Managing Non-Parallel Shift Risk of Yield Curve with Interest Rate Futures

Sang Bin Lee
Seung Hyun Oh

INTRODUCTION

There have been many studies on using interest rate futures to manage interest rate risk in fixed-income portfolios. Kolb and Chiang (1981), Gay and Kolb (1983b) and Hilliard (1984)[1] developed duration based hedging models with interest rate futures. Kolb and Gay (1982), Gay and Kolb (1983a) suggested an immunization strategy using interest rate futures.

It has been claimed that using interest rate futures for bond portfolio management has three advantages [Gay and Kolb (1983a)]. One purported advantage is that the portfolio manager can maintain the composition of the bond portfolio, after rebalancing, by simply trading interest rate futures. Thus, the portfolio manager need not worry about disturbing favored issues or maturities, and can avoid the problem of a lack of marketability for any of the bonds. Also, the transaction costs resulting from rebalancing can be much lower.

However, when the yield curve does not move in a parallel fashion, the use of interest rate futures may not perform as well as anticipated. The above approaches are duration based and this duration measure is derived under the assumption of change in interest rates of equal magnitude for all maturities. Therefore, these approaches are efficient portfolio management tools only when the yield curve moves in a parallel fashion.

The assumption that interest rates can only change by a parallel shift has been one of the major concerns in the bond portfolio immunization field. Bierwag (1977), Cooper (1977), and Khang (1979) have postulated alternative models of interest rate behavior. Each of the models implies a different measure of duration, with immunization attained if this duration measure is equal to the holding period. But, as Fong and Vasicek (1984)

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[1]Hilliard (1984) does not deal with duration explicitly but he extends the study of Kolb and Chiang (1981) into a multivariate model.

Sang Bin Lee is an Associate Professor in the Department of Management Science at the Korea Advanced Institution of Science and Technology.

Seung Hyun Oh is a Doctoral Candidate at the Korea Advanced Institution of Science and Technology.

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noted, these immunization strategies have limitations in that the portfolio is protected only against the particular type of yield curve change assumed.

Fong and Vasicek argue that for an immunization strategy based upon a parallel shift process, there may be many portfolios having the same duration. They suggest that it is appropriate to pick the portfolio which also minimizes a measure of the error resulting if a non-parallel process were in effect. But, in extreme cases, this procedure may result in a degenerate situation in which the target portfolio is composed of only one pure discount bond whose maturity is equal to the holding period.

This article assumes relatively general yield curve movements that allow both parallel shift and slope change of the curve. It shows how interest rate futures contracts can be used to hedge the risk of the yield curve slope change.

The next section presents assumptions about the yield curve changes. The third section outlines the possibility of managing non-parallel shift risk of the yield curve while the fourth section describes a hedging vehicle, a spread position of futures contracts. In the fifth section, a hedging strategy is constructed with interest rate futures. In the sixth section, the performance of this hedging strategy is compared to a duration matching strategy.

YIELD CURVE CHANGES

As mentioned, all hedging or immunization strategies make some implicit assumptions about the way interest rates change. But any kind of assumption about the yield curve changes is accompanied by stochastic process risk or immunization risk.[2] This problem is approached here by generalizing the assumption on the yield curve changes such that the assumption allows slope changes in the yield curve as well as parallel shifts. This generalization can reduce the stochastic process risk.

It is assumed that the yield curve changes immediately after the portfolio construction. Some definitions of notations follow. From the current yield curve, $h(t, t')$ is the implied forward rate spanning the time interval, (t, t'). After the change of yield curve, $h(t, t')$ changes to $h^*(t, t')$. Any change of yield curve can be broken into two parts, the parallel shift part and the pure non-parallel shift part as in eq. (1).

$$h^*(0, t) = h(0, t) + \lambda + \Delta(t) \tag{1}$$

where: $\lambda = h^*(0, m) - h(0, m)$, $\Delta(\cdot)$ represents the pure non-parallel shift; and m is the investor's planning period.

In eq. (1), λ is the magnitude of parallel shift on the basis of the rate $h(0, m)$, and $\Delta(\cdot)$ represents the pure non-parallel shift or rotation of yield curve around the time point m. Figure 1 shows this decomposition of yield curve shifts. It is assumed that $\Delta(t)$ is a linear function of t such that $\Delta(t) = \alpha \cdot (t - m)$.[3] Positive α means upward rotation of the yield curve and negative α means downward rotation. By adopting the equilibrium condition,

$$h(t, t') \cdot (t' - t) = h(0, t') \cdot t' - h(0, t) \cdot t^4 \tag{2}$$

[2]If the interest rate changes arbitrarily, the immunization strategy derived from a particular type of interest rate change cannot protect portfolio value against the arbitrary type of rate change.

[3]A change in curvature of yield curve at some time point is approximated as $(\partial^2 h^*/\partial t^2 - \partial^2 h/\partial t^2)$. If $\Delta(t)$ is $\alpha \cdot (t - m)$, the change in curvature, $(\partial^2 h^*/\partial t^2 - \partial^2 h/\partial t^2)$, is zero. Hence, the assumption of linear non-parallel shift, $\alpha \cdot (t - m)$, means that the change in the curvature of the yield curve is not considered.

[4]In a world of certainty with no transaction costs, if one employs continuous compounding, no arbitrage condition requires that eq. (2) holds.

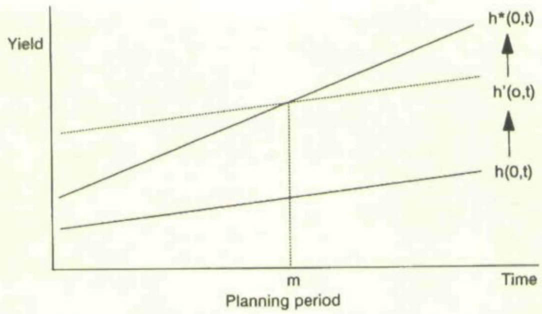

Figure 1.

When the yield curve $h(0,t)$ changes to $h^*(0,t)$, the change can be decomposed into parallel shift part, $h(0,t) \rightarrow h'(0,t)$, and pure non-parallel shift part, $h'(0,t) \rightarrow h^*(0,t)$. Note that $\Delta(t) = h^*(0,t) - h'(0,t)$

and using the assumptions, (1) and $\Delta(t) = \alpha \cdot (t - m)$; then:

$$h^*(t,t') = h(t,t') + \lambda + \alpha \cdot (t + t' - m) \tag{3}$$

HEDGING NON-PARALLEL SHIFT RISK

Under the assumptions of parallel shift of yield curve and zero transaction costs, as Fisher and Weil (1971) have shown, a strategy which matches the investor's planning period and duration ensures that the investor's realized return at the end of the horizon does not fall below the target return. But the assumption of a parallel shift of the yield curve is unrealistic.

Suppose an investor implements Fisher and Weil's duration matching strategy with a bond portfolio which has coupon rate c, principal Z, and maturity of length T. The portfolio's value at time m is $G(\lambda,\alpha) = cZ\int_0^T \exp[-h^*(m,t)(t-m)]\,dt + Z \cdot \exp[-h^*(m,T)(T-m)]$.[5] If there is no change in interest rate, then $G(0,0) = V \cdot \exp[h(0,m)m]$, where V is the present value of the bond such that

[5]If continuous compounding is used, the investor's return at time m from reinvesting the coupon payments is

$$Q(\lambda,\alpha) = cZ\int_0^m \exp[h^*(t,m)(m-t)]\,dt$$

where $h^*(\cdot,\cdot)$ is from (3).
His return at time m from the sale of the bond is

$$T(\lambda,\alpha) = cZ\int_m^T \exp[-h^*(m,t)(t-m)]\,dt + Z \cdot \exp[-h^*(m,T)(T-m)]$$

If notational convenience, $h(t,t') = h(t',t)$, is used, and the equilibrium condition of eq. (2) is applied, his total return at time m is

$$Q(\lambda,\alpha) + T(\lambda,\alpha) = cZ\int_0^T \exp[-h^*(m,t)(t-m)]\,dt + Z \cdot \exp[-h^*(m,T)(T-m)]$$

$$V = cZ \int_0^T \exp[-h(0,t)t]\,dt + Z \cdot \exp[-h(0,T)T] \tag{4}$$

His return at time m is $R(\lambda, \alpha) = G(\lambda, \alpha)/G(0,0)$.
Take the first two partial derivatives of $R(\lambda, \alpha)$ around $(0,0)$,

$$\frac{\partial R(0,0)}{\partial \lambda} = m - D \quad \text{and} \quad \frac{\partial R(0,0)}{\partial \alpha} = m \cdot D - C \tag{5}$$

where

$$D = \frac{1}{V}\left\{cZ \cdot \int_0^T \exp[-h(0,t)t] \cdot t\,dt + Z \cdot \exp[-h(0,T)T] \cdot T\right\}^6$$

and

$$C = \frac{1}{V}\left\{cZ \cdot \int_0^T \exp[-h(0,t)t] \cdot t^2\,dt + Z \cdot \exp[-h(0,T)T] \cdot T^2\right\}$$

If the above two derivatives are set equal to zero simultaneously, it can be shown that immunization is obtained because the Hessian matrix of $R(\lambda, \alpha)$ is positive semi-definite.[7] But in general, C is not less than D^2 and $C = D^2$ only for the pure discount bond.[8] Hence, when considering coupon bonds, C is greater than D^2 and the above two derivatives of (5) can not be equal to zero simultaneously.

If the investor designs the bond portfolio such that the duration of the portfolio is equal to his planning period (i.e., $D = m$), then the derivative about α is negative ($\partial R(0,0)/\partial \alpha = D^2 - C < 0$). Hence, an increase in slope reduces the portfolio value at time m. In the same manner, when the slope decreases, the total portfolio value at the horizon will increase. In other words, although the portfolio is immunized against parallel shift by the duration matching strategy, the portfolio is exposed to slope change risk or non-parallel shift risk.

Consider a portfolio whose value at the horizon increases when the slope of the yield curve increases and whose value decreases when the slope decreases irrespective of the level of the yield curve. If one could design such a hedging portfolio whose change in

[6]For simplicity, continuous compounding is used at the theoretical development stage. Discrete compounding is used in the example of the sixth section.
D and C of eq. (5) may be approximated in discrete terms as:

$$D = \frac{1}{V} \sum_{t=1}^{T} \frac{n(t) \cdot t}{[1 + h(0,t)]^t}$$

$$C = \frac{1}{V} \sum_{t=1}^{T} \frac{n(t) \cdot t^2}{[1 + h(0,t)]^t}$$

where $n(t)$ is cash flow at time t.

[7]Using Cauchy–Schwarz inequality, one can prove the positive semi-definiteness of $R(\lambda, \alpha)$.

[8]For simplicity, let $n(t) = \begin{cases} cZ \cdot \exp[-h(0,t)t]/V, & \text{for } 0 < t < T \\ (1+c)Z \cdot \exp[-h(0,t)t]/V, & \text{for } t = T \end{cases}$
then $D = \int_0^T n(t)t\,dt$, $C = \int_0^T n(t) \cdot t^2\,dt$, and $\int_0^T n(t)\,dt = 1$.
Cauchy–Schwarz inequality says that

$$\left(\int_0^T n(t)t\,dt\right)^2 \leq \left(\int_0^T n(t)\,dt\right) \cdot \left(\int_0^T n(t)t^2\,dt\right)$$

since $n(t) > 0$ for $0 \leq t \leq T$. Left-hand side is D^2 and right-hand side is C. Hence $D^2 \leq C$ and equality holds only for the case that $n(t)$ is zero except one particular time t^*. This special case corresponds to the pure discount bond.

value exactly offsets the change in value of the (duration matched) bond portfolio against the change in the slope of the yield curve, one could manage the non-parallel shift risk of the yield curve. The next sections discuss such a hedging portfolio.

SPREAD POSITION TRADING OF FUTURES CONTRACTS

Consider a speculator who believes that the yield curve will become steeper but is not sure about what the level of the yield curve will be. In other words, he expects that long-term yields will increase more than short-term yields, or that short-term yields will decrease more than long-term yields. Assume that to profit by this expectation he employs a spread position of futures contracts. The spread position is composed of short-term bond futures contracts and long-term bond futures contracts. Assume that both contracts have the same delivery date f. The short-term and long-term bonds underlying the futures contracts have coupon rates c_s and c_L, principals Z_s and Z_L, and mature at time S and L, respectively. The prices of long-term futures and short-term futures at the time of trading are P_L and P_s, respectively. P_L' and P_s' are the prices after the yield curve change.

The forward pricing formula for futures prices is used to analyze the sensitivity of futures prices with respect to the yield curve change.[9] The long-term and short-term futures prices, before the yield curve change, are

$$P_L = c_L Z_L \int_f^L \exp[-h(f,t)(t-f)]\,dt + Z_L \cdot \exp[-h(f,L)(L-f)]$$

$$P_s = c_s Z_s \int_f^S \exp[-h(f,t)(t-f)]\,dt + Z_s \cdot \exp[-h(f,S)(S-f)]$$

After the change of the yield curve, futures prices are

$$P_L' = c_L Z_L \int_f^L \exp[-h^*(f,t)(t-f)]\,dt + Z_L \cdot \exp[-h^*(f,L)(L-f)]$$

$$P_s' = c_s Z_s \int_f^S \exp[-h^*(f,t)(t-f)]\,dt + Z_s \cdot \exp[-h^*(f,S)(S-f)]$$

If the futures' version of duration[10] is defined as

$$D_L = \frac{1}{P_L}\left\{c_L Z_L \int_f^L \exp[-h(f,t)(t-f)](t-f)\,dt \right.$$
$$\left. + Z_L \cdot \exp[-h(f,t)(L-f)](L-f)\right\}$$

$$D_s = \frac{1}{P_s}\left\{c_s Z_s \int_f^S \exp[-h(f,t)(t-f)](t-f)\,dt\right.$$

[9]Under certain conditions, the forward or futures price equals the expected future spot price. Black (1976) shows that if changes in the futures price are independent of the return on the market, the futures price is the expected future spot price.

[10]The discrete approximation of D_L is

$$D_L = \frac{1}{P_L} \sum_{t=f+1}^{L} \frac{g(t) \cdot (t-f)}{[1 + h(f,t)]^{t-f}}$$

where $g(t)$ is the cash flow of underlying instrument at time t.
We call this measure as futures' version of duration because it is similar in form to the duration of a bond.

$$+ Z_s \cdot \exp[-h(f,t)(S-f)](S-f)\Big\} \tag{6}$$

and futures' version of convexity[11] as

$$C_L = \frac{1}{P_L}\left\{c_L Z_L \int_f^L \exp[-h(f,t)(t-f)](t^2-f^2)\,dt + Z_L \cdot \exp[-h(f,t)(L-f)](L^2-f^2)\right\}$$

$$C_s = \frac{1}{P_s}\left\{c_s Z_s \int_f^S \exp[-h(f,t)(t-f)](t^2-f^2)\,dt + Z_s \cdot \exp[-h(f,t)(S-f)](S^2-f^2)\right\} \tag{7}$$

then

$$\frac{\partial P_L}{\partial \lambda} = -P_L D_L, \ \frac{\partial P_s}{\partial \lambda} = -P_s D_s$$

$$\frac{\partial P_L}{\partial \alpha} = -P_L C_L + mP_L D_L \text{ and } \frac{\partial P_s}{\partial \alpha} = -P_s C_s + mP_s D_s$$

Hence, $P_L' - P_L$ and $P_s' - P_s$ can be approximated as[12]

$$P_L' - P_L \approx P_L D_L (m \cdot \Delta\alpha - \Delta\lambda) - P_L C_L \cdot \Delta\alpha$$

and

$$P_s' - P_s \approx P_s D_s (m \cdot \Delta\alpha - \Delta\lambda) - P_s C_s \cdot \Delta\alpha \tag{8}$$

where $\Delta\lambda$ and $\Delta\alpha$ are changes in level and slope, respectively.

If the investor shorts one unit of long-term futures contracts and holds N units of short-term futures contracts, the value of his spread position is $\Delta F = -(P_L' - P_L) + N \cdot (P_s' - P_s)$, after the yield curve change.

By substituting $P_L' - P_L$ and $P_s' - P_s$ from (8), ΔF is approximated as

$$\Delta F \approx (P_L C_L - N \cdot P_s C_s) \cdot \Delta\alpha - (P_L D_L - N \cdot P_s D_s) \cdot (m \cdot \Delta\alpha - \Delta\lambda)$$

According to the investor's expectation of increasing slope, $\Delta\alpha$ is greater than zero. But $\Delta\lambda$ can be positive or negative. If he sets N such that $D_L P_L = N \cdot D_s P_s$, he can ignore the effect of $\Delta\lambda$. Hence, ΔF is determined by $(P_L C_L - N \cdot P_s C_s)$ and $\Delta\alpha$. Assume that he uses T-bond futures and T–bill futures for long-term and short-term futures, respectively, and set $N = D_L P_L / D_s P_s$. In the appendix, it is shown that $(P_L C_L - N \cdot P_s C_s)$ is positive. Therefore, whether the yield curve moves up or not, the

[11]Because C_L and C_s of eq. (7) are similar in form to the convexity of a bond, for convenience' sake, these measures are called futures' version of convexity. The discrete version of C_L is

$$C_L = \frac{1}{P_L} \sum_{t=f+1}^{L} \frac{g(t) \cdot (t^2 - f^2)}{[1 + h(f,t)]^{t-f}}$$

[12]Taking total differentiation of P_L, one obtains $dP_L = (\partial P_L/\partial\lambda) \cdot d\lambda + (\partial P_L/\partial\alpha) \cdot d\alpha$. By substituting $-P_L D_L$ and $(-P_L C_L + mP_L D_L)$ of eq. (7) for $\partial P_L/\partial\lambda$ and $\partial P_L/\partial\alpha$ respectively, $\Delta P_L = P_L' - P_L$ can be approximated as eq. (8).

speculator's spread position has positive value only if the yield curve becomes steeper. By the same logic, if his expectation is not right, i.e., the slope of yield curve decreases, his spread position has negative value.

The spread position of futures contracts, selling one unit of long-term futures and buying $D_L P_L / D_s P_s$ units of short-term futures, has the property of the hedging portfolio considered in the third section. The following sections illustrate how to construct a spread position of futures contracts that can hedge the non-parallel yield curve shift risk of a given bond portfolio.

HEDGING STRATEGY[13]

After the yield curve change, the value of a spread position, selling one unit of long-term futures contracts and buying $N = D_L P_L / D_s P_s$ units of short-term futures contracts, is $\Delta F(\lambda, \alpha) = -(P_L' - P_L) + N \cdot (P_s' - P_s)$. The value of his futures portfolio at time m is $F(\lambda, \alpha) = \Delta F(\lambda, \alpha) \cdot \exp[h^*(0, m)m]$ and the change in value F due to small parallel shift in the yield curve (i.e., $\partial F / \partial \lambda$) is zero since $N = D_L P_L / D_s P_s$. Assume that the bond portfolio's duration matches the investor's planning period m (i.e., $m = D$) and that his total portfolio is composed of the bond portfolio and x units of the above futures spread position.[14] Then the return of his total portfolio at time m is $K(\lambda, \alpha) = (G(\lambda, \alpha) + xF(\lambda, \alpha))/G(0,0)$.

The conditions, $m = D$ and $N = D_L P_L / D_s P_s$, guarantee that $\partial K(0,0)/\partial \lambda = 0$. Hence, if the investor holds x units of the spread position such that $\partial K(0,0)/\partial \alpha = 0$[15], the change in value of the spread position offsets the change in the bond portfolio value against the slope change most effectively. Equation (9) represents the value of x.

$$x = \frac{-\partial G(0,0)}{\partial \alpha} \div \frac{\partial F(0,0)}{\partial \alpha} = \frac{(C - mD) \cdot V}{P_L C_L - N P_s C_s} \tag{9}$$

where: $N = D_L P_L / D_s P_s$ and V is from (4); D and C are defined in (5); D_L and D_s are from (6); and C_L and C_s are from (7)

To hedge a bond portfolio, the investor has to hold x units of the spread position. Equation (9) states that the holding amount of a spread position is an increasing function of $(C - mD)$. This can be interpreted as follows. When the slope of the yield curve changes, the more dispersed the bond portfolio's cash flows are, the more variable the portfolio's value is. Hence, to offset the change in value, the investor should increase the holding amount of the hedging portfolio. Note that the more dispersed the bond portfolio's cash flows, the greater the value of $(C - mD)$. Therefore, the holding amount

[13]The idea of using multiple kinds of futures contracts to hedge interest rate risk can be found in Hilliard (1984) and Chambers (1984). Chambers (1981) developed a vector of risk measures called duration vector which permits interest rate risk control for a two or more factor term structure behavior. Extending the duration vector approach to futures contracts, Chambers (1984) used two or more kinds of futures contracts to hedge interest rate risk.

[14]One unit of the spread position is implicitly defined as the portfolio of shorting one unit of long-term futures contracts and buying N units of short-term futures contracts.

[15]Little (1984) showed that if a portfolio whose duration is equal to the holding period generates both cash inflows and outflows, the portfolio may not satisfy the second order conditions for immunization. A futures (or forward) contract generates both cash inflows and outflows. A long position implies an outflow at the delivery date and subsequent inflows from the delivered instrument; a short position implies an inflow at the delivery date and subsequent outflows equal to the cash flows of the delivered instrument. Since the total portfolio in this section contains futures position, the portfolio includes implicit negative cash flows. Because of these negative cash flows, the strategy cannot guarantee the second order condition for the "immunization" such that the Hessian matrix of $K(\lambda, \alpha)$ is positive semidefinite. Hence, this study uses the term *hedging strategy* instead of *immunization strategy*.

of the spread position is an increasing function of $(C - mD)$. If the bond portfolio is composed of homogeneous zero coupon bonds with maturity equal to the holding period, this portfolio is perfectly immunized against any type of one-time yield change. Hence, he has no need to hedge the slope change risk. In this case of perfect immunization, eq. (9) states that the demand for a spread position is zero, since $C - mD$ is zero.

The strategy to manage the slope change risk is summarized as follows: First, construct a bond portfolio whose duration is equal to the planning horizon. Next, sell x units of long-term futures contracts and buy $x \cdot N$ units of short-term futures contracts where

$$x = \frac{(m \cdot D - C) \cdot V \cdot D_s}{(D_L C_s - D_s C_L) \cdot P_L} \quad \text{and} \quad x \cdot N = \frac{(m \cdot D - C) \cdot V \cdot D_L}{(D_L C_s - D_s C_L) \cdot P_s} \tag{10}$$

APPLYING THE HEDGING STRATEGY

To illustrate this hedging strategy, assume a manager holds a \$100 million bond portfolio whose duration is equal to his planning period of 8 years. Assume that he considers employing futures spread position to protect the value of the portfolio against the slope change of yield curve. Table I presents profiles of two bonds for the bond portfolio and the underlying instruments (a T-bill and a T-bond) of futures contracts for the spread position.

For convenience' sake, assume 12% flat yield curve. Table II represents some possible scenarios of yield curve changes. Note that one should ignore the cases of parallel or non-parallel shifts which result in negative interest rates. For example, if the level shift is –0.01 and the slope changes to –0.0085, some of the yields considered are negative.[16] Hence, in these scenarios, the slope change of $\alpha = \pm 0.001$ can be thought as a small rotation and the change $\alpha = \pm 0.007$ as a large rotation of the yield curve. Table III compares

Table I
PROFILES OF INSTRUMENTS[a]

	Coupon Rate	Maturity	Price	Duration[b]	C[c]
Bond–1	8%	4 Yrs	\$878.51	3.546	13.456
Bond–2	5%	20 Yrs	\$477.14	9.840	145.962
Bond only portfolio[d]			\$1,000,000.00	8.000	107.220
T-bill futures[e]	—	1 Yr[f]	\$892,857.14	1.000	3.000
T-bond futures[e]	8%	20 Yrs	\$70,122.23	8.939	141.307

[a]Annual cash flows and discrete compounding of funds are assumed.

[b]Because of discrete compounding, the discrete version of D of eq. (5) and, that of D_L and D_s of eq. (6) are used. The formulae for these discrete versions of D, D_L, and D_s are contained in footnotes (6) and (10).

[c]The discrete version of C of eq. (5) and that of C_L and C_s of eq. (7) are found in footnotes (6) and (11), respectively.

[d]The bond only portfolio's duration and convexity are obtained by weighted averaging the each bond's duration and convexity.

[e]It is assumed that T-bill and T-bond futures contracts have 1 year remaining until delivery date (i.e., $f = 1$).

[f]A T-bill whose maturity is 1 year is used for computational simplicity.

[16]For example, to discount the principle of the T-bond which is the underlying instrument of the futures contract, one has to consider the rate $h(0, 21)$. However, in this case of yield change, $h^*(0, 21) = 0.12 - 0.01 - 0.0085 \cdot (21 - 8)$ is negative.

Table II
SOME POSSIBLE SCENARIOS OF YIELD CURVE CHANGES $h^*(0,t) = h(0,t) + \lambda + \alpha \cdot (t-m)$

Level λ		0.0	0.01	−0.01
Slope α	0.0	case 0[a]	case 1	case 2
	0.001	case 3	case 4[b]	case 5
	−0.001	case 6	case 7	case 8
	0.007	case 9	case 10	case 11
	−0.007	case 12	case 13	case 14

[a]Case 0 represents no-change in yield curve.
[b]Case 4 represents 0.01 level shift and 0.001 slope change. All other cases are read similarly.

the value preservation performance of a bond only portfolio (portfolio 1) with that of bond plus futures portfolio (portfolio 3). By the duration matching strategy, the bond only portfolio (portfolio 1) is composed of 332.81 units of bond_1 and 1483.05 units of bond_2.[17] This means that the weights of bond_1 and bond_2 are 29.2% and 70.8%, respectively. Then the portfolio's duration matches the manager's planning period and the traditional immunization is accomplished. The third column of Table III shows the values of this portfolio at the end of the holding period. The first value ($2,475,963) of this column (case 0) is the value if there is no change in yields. As anticipated from the immunization theory, if there are only level changes (case 1 and 2), the after-change values ($2,480,121 and $2,480,344) are greater than the no-change value of case 0. However, if the yield curve experiences a large positive slope change (case 9, 10 and 11), the value deteriorates significantly. On the other hand, if the yield curve experiences a large negative slope change (case 12, 13, and 14), the value is significantly enhanced. It is apparent, in spite of the duration matching strategy, how serious the effect of the slope change is on the bond only portfolio. According to our hedge strategy, the futures spread position (portfolio 2) consists of selling 5.384 units of futures contracts on T-bonds and buying 3.780 units of futures contracts on T-bills.[18] The fourth column of Table III shows the value of this spread position (portfolio 2) at the end of the holding period. As discussed in previous sections, the value of this position moves opposite to that of portfolio 1.

The fifth column of Table III is the result of combining portfolio 1 and 2 into portfolio 3. One can see from this column that the variation of portfolio value due to slope change is considerably decreased. Moreover, with only three exceptions (case 1, 5, and 7), the after-change value of portfolio 3 is greater than the no-change value (case 0).

[17]One can compose the duration matching bond portfolio by trading bond_1 and bond_2 under the following conditions:

$$W_1 D_1 + W_2 D_2 = 8 \text{ years}$$

$$W_1 + W_2 = 1$$

where

W_i = Percent of portfolio funds committed to bond i

D_i = Duration of bond i

[18]From Table I, it is known that $D = 7.143$, $C = 95.732$, $V = \$1000{,}000$, $D_s = 0.893$, $C_s = 2.679$, $D_L = 7.981$, and $C_L = 126.167$. Substituting the data into eq. (10), the numbers 5.384 and 3.780 are obtained.

Table III
EFFECTS OF YIELD CURVE SCENARIOS ON REALIZED VALUE OF PORTFOLIOS

Scenario	(λ, α)	Portfolio 1 (Bond only)	Portfolio 2 (Futures)	Portfolio 3 (Bond + Futures)
case 0	(0%, 0)	2,475,963 (12.00)	0	2,475,963 (12.00)
case 1	(1%, 0)	2,480,121 (12.02)	−4310	2,475,811 (12.00)[a]
case 2	(−1%, 0)	2,480,344 (12.02)	−4426	2,476,118 (12.00)
case 3	(0%, 0.001)	2,391,037 (11.51)[a]	89,835	2,480,872 (12.03)
case 4	(1%, 0.001)	2,405,950 (11.60)[a]	82,536	2,488,486 (12.07)
case 5	(−1%, 0.001)	2,383,401 (11.47)[a]	89,280	2,472,681 (11.98)[a]
case 6	(0%, −0.001)	2,583,807 (12.60)	−102,316	2,481,491 (12.03)
case 7	(1%, −0.001)	2,574,656 (12.55)	−102,217	2,472,439 (11.98)[a]
case 8	(−1%, −0.001)	2,603,118 (12.70)	−111,829	2,491,289 (12.09)
case 9	(0%, 0.007)	2,167,351 (10.15)[a]	485,366	2,652,717 (12.97)
case 10	(1%, 0.007)	2,216,676 (10.46)[a]	476,038	2,692,714 (12.97)
case 11	(−1%, 0.007)	2,121,795 (9.86)[a]	488,912	2,610,707 (12.74)
case 12	(0%, −0.007)	4,171,511 (19.55)	−1,290,344	2,881,167 (14.14)
case 13	(1%, −0.007)	3,970,187 (18.81)	−1,194,283	2,775,904 (13.61)
case 14	(−1%, −0.007)	4,409,544 (20.38)	−1,411,414	2,998,130 (14.71)

[a]These numbers are less than the no-change value of case 0.
The numbers in parenthesis are annualized holding period yields (%) over 8 years.

Little (1984) noted that the strategies using financial futures contracts may generate negative cash flows. Because of these negative cash flows, the strategy proposed here cannot guarantee that the after-change value is always greater than the no-change value. In cases 1, 5, and 7, the after-change values are less than the no-change value and this result confirms Little's argument. But these cases are not so serious because the after-change values are almost the same as the no-change value (the minimal annual return of these three cases is about 11.98%).

Table III shows that, among the 14 scenarios of yield curve change, the minimal annual return of bond only portfolio is 9.86% (case 11). However, by using futures spread positions, the manager can achieve a minimal annual return 11.98% (case 7) among the 14 scenarios. This example illustrates that a manager could manage the slope change risk of yield curve using futures spread positions.

CONCLUSION

This article proposes a general yield curve change pattern which allows both parallel and non-parallel shift of the yield curve. It investigates the property of a hedging portfolio whose value moves contrary to the value of the bond portfolio. Irrespective of the level change, the hedging portfolio value should be positive if the slope of yield curve increases and negative if the slope decreases. The spread position of futures contracts, selling one unit of futures contracts on long-term bonds and buying $D_L P_L / D_s P_s$ units of futures contracts on short-term bonds, has the property of this hedging portfolio. A simple example of a hedging strategy utilizing a futures spread position is presented. The example shows that, in spite of duration matching, slope change can seriously deteriorate the value of the bond portfolio. The example also shows that the spread position strategy can be an effective tool for hedging the slope change risk. This study concludes that interest rate futures contracts can be a useful means of managing the non-parallel shift risk of yield curve.

Appendix

Let

$$g(t) = \begin{cases} c_L Z_L \cdot \exp[-h(f,t)t]/P_L, & \text{for } f < t < L \\ (1 + c_L) Z_L \cdot \exp[-h(f,t)t]/P_L, & \text{for } t = L \end{cases}$$

Then

$$D_L = \int_f^L g(t) \cdot (t - f)\, dt = \int_f^L g(t) \cdot t\, dt - f,$$

and

$$C_L = \int_f^L g(t) \cdot (t^2 - f^2)\, dt = \int_f^L g(t) \cdot t^2\, dt - f^2,$$

since

$$\int_f^L g(t)\, dt = 1$$

Cauchy–Schwarz Inequality says that

$$\left(\int_f^L g(t)t\, dt\right)^2 \leq \left(\int_f^L g(t)\, dt\right) \cdot \left(\int_f^L g(t)t^2\, dt\right)$$

First, note that $D_L^2 \leq C_L$, because

$$\begin{aligned} D_L^2 &= \left(\int_f^L g(t)t\, dt\right)^2 - 2f\left(\int_f^L g(t)t\, dt\right) + f^2 \\ &\leq \left(\int_f^L g(t)t\, dt\right)^2 - f^2, \quad \text{since } \int_f^L g(t)t\, dt \geq f \\ &\leq \left(\int_f^L g(t) \cdot t^2\, dt\right) - f^2, \quad \text{by Cauchy–Schwarz Inequality} \\ &= C_L \end{aligned}$$

Next, if T-bill futures are considered, then $D_s = S - f$ and $C_s = S^2 - f^2$, because a T-bill provides cash flow only at maturity S.

When

$$N = \frac{P_L D_L}{P_s D_s}$$

$(P_L C_L - N \cdot P_s C_s)$ is positive if and only if $(C_L D_s - C_s D_L)$ is positive. Substituting D_s and C_s results in $(C_L D_s - C_s D_L) = (S - f) \cdot [C_L - D_L(S + f)]$. Note that $S - f > 0$. Hence, $(P_L C_L - N \cdot P_s C_s)$ is positive if and only if $[C_L - D_L(S + f)]$ is positive. Since $C_L \geq D_L^2$, the condition that $D_L > (S + f)$ is sufficient to guarantee that $[C_L - D_L(S + f)] > 0$.

The futures contracts traded in real futures markets have at most 2 years remaining until delivery (i.e., $f \leq 2$). The deliverable T-bills mature in 1/4 year (i.e., $S - f = 1/4$) and the deliverable T-bonds should have at least 15 years remaining until maturity or first permissible call date. Hence, in general, D_L is greater than $S + f$ where $S - f$ is 1/4 year and f is less than 2 years. In other words, the condition $[C_L - D_L(S + f)] > 0$ corresponds to real market conditions.

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