ACCURACY OF THE MODIFIED METHODS

The numerical accuracy and convergence properties of the modified approximation methods, as compared to the CRR and Boyle models are examined in this section. Altogether five lattice procedures are investigated and compared. Numerical simulations are carried out in FORTRAN to calculate the value of an option on a non-dividend paying stock using the input parameter values given in Table I.

Table II reports the value of a European[7] call option obtained from all five lattice procedures using time steps ranging from $5, 10, 20, \cdots, 100$. It is clear that call values obtained from all methods converge fairly quickly to the Black–Scholes price. With 100 time steps used, all methods produce prices that are within 2 cents of the Black–Scholes price which varies from \$13.7670 to \$3.8056. Similar results are obtained from a study

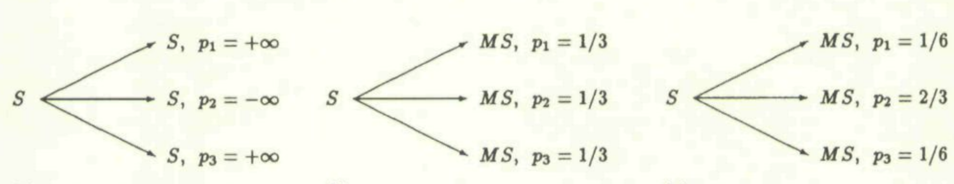

Figure 3

[6]The same argument also applies to the comparison of the binomial procedures, i.e., the BIN model behaves properly in the limit as $\sigma \rightarrow 0$ while the CRR model does not.

[7]Since it is assumed that the stock does not pay dividends, American call options will have the same value as the European call options.

Table I
OPTION PARAMETER VALUES

Parameter Values	
Stock price:	$S_0 = 100$
Exercise price:	$X = 90, 100, 110$
Time to maturity:	$T = 4$ months
Riskless rate:	$r = 5\%$ p.a.
Standard deviation:	$\sigma = 30\%$ p.a.

of European put options using the same option parameter values, and are summarized in Table III.

The valuation of an American put option on a non-dividend paying stock is quite different from an American call option on the same stock. It may become optimal to exercise an American put prior to maturity, with positive probability. As a result,

Table II
CALL PRICES CALCULATED BY VARIOUS LATTICE METHODS USING THE FOLLOWING PARAMETER VALUES: $S_0 = 100$, $X = 90, 100, 110$, $r = 5\%$ p.a., $\sigma = 30\%$, AND $T = 4$ MONTHS

	Numerical Method					
Time Steps (N)	CRR	BIN	BOYLE	TRIN1	TRIN2	BS
			Exercise price, $X = 90$			
5	13.7545	13.9349	13.7864	13.7733	13.8499	13.7670
10	13.6708	13.8282	13.8267	13.8268	13.7087	13.7670
20	13.8113	13.7030	13.7869	13.7840	13.7860	13.7670
40	13.7513	13.7708	13.7741	13.7719	13.7636	13.7670
60	13.7820	13.7780	13.7663	13.7676	13.7737	13.7670
80	13.7758	13.7557	13.7747	13.7746	13.7741	13.7670
100	13.7570	13.7733	13.7663	13.7645	13.7581	13.7670
			Exercise price, $X = 100$			
5	8.0244	7.7638	7.6362	7.6329	7.8534	7.6823
10	7.5128	7.8347	7.6555	7.6581	7.6741	7.6823
20	7.5969	7.7565	7.6683	7.6717	7.6563	7.6823
40	7.6394	7.6820	7.6752	7.6782	7.7008	7.6823
60	7.6537	7.6597	7.6775	7.6802	7.0590	7.6823
80	7.6608	7.6862	7.6787	7.6811	7.6863	7.6823
100	7.6651	7.6945	7.6794	7.6817	7.6906	7.6823
			Exercise price, $X = 110$			
5	3.6748	3.9031	3.7401	3.7213	3.6803	3.8056
10	3.7759	3.9615	3.8764	3.8726	3.7889	3.8056
20	3.8236	3.8777	3.7944	3.7859	3.8204	3.8056
40	3.8229	3.7966	3.8052	3.8129	3.8229	3.8056
60	3.7985	3.7968	3.8174	3.8172	3.8190	3.8056
80	3.8225	3.8164	3.8052	3.8016	3.8152	3.8056
100	3.8115	3.8206	3.8124	3.8126	3.8120	3.8056

Table III
EUROPEAN PUT PRICES CALCULATED BY VARIOUS LATTICE METHODS USING THE FOLLOWING PARAMETER VALUES: S_0 = 100, X = 90, 100, 110, r = 5% p.a., σ = 30%, AND T = 4 MONTHS

	Numerical Method					
Time Steps (N)	**CRR**	**BIN**	**BOYLE**	**TRIN1**	**TRIN2**	**BS**
			Exercise price, $X = 90$			
5	2.3027	2.4830	2.3345	2.3215	2.3980	2.3152
10	2.2190	2.3763	2.3749	2.3749	2.2568	2.3152
20	2.3595	2.2511	2.3351	2.3322	2.3342	2.3152
40	2.2995	2.3189	2.3223	2.3200	2.3118	2.3152
60	2.3301	2.3261	2.3145	2.3157	2.3218	2.3152
80	2.3239	2.3039	2.3228	2.3227	2.3222	2.3152
100	2.3052	2.3214	2.3145	2.3127	2.3062	2.3152
			Exercise price, $X = 100$			
5	6.4112	6.1506	6.0231	6.0197	6.2402	6.0691
10	5.8996	6.2215	6.0423	6.0449	6.0609	6.0691
20	5.9837	6.1433	6.0552	6.0585	6.0431	6.0691
40	6.0263	6.0689	6.0620	6.0650	6.0876	6.0691
60	6.0405	6.0465	6.0643	6.0670	6.0458	·6.0691
80	6.0476	6.0730	6.0655	6.0680	6.0731	6.0691
100	6.0519	6.0813	6.0662	6.0685	6.0774	6.0691
			Exercise price, $X = 110$			
5	11.9003	12.1286	11.9656	11.9468	11.9058	12.0311
10	12.0014	12.1870	12.1019	12.0981	12.0144	12.0311
20	12.0491	12.1032	12.0199	12.0114	12.0459	12.0311
40	12.0483	12.0221	12.0307	12.0384	12.0484	12.0311
60	12.0240	12.0223	12.0429	12.0427	12.0445	12.0311
80	12.0480	12.0419	12.0307	12.0271	12.0407	12.0311
100	12.0370	12.0461	12.0379	12.0381	12.0375	12.0311

no closed-form solution for the value of an American put is available. To study the convergence properties of various lattice procedures when applied to American puts, the accurate price of an American put is calculated using a 400-step TRIN1 trinomial procedure. The results are presented in Table IV. Similar convergence properties are exhibited by these results. With 100 time steps used, all methods report prices that are within 2 cents of the "accurate" price which varies from \$12.3918 to \$2.3551.

However, to further investigate the comparative accuracy of the five lattice procedures, including the three modified versions and the CRR and Boyle procedures, there is a need to state exactly what is meant by "accuracy." The accuracy of an approximation method is measured by the minimum number of steps needed to achieve a given precision level. For instance, if it is required that the approximation error be less than 5 cents, and if it is found that approximation method (I) needs at least 60 steps to achieve such a required accuracy, compared to 40 steps for approximation method (II), then method (II) is deemed more accurate than (I).

Table IV
AMERICAN PUT PRICES CALCULATED BY VARIOUS LATTICE METHODS USING THE FOLLOWING PARAMETER VALUES: $S_0 = 100$, $X = 90, 100, 110$, $r = 5\%$ p.a., $\sigma = 30\%$, AND $T = 4$ MONTHS

	Numerical Method					
Time Steps *(N)*	CRR	BIN	BOYLE	TRIN1	TRIN2	ACCURATE[a]
			Exercise price, $X = 90$			
5	2.3205	2.5109	2.3642	2.3494	3.4109	2.3545
10	2.2870	2.4085	2.4148	2.4137	2.2926	2.3545
20	2.3975	2.3029	2.3733	2.3701	2.3695	2.3545
40	2.3819	2.3617	2.3636	2.3615	2.3444	2.3545
60	2.3696	2.3650	2.3560	2.3569	2.3598	2.3545
80	2.3654	2.3464	2.3627	2.3624	2.3608	2.3545
100	2.3464	2.3606	2.3546	2.3531	2.3452	2.3545
			Exercise price, $X = 100$			
5	6.5504	6.3359	6.2019	6.1926	6.3367	6.2076
10	6.1041	6.3455	6.2031	6.1991	6.1801	6.2076
20	6.1558	6.2825	6.2056	6.2044	6.1667	6.2076
40	6.1811	6.2149	6.2065	6.2064	6.2184	6.2076
60	6.1898	6.1890	6.2066	6.2072	6.1838	6.2076
80	6.1942	6.2104	6.2068	6.2076	6.2111	6.2076
100	6.1967	6.2181	6.2068	6.2077	6.2119	6.2076
			Exercise price, $X = 110$			
5	12.4060	12.4289	12.3738	12.3494	12.2683	12.3914
10	12.3681	12.5350	12.4614	12.4584	12.3400	12.3914
20	12.4295	12.4502	12.3952	12.3857	12.3817	12.3914
40	12.4081	12.3848	12.3937	12.3984	12.3935	12.3914
60	12.3943	12.3908	12.4025	12.4023	12.3961	12.3914
80	12.4084	12.4034	12.3937	12.3899	12.3936	12.3914
100	12.3977	12.4050	12.3978	12.3979	12.3910	12.3914

[a]The "Accurate" price is calculated using a 400-step TRIN1 trinomial method, since no closed-form solution is available.

Formally, the accuracy of an approximation method may be defined as follows:[8]

Consider an infinite sequence $\{P(n)\}$, which converges to a positive limit P^*. Given a precision level, i.e., a small positive number ϵ, define a positive integer N_ϵ,

$$N_\epsilon = \mathrm{Min}\left\{N : \left| \frac{P(n) - P^*}{P^*} \right| < \epsilon, \qquad \text{for all } n \geq N\right\} \tag{29}$$

This number N_ϵ is referred to as the *minimum convergence step (MCS)* relative to ϵ. If P^* is the accurate price of an option, and $P(n)$ is the price of the option obtained from an n-step approximation procedure, then the MCS (i.e. N_ϵ) of the price sequence measures the accuracy of the approximation method at the ϵ precision level. The smaller the MCS, the more accurate the procedure.

[8]Credit for this definition must be given to Alan White, University of Toronto.

This definition of accuracy emphasizes the importance of the stability of convergence. It identifies the minimum number of steps that are needed to ensure that the relative approximation error be less than the required precision level for all subsequent steps. The smaller the MCS, the more accurate the approximation. An example may help illustrate this concept. Consider two sequences: $\{a_n = 1 - \frac{1}{2^n}\}$ and $\{b_n = 1 - \frac{1}{n}\}$. Both sequences converge to 1. It is clear that the sequence $\{a_n\}$ converges to 1 much faster than the other sequence. To apply the concept of MCS to those two sequences, consider $\epsilon = 0.05$. The MCS is 5 for the sequence $\{a_n\}$ and 21 for the sequence $\{b_n\}$.

Now, the concept of *minimum convergence step* is applied to investigate the comparative accuracy of the five lattice procedures considered in this article.[9] The option parameter values shown in Table I are used for the study below. The value of an option is calculated for each lattice procedure using time steps ranging from $1, 2, 3, \cdots, 400$. By comparing these option values with the accurate value, the MCS is then obtained given a precision level. Call options on a non-dividend paying stock are examined first. A close-form solution to the value of such an option is available and it is given by the well-known Black–Scholes formula. Table V reports the MCS values at three precision levels, $5\%, 1\%$, and 0.5%. At the 5% precision level, all methods are very accurate. TRIN1 reports the lowest average MCS of 2.0, while both CRR and BIN report the highest

Table V

THE ACCURACY OF THE PROPOSED METHODS AS COMPARED TO THE CRR BINOMIAL AND THE BOYLE TRINOMIAL METHODS: CALL OPTION ON A NON-DIVIDEND PAYING STOCK

Exercise Price (X)	Numerical Method				
	CRR	BIN	BOYLE	TRIN1	TRIN2
			Precision level, $\epsilon = 5\%$		
90	1	2	2	2	2
100	5	2	2	1	1
110	5	7	3	3	5
Average	3.7	3.7	2.3	2.0	2.7
			Precision level, $\epsilon = 1\%$		
90	5	7	3	3	4
100	23	19	3	3	17
110	33	27	14	15	9
Average	20.3	17.7	6.7	7.0	10.0
			Precision level, $\epsilon = 0.5\%$		
90	14	14	10	9	12
100	45	27	7	7	20
110	84	68	34	35	10
Average	47.7	36.3	17.0	17.0	14.0

The table shows calculated MCS values which measure the accuracy of an approximation procedure.

[9]The concept of "minimum convergence step" can be applied to the five lattice procedures since the option value from all methods converge to the accurate value (e.g., the Black–Scholes price) in the limit.

average MCS of 3.7. At the 1% precision level, BOYLE is the most accurate method and reports the lowest average MCS of 6.7, although TRIN1 is almost as accurate with an average MCS of 7.0. CRR is the least accurate and reports an average MCS of 20.3, while BIN is more accurate with an average MCS of 17.7. At the 0.5% precision level, TRIN2 is the most accurate and reports the lowest average MCS of 14.0. BOYLE and TRIN1 are less accurate with an identical average MCS of 17.0. CRR is the least accurate and reports the highest average MCS of 47.7, compared to a value of 36.3 reported by the BIN method. Summarizing the results at all three precision levels, BOYLE and TRIN1 are the most accurate and very little differences are observed between them. CRR is the least accurate method and is significantly less accurate than BIN. It is also clear that all trinomial methods are much more accurate than binomial methods.

European put options on a non-dividend stock are considered next. The accurate value of such a put is calculated again using the Black–Scholes formula. Table VI presents the MCS values using the same levels of precision as before. The results are slightly different from the case of call options, although trinomial methods are still much more accurate than binomial methods. At the 5% precision level, BOYLE is the most accurate and reports the lowest average MCS of 2.3, while TRIN1 and TRIN2 are slightly less accurate reporting an average MCS of 2.7 and 3.0, respectively. CRR is the least accurate and reports the highest average MCS of 5.3, while BIN is more accurate reporting an average MCS of 3.7. At the 1% precision level, BOYLE is again the most accurate and

TABLE VI
THE ACCURACY OF THE PROPOSED METHODS AS COMPARED TO THE CRR BINOMIAL AND THE BOYLE TRINOMIAL METHODS: EUROPEAN PUT ON A NON-DIVIDEND PAYING STOCK

Exercise Price (X)	Numerical Method				
	CRR	BIN	BOYLE	TRIN1	TRIN2
			Precision level, $\epsilon = 5\%$		
90	7	7	3	3	4
100	6	2	2	1	3
110	3	2	2	2	2
Average	5.3	3.7	2.3	2.7	3.0
			Precision level, $\epsilon = 1\%$		
90	48	49	26	26	45
100	29	23	5	5	18
110	7	13	3	4	6
Average	28.0	28.3	11.3	11.7	23.0
			Precision level, $\epsilon = 0.5\%$		
90	99	131	51	51	99
100	57	57	9	9	20
110	18	23	12	13	7
Average	58.0	70.3	24.0	24.3	42.0

The table shows calculated MCS values which measure the accuracy of an approximation procedure.

reports the lowest average MCS of 11.3, while TRIN1 is slightly less accurate with an average MCS of 11.7. TRIN2 is much less accurate with an average MCS of 23.0. BIN is the least accurate with the highest average MCS of 28.3, while CRR is not much better with an average MCS of 28.0. At the 0.5% precision level, BOYLE remains the most accurate with the lowest average MCS of 24.0, although TRIN1 is only slightly worse with an average MCS of 24.3. TRIN2, on the other hand, is much less accurate with an average MCS of 42.0. BIN is the least accurate with the highest average MCS of 70.3, while CRR performs much better with an average MCS of 58.0. Summarizing the results of the European put options, BOYLE and TRIN1 are the most accurate with almost identical MCSs, while BIN is the least accurate with the highest average MCSs. CRR and TRIN2 are mediocre.

Finally, American put options on a non-dividend paying stock are examined. Recall that a key advantage of the lattice approach is that it is capable of handling American type options. The performance of a lattice procedure when applied to American put options is, therefore, crucial. No closed-form solution is available for such a put option. The accurate value is calculated by using a 400-step TRIN1 procedure.[10] Table VII presents the MCS values calculated using the same levels of precision as before. The results are

Table VII
THE ACCURACY OF THE PROPOSED METHODS AS COMPARED TO THE CRR BINOMIAL AND THE BOYLE TRINOMIAL METHODS: AMERICAN PUT ON A NON-DIVIDEND PAYING STOCK

Exercise Price (X)	Numerical Method				
	CRR	BIN	BOYLE	TRIN1	TRIN2
			Precision level, $\epsilon = 5\%$		
90	7	7	3	3	4
100	6	4	2	1	1
110	3	2	1	1	3
Average	5.3	4.3	2.0	1.7	2.7
			Precision level, $\epsilon = 1\%$		
90	48	49	27	26	45
100	28	23	3	3	19
110	12	11	3	3	5
Average	29.3	27.7	11.0	10.7	23.0
			Precision level, $\epsilon = 0.5\%$		
90	102	108	52	52	99
100	54	31	3	4	21
110	18	19	11	11	10
Average	58.3	52.7	22.0	22.3	43.3

The table shows calculated MCS values which measure the accuracy of an approximation procedure.

[10]Since the maturity of the put option is only 4 months, a 400-step lattice procedure is used with a step size of 0.30 day. Thus, the value of the put option obtained can be regarded as very "accurate". If a 400-step BOYLE procedure is used instead, the results are almost identical and, therefore, is not reported.

remarkably similar to the case of call options on a non-dividend paying stock (Table V). All trinomial methods are more accurate than binomial ones. Of the trinomial procedures, TRIN1 and BOYLE are the most accurate and report almost identical MCSs. TRIN2 is less accurate than its trinomial counterparts. Of the binomial procedures, BIN is more accurate than CRR.

In summary, it appears that all trinomial methods are more accurate than binomial ones. Although an N-step trinomial procedure involves more computations than an N-step binomial procedure,[11] the significant improvement in accuracy seems to justify the additional computations that are needed on a cost–benefit basis. In other words, to achieve a given level of precision, the trinomial procedure seems to require less computations than its binomial counterpart. For instance, to obtain the price of a call option on a non-dividend paying stock allowing an error of 0.5%, a 17-step TRIN1 or BOYLE trinomial procedure is generally needed which requires checking 324 nodes and evaluating an expression at each node, while a 36-step BIN binomial procedure is generally needed which requires checking 703 nodes and evaluating an expression at each node. Thus, the trinomial procedure appears to be more cost-efficient than the binomial procedure. Furthermore, of the two binomial methods, BIN is more accurate when applied to call options and American put options, while CRR is more accurate when applied to European put options. Finally, of the three trinomial methods, TRIN1 and BOYLE are almost identically accurate while TRIN2 is less accurate.

CONCLUSIONS

A modified lattice approach to option pricing is developed in this study. The approach is applied to modify the CRR binomial model and the Boyle trinomial model. It is recognized that the convergence of a discrete-time lattice model to the continuous-time Black–Scholes model is ensured as long as the first two moments of the discrete-time process are correct according to the continuous-time process, a Geometric Brownian Motion. These two conditions do not in principle provide a unique solution to lattice parameters including probabilities and jumps. The choice of additional condition(s) may lead to different parameter values, which in turn leads to different lattice procedures. In this article, these choices are made such that the accuracy of the approximation process is improved over previous choices.

The numerical results show that the modified binomial model (BIN) is more accurate than the CRR binomial model when applied to call and American put options on a non-dividend paying stock, while CRR is more accurate when applied to European put options. For the three trinomial procedures, TRIN1 and BOYLE are almost identically accurate, while TRIN2 is less accurate. Furthermore, it appears that all trinomial methods examined in this article are more accurate than binomial ones. It also appears that the best trinomial procedure is better than the best binomial one on the cost-accuracy basis.

In conclusion, the results in this study seem to point to the following recommendations: (1) If one is to use a binomial lattice procedure, BIN is the best procedure; (2) If one is to use a trinomial lattice procedure, TRIN1 or BOYLE may be used, although one may argue for TRIN1 since it involves simpler computations of trinomial parameters.

[11]An N-step trinomial lattice has a total of $(N + 1)^2$ nodes, compared to $(N + 1)(N + 2)/2$ nodes of an N-step binomial lattice. As well, in a trinomial procedure, a slightly more complex expression needs to be evaluated at each step.

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