European Options on Bond Futures: A Closed Form Solution

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INTRODUCTION

An important segment of derivative assets research involves the pursuit of closed form solutions for the price function of the various instruments. The best known example is the Black–Scholes formula for the price of European stock options. While the exchanges usually trade American options, closed form solutions for the prices of American options do not usually exist.[1] Closed form solutions for the prices of the corresponding European options, however, provide useful reference values and bounds for American option prices, help the study of early exercise premiums, and enhance the understanding of options prices' components and determinants. Moreover, if early exercise is never optimal, the price of American options is, of course, equal to the price of the corresponding European options.

About 10 years ago, options on bond futures were introduced on several exchanges. Since that time, they have been traded continuously with substantial volume. This article provides a closed form solution for the price of a European call option written on a bond futures contract. While previous futures options models, such as Black's (1976), exogenously assume a futures price with a non-stochastic instantaneous variance, the model in this article uses endogenously priced bond and futures contracts and allows a stochastic instantaneous variance for the underlying interest rate. Recently, Chen (1992) and Musiela, Turnbull, and Wakeman (1991) priced a bond futures option where the underlying interest rate evolves with a deterministic instantaneous variance.

Multiperiod default-free bonds were first priced in Vasicek (1977), Dothan (1978), and Richard (1978), who used arbitrage arguments. Cox, Ingersoll, and Ross (1985a) developed an intertemporal equilibrium model, which they used in another article to price bonds and European bond options[2] [Cox, Ingersoll, and Ross (1985b)]. Their main example used a stochastic instantaneous spot rate of interest that follows a mean reverting Itô diffusion with an instantaneous variance proportional to its level. In an earlier article,

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[1]The price of an American option is the solution of a free boundary problem. In most, if not all, cases these solutions are determined numerically.

[2]Jamshidian (1989) and Rabinovitch (1989) priced a European bond option on the bond priced in Vasicek (1977). There, the instantaneous spot rate of interest evolves as the Ornstein–Uhlenbeck process, a mean reverting process with a deterministic instantaneous variance.

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Cox, Ingersoll, and Ross (1981) priced a futures contract written on this same default-free bond.[3] While the bond, bond option, and bond futures prices were provided by the above works, this article provides the price of the bond futures option.

It is shown that the bond futures option price is a difference of two terms. As in the case of the corresponding bond option price, each term is the product of a price and a probability. The price in the subtrahend (the present value of the exercise price) and both probabilities are similar to those in the bond option price. Unlike the case of the bond option, however, the price in the minuend is not the price of the underlying asset, the futures contract. A reason for this difference is explained.

In addition, it is demonstrated that the components of the bond futures price can be determined recursively. This quality allows the decomposition of the futures price into a separable function of the components of the price of a similar futures contract with a shorter term to maturity. In particular, when this shorter term goes to zero, the futures price becomes a separable function of the components of the price of the bond on which it is written. This decomposition aids in the interpretation of the bond futures option price.

The next section outlines the assumptions and introduces definitions and notations that are used in this study. The third section contains the main results of the article and the last section discusses the results and interprets the difference in the form of a bond option price and a bond futures option price.

Assumptions and Notations

Assume frictionless competitive markets, where investors continuously and without restrictions trade a variety of interest rate contingent claims. In particular, they trade bonds, bond futures, and bond futures options. The stochastic instantaneous spot rate of interest r (henceforth, the spot rate or the interest rate) evolves as the stochastic differential equation

$$dr = k(\mu - r)\,dt + \sigma\sqrt{r}\,dW_t \tag{1}$$

where W_t is a scalar Wiener Process, μ is the asymptotic value of the spot rate, k is the rate in which the spot rate reverts to its asymptotic value, and $\sigma^2 r$ is the instantaneous variance of the spot rate. The parameters μ, k, and σ^2 are scalar constants.

The following definitions describe the bond, bond futures, and bond futures option contracts.

Definition 1. $\mathrm{P(r, u; v)}$ *is the price at time* u *of a default-free unit discount bond that matures at time* v. *The prevailing spot rate of interest at time* u *is* r, *and* $\mathrm{u \leq v}$. *This bond is also called the underlying bond.*

Definition 2. $\mathrm{H(r, t; u, v)}$ *is the price at time* t *of a futures contract that matures at time* u. *The contract is written on a default-free unit discount bond that matures at time* v, *the prevailing spot rate at* t *is* r, *and* $\mathrm{t \leq u \leq v}$. *This futures contract is called the underlying futures contract.*

Definition 3. $\mathrm{D(r, s; ,t, u, v, E)}$ *is the price at time* s *of a European call option that expires at time* t. *The option is written on a futures contract that matures at time* u, *and*

[3]See also Jarrow and Oldfield (1981). For the price of a similar futures contract where the interest rate follows the deterministic variance Ornstein–Uhlenbeck process, see Carr (1988), Musiela, Turnbull, and Wakeman (1991), and Chen (1992). See also Turnbull and Milne (1991), who provide a closed form solution for the price of a bond futures option in a discrete time framework in which the spot rate is normally distributed.

the futures contract is written on a default-free discount bond that matures at time v. *The prevailing spot rate at time* s *is* r, *the exercise price of the option is* E *and* s ≤ t ≤ u ≤ v.

For notational simplicity, arguments of functions are omitted sometimes. Lower case subscripts of functions denote partial derivatives.

THE PRICE OF AN OPTION ON BOND FUTURES

Let $F(r,t)$ be the value at time t of a claim contingent on the interest rate r. Assume that the claim expires at time u, $t \leq u$. Several authors, including Cox, Ingersoll, and Ross (1985b),[4] have demonstrated an equilibrium where the price of the claim obeys the general valuation equation

$$\frac{1}{2}\sigma^2 rF_{rr} + [k(\mu - r) - \Lambda(r,t)]F_r + F_t - rF + \delta(r,t) = 0,$$
$$F(r,u) = G(r). \qquad (2)$$

The value of the claim at maturity is $G(r)$. $\delta(r,t)$ is the continuous payment generated by the claim, and $\Lambda(r,t)$ is the market risk premium. The general valuation equation is derived within both arbitrage models and equilibrium models. While the market risk premium is assumed exogenously in arbitrage models—either directly, as in Vasicek (1977), or through prices of spanning securities, as in Dothan and Williams (1980)—it is endogenously determined within general equilibrium models. Cox, Ingersoll, and Ross (1985b) identified $\Lambda(r,t)$ as the covariance of changes in the interest rate with percent changes in optimally invested wealth.[5] They demonstrated also that under logarithmic preferences $\Lambda(r,t) = \lambda r$, where the market risk parameter λ is a constant. This functional form of the market risk premium is assumed here.

The following two propositions from the works of Cox, Ingersoll, and Ross present results used in the construction of the price of the bond futures option. To simplify the presentation of results, this article uses notation and functional definitions that are different from those used by Cox, Ingersoll, and Ross.

Proposition 1 [Cox, Ingersoll, and Ross (1985b)]. *The price of the default-free unit discount bond $P(r,u;\nu)$, as described in Definition 1, is*

$$P(r,u;\nu) = A(u,\nu)e^{-rB(u,\nu)} \qquad (3)$$

where

$$A(u,\nu) \triangleq \left\{\frac{\phi(u,\nu)}{\psi(u,\nu)}\exp\left[\frac{1}{2}\sigma^2\theta(\nu - u)\right]\right\}^{q+1} \qquad (4)$$

$$B(u,\nu) \triangleq 2[\sigma^2\psi(u,\nu)]^{-1} \qquad (5)$$

$$\psi(u,\nu) \triangleq \phi(u,\nu) + \theta \qquad (6)$$

$$\phi(u,\nu) \triangleq \frac{2\gamma/\sigma^2}{e^{\gamma(\nu-u)} - 1} \qquad (7)$$

$$\theta \triangleq (k + \lambda + \gamma)/\sigma^2, \qquad (8)$$

$$\gamma \triangleq \sqrt{(k + \lambda)^2 + 2\sigma^2} \qquad (9)$$

[4]For additional references see Cox, Ingersoll, and Ross (1985a and 1985b).

[5]They showed that under constant relative risk aversion, the market risk premium does not depend on wealth.

$$q \overset{\Delta}{=} 2k\mu/\sigma^2 - 1 \tag{10}$$

The bond price is the solution of the system

$$\frac{1}{2}\sigma^2 rP_{rr} + [k(\mu - r) - \lambda r]P_r + P_u - rP = 0, \qquad P(r, \nu; \nu) = 1 \tag{11}$$

Proposition 2 [Cox, Ingersoll, and Ross (1981)]. *The price of the futures contract* H(r, t; u, ν), *as described in Definition 2, is*

$$H(r, t; u, \nu) = A_H(t, u, \nu)e^{-rB_H(t,u,\nu)} \tag{12}$$

where

$$A_H(t, u, \nu) \overset{\Delta}{=} A(u, \nu)\left\{\frac{1}{1 + \frac{1}{\alpha}B(u, \nu)[1 - e^{-(k+\lambda)(u-t)}]}\right\}^{q+1} \tag{13}$$

$$B_H(t, u, \nu) \overset{\Delta}{=} B(u, \nu)\left\{\frac{e^{-(k+\lambda)(u-t)}}{1 + \frac{1}{\alpha}B(u, \nu)[1 - e^{-(k+\lambda)(u-t)}]}\right\}^{q+1} \tag{14}$$

$$\alpha \overset{\Delta}{=} 2(k + \lambda)/\sigma^2 \tag{15}$$

A(u, v), B(u, v), *and* q *are defined in Proposition 1.*

The following corollaries establish two properties of the bond futures price. First, the components of the futures price, A_H and B_H [see eqs. (12), (13), and (14)], can be determined recursively. Second, the bond futures price is a separable function of the components of the price of another bond futures price, one similar to the underlying futures contract but with an intermediate maturity, i.e., any initiation date between the current date t and the underlying futures contract's expiration date u. A special case of this decomposition is the presentation of the bond futures price as a separable function of the components of the price of the underlying bond [see eq. (21)]. The decomposition property of the bond futures price aids in the interpretation, (in the last section), of the bond futures option price.

Corollary 1 *Let* t_u *represent an intermediate time point,* $t \le t_u \le u$; *then the expressions* $A_H(t, u, v)$ *and* $B_H(t, u, v)$ *obey the recursive relations*

$$A_H(t, u, \nu) = \overline{A}_H(t, t_u, u, \nu)A_H(t_u, u, \nu) \tag{16}$$

$$B_H(t, u, \nu) = \overline{B}_H(t, t_u, u, \nu)B_H(t_u, u, \nu) \tag{17}$$

where[6]

$$\overline{A}_H(t, t_u, u, \nu) \overset{\Delta}{=} \left\{\frac{1}{1 + \frac{1}{\alpha}B_H(t_u, u, \nu)[1 - e^{-(k+\lambda)(t_u-t)}]}\right\}^{q+1} \tag{18}$$

[6]Another presentation of eq. (18) is

$$\overline{A}_H(t, t_u, u, \nu) \overset{\Delta}{=} \left\{\frac{e^{(k+\lambda)(t_u-t)}}{\overline{B}_H(t_u, u, \nu)}\right\}^{q+1}$$

and

$$\overline{B}_H(t, t_u, u, \nu) \triangleq \left\{ \frac{e^{-(k+\lambda)(t_u - t)}}{1 + \dfrac{1}{\alpha} B_H(t_u, u, \nu)[1 - e^{-(k+\lambda)(t_u - t)}]} \right\} \tag{19}$$

Proof: For any time t_u, $t \leq t_u \leq u$, substitution of eq. (18) and then eqs. (14) and (13) in the left-hand side of eq. (16) demonstrates the equivalence of eqs. (13) and (16). For times t_u, $t_u = u$, this equivalence follows from the equalities $A_H(u, u, \nu) = A(u, \nu)$ and $B_H(u, u, \nu) = B(u, \nu)$. These equalities are indicated by eqs. (13) and (14). Analogous substitutions demonstrate the equivalence of eqs. (14) and (17).

Corollary 2. *Let* t_u *be the same as in Corollary 1. The price of the futures contract* $H(r, t; u, v)$ *can be expressed as a separable function of the components of the price of* $H(r, t_u; u, v)$*, a similar futures contract with an intermediate term to maturity* $u - t_u$*;*

$$H(r, t; u, \nu) = \overline{A}_H(t, t_u, u, \nu) A_H(t_u, u, \nu) e^{-rB_H(t_u, u, \nu)\overline{B}_H(t, t_u, u, \nu)} \tag{20}$$

where $\overline{A}_H(t, t_u, u, v)$ *and* $\overline{B}_H(t, t_u, u, \nu)$ *are as defined in Corollary 1. In particular, when* $t_u = u$*, the price of the futures contract becomes a separable function of the components of* $P(r, u; \nu)$*, the price of the bond on which it is written:*

$$H(r, t; u, \nu) = \overline{A}_H(t, u, u, \nu) A(u, \nu) e^{-rB(u, \nu)\overline{B}_H(t, u, u, \nu)} \tag{21}$$

Proof: Substitution of Eqs. (16) and (17) into eq. (12) yields eq. (20). An explanation of why eq. (21) follows from eq. (20) is in the proof of Corollary 1.

The bond futures price is the solution to the system

$$\frac{1}{2}\sigma^2 r H_{rr} + [k(\mu - r) - \lambda r]H_r + H_t - rH + \delta(r, t) = 0 \tag{22}$$

$$\delta(r, t) = rH(r, t; u, \nu), \qquad H(r, u; u, \nu) = P(r, u; \nu) \tag{23}$$

The following is the main proposition of this article.

Proposition 3. *The price of the option on bond futures* $D(r, s; t, u, v, E)$*, as described in Definition 3, is*

$$D(r, s; t, u, \nu, E) = \overline{D}(r, s, t, u, \nu) H(r, t; u, \nu) \chi^2(d_1; d_2, d_3) - P(r, s; t) E \chi^2(e_1; e_2, e_3) \tag{24}$$

where

$$\overline{D}(r, s, t, u, \nu) \triangleq A_D(s, t, u, \nu) e^{-rB_D(s, t, u, \nu)} \tag{25}$$

$$A_D(s, t, u, \nu) \triangleq \left\{ \frac{\phi(s, t) \exp\left[\dfrac{1}{2}\sigma^2 \theta(t - s)\right]}{\psi(s, t) + B_H(t, u, \nu)} \right\}^{q+1} \tag{26}$$

$$B_D(s, t, u, \nu) \triangleq \left\{ \frac{\dfrac{2}{\sigma^2} - \alpha B_H(t, u, \nu) - B_H^2(t, u, \nu)}{\psi(s, t) + B_H(t, u, \nu)} \right\} \tag{27}$$

$$d_1 = 2r_H[\psi(s, t) + B_H(t, u, \nu)] \tag{28}$$

$$r_H \triangleq \frac{1}{B_H(t,u,\nu)} \ln[A_H(t,u,\nu)/E] \tag{29}$$

$$d_2 = 2(q + 1) \tag{30}$$

$$d_3 \triangleq \frac{2\phi^2(s,t)re^{\gamma(t-s)}}{\psi(s,t) + B_H(t,u,\nu)} \tag{31}$$

$$e_1 = 2r_H\psi(s,t) \tag{32}$$

$$e_2 = 2(q + 1) \tag{33}$$

$$e_3 = 2\frac{\phi^2(s,t)}{\psi(s,t)} re^{\gamma(t-s)} \tag{34}$$

where $\chi^2(z_1; z_2, z_3)$ is a noncentral chi-square function[7] with argument z_1, with z_2 degrees of freedom, and with noncentrality parameter z_3. Proposition 1 specified $P(r,s;t)$, q, θ, γ, $\phi(s,t)$, and $\psi(s,t)$; and Proposition 2 defined $H(r,t;u,\nu)$, $A_H(t,u,\nu)$, $B_H(t,u,\nu)$, and α. For a description of the noncentral χ^2 distribution and density functions, see Cox, Ingersoll, and Ross (1985b) and Johnson and Kotz (1970).

The Bond futures option price is the unique solution of the system,

$$\frac{1}{2}\sigma^2 rD_{rr} + [k(\mu - r) - \lambda r]D_r + D_s - rD = 0 \tag{35}$$

$$D(r,t;t,u,\nu,E) = \max[H(r,t;u,\nu) - E, 0] \tag{36}$$

The price of the option on bond futures is obtained by methods described in Friedman (1975). Because the proof of Proposition 3 is long, it is not presented here.

Discussion

With one exception, the form of the bond futures option price is similar to that of the bond option price analyzed by Cox, Ingersoll, and Ross (1985b, p. 396). Both include two terms, each multiplied by a cumulative probability, drawn from a noncentral $\chi 2$ distribution function.[8] The parameters of the distribution functions here are identical to or the direct equivalents of the parameters there. Furthermore, in both cases the first part of the second term is the present value of the option's exercise price.

The difference between the form of the bond futures option price and that of the bond option price lies in the first part of the first term. In the case of the bond option, this part is the price of the bond on which the option is written.[9] In the case of the bond futures option, however, this part is different from the price of the underlying futures contract. The relevant part of the bond futures option price is $\overline{D}(r,s,t,u,\nu)H(r,t;u,\nu)$ [see eq. (24)], while the price of the underlying futures contract is $H(r,s;u,\nu)$.

According to Corollary 2, the price of the underlying futures contract can be decomposed. It can be written as a separable function of the components of the price of a similar futures contract that exists at any time between the current time s and the

[7]The noncentral chi-square distribution function is the distribution function of the sum of the squares of shifted independent standard normal random variables. The degrees of freedom are equal to the number of variables in the sum, and the noncentrality parameter is equal to the sum of squares of the "shifts".

[8]This is the probability distribution function of the rate of interest induced by the assumption of its evolution as specified in eq. (1).

[9]Therefore, the form of the bond option price is similar to the form of Black and Scholes's (1973) stock option price.

maturity date u of the underlying futures contract. In particular, this intermediate time can be the option's maturity date t. In this case, the decomposition of the underlying futures price $H(r, s; u, \nu)$ is in terms of the components of the futures price $H(r, t; u, \nu)$. If this presentation of the underlying futures price is compared to the relevant expression in the bond futures option price $\overline{D}(r, s, t, u, \nu)H(r, t; u, \nu)$, it can be seen that both are functions of the components of the same intermediate futures price but they differ in the type of function. While the underlying futures price is a separable function of the price components, the relevant term in the futures option price is a multiplicatively separable function of the price itself.

The familiar part of the decomposed bond futures option price is equal to $H(r, t; u, \nu)$. This part pertains to the period in the life of the underlying futures contract that starts with the option's expiration date t and ends with the futures maturity date u. This suggests that the difference in the forms of the bond option price and bond futures option price might stem from differences that prevail throughout the life of the futures option, i.e., the period that starts at the time of the valuation s and ends at the option's expiration date t. Although the form of the corresponding functionals $\overline{D}(r, s, t, u, \nu)$, $\overline{A}_H(s, t, u, \nu)$, and $\overline{B}_H(s, t, u, \nu)$ is too complex to suggest an interpretation of these differences, the reason might be the different nature of the contracts' early exercise policies. Clearly, the early exercise policies are relevant only during the life of the option, the period of the expressed difference.

Since the underlying bond makes no payments through the life of the option, a bond option is never exercised early.[10] By contrast, the bond futures option may be exercised early. As eqs. (22) and (23) demonstrate, the underlying futures price is the value of a contract that pays a continuous cash flow equal to the prevailing interest rate times the prevailing futures price, plus a single payment at a maturity of the value of the bond on which it is written.[11] The continuous cash flow associated with the bond futures price may induce the early exercise of the bond futures option just as stock dividends may induce early exercise of stock options.

The above idea may be demonstrated as follows. Let the intrinsic value of a call option be the maximum of zero and the difference between the price of the underlying asset and the exercise price. American options prices are greater than the corresponding European options prices if and only if there is a possibility of early exercise. To have this possibility with American bond futures options, it is necessary and sufficient to have instances where corresponding European bond futures options prices are lower than their intrinsic values.[12] The options prices here satisfy this condition: we identify parameter values for which prices of European call bond futures options are below their intrinsic values.[13] Clearly, the gap between the intrinsic value of the futures option and its price is

[10] See also Cox, Ingersoll, and Ross (1985b), footnote 12.

[11] See also Cox, Ingersoll, and Ross (1981), Proposition 7.

[12] Merton (1973), for example, showed the necessity of this condition: If European call options prices are never lower than their intrinsic value, the cash flow from selling the call option is never lower than the cash flow from exercising the corresponding American option. Since American option prices are never lower than the corresponding European option prices, there will never be an early exercise of the corresponding American options. The sufficiency of the condition is shown by contradiction. Suppose there are cases where European call options are below their intrinsic value and there is no possibility that the corresponding American options would be exercised early. Then the corresponding American options prices would equal the European options prices. Therefore, the corresponding American option prices will fall below their intrinsic value. This is, clearly, a contradiction.

[13] For example, consider the case where $k = 1.0$, $\lambda = 0.5$, $\sigma^2 = 0.1$, $\mu = 0.1$, $E = 0.9408$, $\nu - u = u - t = t - s = 0.3$, and $r = 0.1$. Then, the intrinsic value of the bond futures option (for a one dollar bond) is 0.0363, but the bond futures option price is only 0.0353.

a lower bound for the gap between American and European bond futures options prices. For most parameter values, however, European bond futures options prices are greater than their intrinsic values.

A closed form solution probably does not exist for the price of the American option that corresponds to the European bond futures option priced here. As in the Black–Scholes model for the stock options, one mus resort to numerical methods to price the corresponding American options. One popular numerical procedure applies the implicit finite difference method to a grid and, beginning at the terminal conditions, solves the valuation equation recursively. The early exercise boundary, then, is the locus of futures prices and term to maturity pairs at which the option price is equal to its intrinsic value. Another numerical procedure is the binomial approximation to diffusion processes suggested by Nelson and Ramaswamy (1990).

Some of the currently traded contracts for which this model is relevant are Treasury Bill futures options, Municipal Bond Index futures options, Eurodollar futures options, and LIBOR futures options. Usually, both serial and quarterly contracts are traded for each of those contracts. While the quarterly futures options expire near the expiration date of the underlying futures, the serial futures options expire between the maturity dates of the futures contracts.

The closer bond futures are to maturity, the closer their prices are to those of the underlying bonds. Therefore, prices of quarterly bond futures options are closer to the prices of the corresponding bond options than prices of similar serial options are. In particular, when the futures options and the underlying futures contracts expire together, the bond futures options become bond options. Indeed, in this borderline case, the solutions for the bond futures prices that are developed here are identical to the solutions of Cox, Ingersoll, and Ross (1985b) for the corresponding bond option prices. This model, therefore, provides a more significant contribution to the pricing of the serial bond futures options than to the pricing of quarterly bond futures options.

Future research could pursue closed form solutions for bond futures option prices under different interest rate dynamics. It could compare the analytical models, and contrast the closed form solutions with those of American futures option prices determined through numerical procedures.

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