Optimal Hedging under Indivisible Choices

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INTRODUCTION

It is widely recognized that financial futures contracts provide potential risk reduction benefits to stock portfolios, foreign currency portfolios, and bond portfolios. The hedging performance of financial futures contracts was first investigated by Ederington (1976). Anderson and Danthine (1980) extended optimal hedging to portfolios involving multiple spot and futures positions. Figlewski (1984) addressed the hedging performance of stock index futures. Hilliard (1984) addressed hedging interest rate risk with futures portfolios. Koppenhaver (1985) and Morgan, Shome, and Smith (1988) investigated optimal futures hedging by banks. Kamara and Siegel (1987) studied the optimal hedging strategy with futures contracts having multiple delivery specifications.

Hedging can be undertaken by individuals, financial institutions, and industrial corporations. Individuals are likely to possess spot positions of small sizes to be hedged while financial institutions and corporations are more likely to possess large spot positions to be hedged. For instance, Grammatikos, Saunders, and Swary (1986) found that the average monthly net position of an average U.S. bank in five currencies between 1976 and 1981 were 265.95 million British pounds, 924.49 million Canadian dollars, 5296.56 million Deutschemarks, 509,613.95 million Japanese yen, and 19,015.51 million Swiss francs. These institutions have substantial foreign exchange risk exposure.

In general, hedging may be accomplished by using futures contracts or forward contracts. Futures contracts are characterized by standardized contract terms. For example, the size of the British pound futures contract traded on the International Money Market (IMM) of the Chicago Mercantile Exchange is 25,000 British pounds. This poses a problem to large and small hedgers alike in that the size of the contract may not fit the specific hedging needs of the hedger. The advantage of standardized contract terms, however, is the higher liquidity of the futures contract over the forward contract and the resulting lower transactions costs.

This article investigates the effect of the standard size specifications of futures contracts upon hedging performance. The optimal hedging model is extended by incorporating the constraint of indivisible choices. Empirical evidence on hedging foreign exchange risk is presented that shows that the hedger is disadvantaged by the fixed sizes of futures contracts.

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There are two issues that motivate the investigation conducted by this article. These are (1) whether corporations should undertake hedging activities and (2) the determinants of a financial contract's success.

Finance literature has debated whether corporations are justified in hedging risks on behalf of shareholders. An answer offered by Dufey and Srinivasulu (1984) is that hedging by corporations is justified if corporations can hedge more efficiently than shareholders. Therefore, if contrast size specifications deter small hedgers (shareholders) but are no impediment to large hedgers (corporations); then the corporation is better able to hedge (or hedges more efficiently) than the shareholders. The corporation is then justified in hedging on behalf of shareholders.

There are alternatives to hedging with fixed-size futures contracts. However, these have their own disadvantages. Small and large hedgers alike may hedge specific exposures in foreign currency through the interbank forward market, where the indivisibility problem is absent. Some disadvantages of this market for small hedgers are that banks may: limit access to these markets to customers engaged in foreign trade or those with foreign subsidiaries; insist on compensating balances (which are costly for the individual, but part of ongoing business activities of the corporation); and charge bid–ask spreads that are larger for small volume customers. Small hedgers may also hedge through the futures contracts of the MidAmerica Commodity Exchange, whose contract sizes are much smaller than those offered by the Chicago Board of Trade and the IMM. For example, the sizes of the foreign currency futures contracts offered by the MidAmerica Commodity Exchange are half the size of those offered by the IMM.

In the early years of futures trading, U.S. futures exchanges dominated the industry. However, increasing competition from emerging futures exchanges in Europe and the Far East has caused a fall in the market share of U.S. exchanges. In 1989, this dropped to 62.21% from 69.11% in 1988 [See Kolb (1991, p. 52)]. The competition increases the importance of determining the characteristics responsible for the success or failure of a futures contract. The characteristics that could affect the success or failure of a futures contract may be classified into exchange characteristics, spot commodity characteristics, and contract characteristics. Contract characteristics include size, maturity, delivery procedure, and others. This study focuses on the importance of the size of the contract as it affects the performance of hedged portfolios and its acceptability by hedgers.

The next section of this article discusses the optimal hedging model under the indivisible choice constraint, proposes solution methods from the optimization literature, and compares it to the unconstrained model. In Section II, the hedging performance of the constrained model is compared to that of the unconstrained model in hedging foreign exchange risk with currency futures contracts of fixed sizes on an ex post and ex ante basis. It is shown that if the spot positions are of small sizes, then the hedger is disadvantaged by the fixed sizes of futures contracts. This disadvantage disappears if the spot positions are of large sizes.

Common practice is to deal with indivisible fixed sizes of futures contracts by rounding the solution to the unconstrained choice problem to the nearest integer and initiating a position in the rounded number of futures contracts. The hedging performance of the constrained model is, therefore, compared to that of the rounded unconstrained model both on an ex post and ex ante basis. As expected, the constrained model performs as well or better than the rounded unconstrained model on an ex post basis. The performance of the constrained model depends on the size of the spot position. The lower the size the better the performance of the constrained model. Parameter instability reduces the superiority of the model on an ex ante basis.

OPTIMAL HEDGING MODEL WITH INDIVISIBLE CHOICES

Hedger's Integer Constrained Choice Model versus Unconstrained Choice Model

The Anderson and Danthine (1980) model is used with a few modifications to that of an individual agent's choice among n futures contracts of fixed size which differ in delivered commodity, delivery location or delivery time. It is assumed that there are two dates, time 0 and time 1. It is assumed that the spot positions C_i to be sold (or bought) at time 1 in the m cash goods are known with certainty at time 0. At time 0 the agent chooses the number of futures contracts N_i of fixed size X_i to buy (or sell) at time 0. The agent's profit at time 1 is a random variable given by

$$\tilde{\pi} = \sum_{i=1}^{m} C_i(\tilde{S}_{i,1} - S_{i,0}) + \sum_{i=1}^{n} N_i X_i(\tilde{F}_{i,1} - F_{i,0}) \tag{1}$$

where

$\tilde{S}_{i,1}$ = spot price of the cash good i at time 1 (a random variable)
$S_{i,0}$ = spot price of the cash good i at time 0
$\tilde{F}_{i,1}$ = futures price of the ith futures contract at time 1 (random variable)
$F_{i,0}$ = futures prices of the ith futures contract at time 0

The hedging agent's problem is to minimize the variance of profit $V\ (\tilde{\pi})$ with respect to N_i, $i = 1 \dots n$, subject to the constraint that N_i = an integer, $i = 1 \dots n$. This is called the constrained choice problem.

The problem may be recognized as one which involves integer nonlinear optimization. There is no closed form solution available to the above problem as there is for the unconstrained choice problem, which is to minimize $V\ (\pi)$ with respect to N_i, $i = 1 \dots n$, whose solution is obtained by solving the following system of n linear equations, for $j = 1 \dots n$,

$$\sum_{i=1}^{n} C_i \sigma_{Si,Fj} + \sum_{i=1}^{n} N_i X_i \sigma_{Fi,Fj} = 0$$

where

$\sigma_{Si,Fj}$ = covariance between the ith cash price and the jth futures price at time1
$\sigma_{Fi,Fj}$ = covariance between the ith futures price and the jth futures price at time 1

The constrained choice problem is much more difficult to solve than the unconstrained choice problem. This may partially explain why it has not been attempted heretofore.[1]

Solution Methods of Constrained Choice Model

There are three possible methods of solving the constrained choice problem. These are

1) Enumeration of all of the possible integer solutions. This becomes infeasible if the number of decision variables is large and if they could take on large values.

[1]Integer Programming has been used in the finance literature in capital budgeting [Pettway (1973); Rychel (1977); McBride (1981)], in leasing [Capettini (1981)], and in credit investigation [Stowe (1985)]. However, all of the above applications were confined either to the solution of integer *linear* programming or to the solution of problems with binary *0–1* decision variables. The solution of *nonlinear optimization* with integer variables (not confined to binary 0–1 decision variables) has not heretofore been applied in the finance literature to the author's knowledge.

2) Transformation of the integer variables N_i into binary variables which take on values 0 or 1. This is described in detail in Rao, pp. 557–568 and Appendix I. This transformation will effectively change the objective function into a linear function of the binary variables. However constraints are to be added. The transformed problem may then be solved using integer 0–1 linear programming which has been more commonly used in the finance literature and is already readily solved by optimization packages such as LINDO.[2] This method is infeasible if the variables N_i can take on large values. For example, if $N_i \leq 4$, $i = 1 \ldots n$ and $n = 2$, the number of constraints added in the transformation process will be 132.
3) Transformation of the objective function by adding (1) a penalty function, which is a function of N_i and its two neighboring integer values, (2) a function of constraints and use of iteration to solve for the N_i. This is described in detail in Rao, pp. 557–568 and Appendix I. The advantage of this approach is that it can handle large values of N_i and could quickly lead to an optimum solution. The disadvantage is that it could settle down at a "false optimum" which is a solution other than an integer solution, and steps have to be taken to recover from this situation.

The optimization and recovery processes are briefly described below. The transformed objective function is given by

Transformed Objective Function $= V(\tilde{\pi}) + r_k$. Function of Constraints $+ s_k$. Penalty Function

The function of the constraints is designed to have very large values when the constraint boundary is approached. For instance, to minimize the optimization time one might wish to specify upper (U_i) and lower (L_i) limits for N_i. The function of the constraints would then be given by $\Sigma_i 1/(U_i - N_i) + 1/(L_i + N_i)$.

The penalty function is designed to have large values when the N_i take on noninteger values and 0 if the N_i take on integer values.

r_k and s_k are arbitrary weighting factors and k stands for the iteration number. Details of the choice of r_k and s_k are described in Rao, pp. 557–568.

The transformed objective function is minimized. It is checked if the process converges to a minimum value of $V(\tilde{\pi})$. If not, r_k and s_k are updated and the above process is repeated.

If the process converges to a minimum, it is checked to determine if the N_i are integers. If yes, this is the solution to the constrained choice problem.

If no, this is a "false optimum." A false optimum results if one of the discretization points lies very close to the constraint boundary but on the infeasible side. The solution is to move away from the constraint boundary to the true optimum, by increasing the value of the weighting factor r_k on the function of the constraints. This is done recursively until the true optimum (a discrete optimum solution) is reached.

Note that if a closed form solution to the constrained choice problem were possible, one could theoretically compare the optimal position in the futures contracts under the constrained and unconstrained approaches as well as their hedging performance. For the present however, all that may be said is that the reduction in risk of the cash position

[2]LINDO (Linear interactive discrete optimizer) is a mathematical programming software package, designed to solve linear programming problems which can be used on the IBM/PC. The copyright (1984, 1985) is owned by Lindo Systems, Inc. Portions of the copyright (1981) are owned by Microsoft Corporation. Information on the use of LINDO is provided in Schrage (1986).

possible under the constrained choice cannot be higher than that under the unconstrained choice. At best, the reduction in risk under the two approaches would be equal, if the unconstrained choice naturally gives rise to an integer number of futures contracts in the solution anyway, which is highly unlikely. Note that the solution to the unconstrained choice may be implementable, in reality, because of the size constraint. Therefore, the agent is worse off, in reality, due to having to conform to the institutional requirement of fixed sizes of futures contracts.

Rounded Unconstrained Choice Model versus Constrained Choice Model

The optimum number of futures contracts under the constrained choice problem need not necessarily be the same as the rounded solution to the unconstrained choice problem, when the spot portfolio has two or more commodities and futures contract on two or more commodities are used in hedging. This is illustrated below with an example and then explained.

Assume the hedger holds a currency portfolio of $C_1 = 24,156$ British pounds and $C_2 = 95,783$ Deutschemarks. The sizes of the futures contracts are $X_1 = 25,000$ British pounds and $X_2 = 125,000$ Deutschemarks. The following parameters are input into the hedging decision.

σ_{S_1} = standard deviation of the percentage return on British pounds
= 0.05

σ_{S_2} = standard deviation of the percentage return on Deutschemarks
= 0.017

σ_{F_1} = standard deviation of the percentage return on the British pound futures contract
= 0.054

σ_{F_2} = standard deviation of the percentage return on the Deutschemark futures contract
= 0.017

$\rho_{S_1S_2}$ = correlation between the return on British pounds and Deutschemarks
= 0.8851

$\rho_{S_1F_1}$ = correlation between the return on British pounds and the British pound futures contract
= 0.9946

$\rho_{S_1F_2}$ = correlation between the return on British pounds and the Deutschemark futures contract
= 0.8918

$\rho_{S_2F_1}$ = correlation between the return on Deutschemarks and the British pound futures contract
= 0.9036

$\rho_{S_2F_2}$ = correlation between the return on Deutschemarks and the Deutschemark futures contract
= 0.9982

$\rho_{F_1F_2}$ = correlation between the return on the British pound futures contract and Deutschemark futures contract
= 0.9131

The hedger's problem is to minimize $V(\tilde{\pi})$, where

$$V(\tilde{\pi}) = C_1^2\sigma_{S_1}^2 + C_2^2\sigma_{S_2}^2 + 2C_1C_2\rho_{S_1S_2}\sigma_{S_1}\sigma_{S_2} + N_1^2X_1^2\sigma_{F_1}^2 + N_2^2X_2^2\sigma_{F_2}^2 + 2X_1\sigma_{F_1}(C_1\rho_{S_1F_1}\sigma_{S_1} + C_2\rho_{S_2F_1}\sigma_{S_2})N_1 + 2X_2\sigma_{F_2}(C_1\rho_{S_1F_2}\sigma_{S_1} + C_2\rho_{S_2F_2}\sigma_{S_2})N_2 + 2X_1X_2\rho_{F_1F_2}\sigma_{F_1}\sigma_{F_2}N_1N_2 \quad (2)$$

Substitution of the numerical values above, the objective function is

$$V(\tilde{\pi}) = 759.16 + 182.25\ N_1^2 + 451.56\ N_2^2 + 721.61\ N_1 + 1148.56\ N_2 + 523.89\ N_1N_2$$

Minimizing the objective function with respect to N_1 and N_2, leads to the unconstrained solution

$N_{1,u}$ = optimal number of British pound futures contracts under the unconstrained choice
= -0.91

$N_{2,u}$ = optimal number of Deutschemark futures contracts under the unconstrained choice
= -0.74

The rounded solutions to the unconstrained choice problem are $N_{1,r} = -2$ and $N_{2,r} = -1$. The value of the objective function at these values of N_1 and N_2 is $V(\tilde{\pi}) = 46.69$.

Minimizing the objective function with respect to N_1 and N_2, with the addition of the constraint that N_1 and N_2 be integers, leads to the solution

$N_{1,c}$ = optimal number of British pound futures contracts under the constrained choice
= -2

$N_{2,c}$ = optimal number of Deutschemark futures contracts under the constrained choice
= 0

The value of the objective function at these values of N_1 and N_2 is $V(\tilde{\pi}) = 44.94$, which is lower. This illustrates that the rounded solution to the unconstrained choice problem need not be the same as the solution to the constrained choice problem.

When the hedged portfolio contains one spot commodity and futures contract on one commodity, the rounded solution to the unconstrained choice problem is the same as the solution to the constrained choice problem. This is shown mathematically in Appendix II. Figure 1 illustrates this result. $N_{1,u}^*$ is the value of N_1, which minimizes $V(\tilde{\pi}; N_1)$ which is a function of N_1 alone. $N_{1,r}^*$ is the rounded value of $N_{1,u}^*$. $N_{1,r}^*$ is the solution to the integer value of N_1, which minimizes $V(\tilde{\pi}; N_1)$.

Figure 2 explains why in the two commodity case, the rounded solution to the unconstrained choice problem may not be the same as the constrained choice solution. The figure graphs $V(\tilde{\pi}; N_1 \,|\, N_2)$ as a function of N_1. Fixing the value of N_2, we can find the value of N_1 which minimizes $V(\tilde{\pi}; N_1 \,|\, N_2)$. By doing this for all possible values of N_2, assume that we determine that the solution to the unconstrained choice problem is given by $N_{1,u}^*$, $N_{2,u}^*$. Curve A represents $V(\tilde{\pi}; N_1 \,|\, N_2)$ with $N_2 = N_{2,u}^*$. Curve B represents $V(\tilde{\pi}; N_1 \,|\, N_2)$ with $N_2 = N_{2,r}^*$ where $N_{2,r}^*$ is the rounded integer value of $N_{2,u}^*$. In this case, the optimal unconstrained value of $N_1 = N_{1,u}^{**}$, which could be quite different from $N_{1,u}^*$ since curve A and B are different curves. Therefore, the closest integer value of $N_{1,u}^*$ will not be the same as the closest integer value of $N_{1,u}^{**}$. Extending this analysis to values of N_2, other than rounded values of $N_{2,u}^*$, it is quite possible that

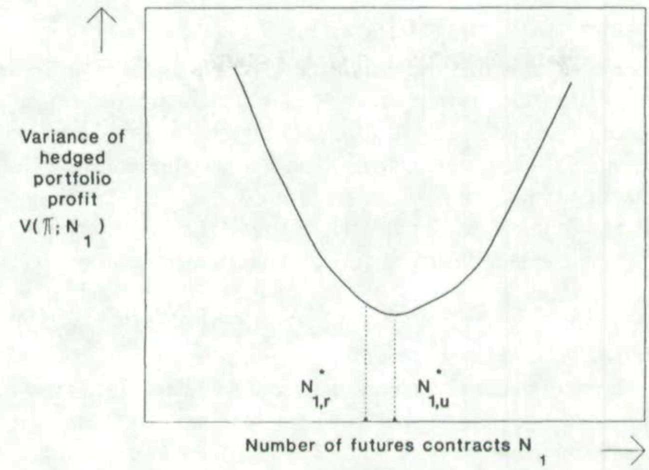

Figure 1
Rounded unconstrained versus integer constrained in the single commodity case.

the solution to the constrained choice could have values of N_1 and N_2, which are quite different from the rounded values of $N^*_{1,u}$ and $N^*_{2,u}$.

Appendix II explains mathematically the conditions under which the rounded unconstrained choice solution and the constrained choice solution in the two commodity case could be the same and the conditions under which they could differ. It should be noted that "natural hedges" provided by negatively correlated cash currencies in the portfolio are implicitly accounted for in choosing N_1 and N_2 by the respective correlation of the spot currencies with the different futures contracts, in the sixth and seventh terms on the right-hand side in eq. (2).

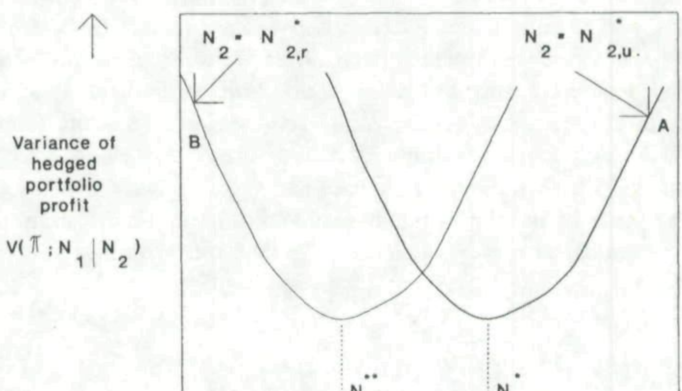

Figure 2
Rounded unconstrained versus integer constrained in the two commodity case.

Instability of Parameters of the Hedging Decision

Note that the parameters on which the hedging decision is based in both the constrained choice approach and the unconstrained choice approach are those contained in the variance–covariance matrix of spot and futures price changes over the hedging period. In practice, the parameters are estimated over a past period, termed the estimation period, and the hedge is initiated in the future. The performance of the hedge may not be as high as forecasted due to the instability in the parameters. The instability is due to various reasons, such as, the difference in the maturity of the futures contrast in the estimation period and hedging period.

There is a wide literature on testing the stability of the hedge ratio and adjusting the hedge ratio for instability, with respect to a hedged portfolio of one spot commodity and a futures contract on one commodity. Grammatikos and Saunders (1983) use three different procedures for testing if such a hedge ratio is stable. They find that, in general, when hedging foreign exchange risk with currency futures, there is evidence of shifts in the hedge ratio, though the shifts are not statistically significant for all currencies. Malliaris and Urrutia (February 1991) use two different tests to determine if the hedge ratio and measure of hedging effectiveness follow a random walk. They find that currency and stock index futures contracts do follow a random walk. Malliaris and Urrutia (June 1991) study the effect of the length of the estimation period and the hedging horizon on the hedge ratio and effectiveness of the hedge. On an ex post basis, the hedge ratio and hedging effectiveness are not affected by the length of the estimation period. On an ex ante basis, the hedging effectiveness increases when the length of the estimation period is reduced. On an ex post basis, the hedging effectiveness is higher the larger the hedging horizon. On an ex ante basis, the effectiveness of hedges improves as the hedging horizon is reduced. Castelino (1990) argues that since the volatility of the basis is high far from contract expiration and low close to it, the minimum variance hedge ratio should depend on when the hedge is lifted and provides a model to calculate such a hedge ratio.

This model is more complicated than those in the above cited papers because more than one spot commodity is considered. In the case of the constrained choice approach, the optimal number of futures contracts cannot be adjusted for instability. The hedge ratio was adjusted in the above papers, using some sort of regression procedure such as random coefficient regression. The optimal number of futures contracts has to be solved for using quadratic integer programming. The variance–covariance matrix has to be adjusted to recognize that the variance–covariance matrix differs between the estimation period and the hedge period. However, since the hedge period is in the future, the adjustment can only be estimated. While there is evidence that the volatility of the futures contract falls as expiration nears, one cannot be sure of what happens to the volatility of the spot commodities, or the covariability of a spot commodity and a futures contract on another commodity. The issue of instability is addressed in the empirical tests by considering ex post and ex ante tests in order to determine the significance of parameter instability on the results.

EVIDENCE OF HEDGING PERFORMANCE OF CURRENCY FUTURES

The Data

Daily data on futures settlement prices and the corresponding spot prices on five major currencies—the British pound, Canadian dollar, Deutschmark, Japanese yen, and Swiss franc—for the period 1986 through 1988 are from the Quotes data tapes of the IMM. These contracts have fixed sizes of British pounds 25,000, Canadian dollar 100,000, Deutschemarks 125,000, Japanese yen 12.5 million, and Swiss francs 125,000, respectively. Currency futures contracts have delivery dates in March, June, September, and December. To eliminate the confounding effect of having time series of futures prices with different times to maturity, the time series of futures prices for each contract is ensured to span the same maximum time to maturity.

For each contract maturity the time series of futures prices is divided into two periods with the same number of observations in each. The first and earlier period, Period I, is used to estimate parameters such as the variance–covariance matrix of spot and futures prices. From these, the optimal futures position is determined. The ex post risk under the constrained choice is compared to the unconstrained choice. There are, therefore, 12 sets of parameter estimates (4 contracts maturing in each of 3 years). The second and later period, Period II, is treated as an out-of-sample period, to initiate the hedges and compare the ex ante risk under the constrained and unconstrained approaches.

Tests and Results

It is assumed that the agent holds portfolios of two foreign currencies, whose value at time 1 are to be hedged. Ten such portfolios of the five currencies is investigated. It is of course possible that the agent could have short positions in the two foreign currencies, whose values at time 1 are to be hedged. Results for this would obviously be similar to the one for long positions in the two currencies, though the optimal positions in the futures contracts would be opposite in sign. A third possibility is that the hedger takes a long position in one currency and a short position in the other. Though not obviously so, the results are similar to those of the long positions in the two foreign currencies. Only the results of the long positions in the two foreign currencies are reported.

An objective of the empirical tests is to determine if the size of the agent's spot positions influences the magnitude of the disadvantage of having to make the constrained choice. Therefore, the empirical tests are conducted for spot positions in different size categories one through eight. The size of futures contracts is used as a benchmark to classify the spot position size categories. Size category 1 refers to spot positions of size $< 1\times$ size of the corresponding futures contract. For example in size category 1, the spot position in British pounds would be less than 25,000 British pounds, the spot position in Canadian dollars would be less than 100,000 Canadian dollars, the spot position in Deutschemarks would be less than 125,000 Deutschemarks, the spot position in Swiss francs would be less than 125,000 Swiss francs, and the spot position in Japanese yen would be less than 12.5 million Japanese yen. In size category n, the spot position would lie between $(n - 1)$ and n times the size of the corresponding futures contract. The spot positions are generated by picking random numbers from uniform distributions with ranges $(n - 1)$ to n times the size of each futures contract using a random number generator. For example, in size category 1 for the British pound–Deutschemark combination, the agent's spot position consists of 24,156 British pounds and 95,783 Deutschemarks. Under each size category, there are ten possible combinations of the five currencies. Since 12 sets of parameters are estimated, there are 120 sets of comparisons in each size category of the constrained choice and unconstrained choice approaches.

The optimal number of futures contracts under the constrained choice and the unconstrained choice is calculated. Table I provides the results for the March 1986 contract. For 36 of the 80 combinations (45%) reported in Table I, the optimal futures position under the constrained choice approach is not the same as the rounded (to the

Table I

OPTIMAL NUMBER OF CURRENCY FUTURES CONTRACTS UNDER THE CONSTRAINED CHOICE AND UNCONSTRAINED CHOICE USING PARAMETERS ESTIMATED FOR THE MARCH 1986 DELIVERY DATE

Currency Combination	Size 1		Size 2		Size 3		Size 4		Size 5		Size 6		Size 7		Size 8	
	C	U	C	U	C	U	C	U	C	U	C	U	C	U	C	U
BP	−1	−0.82	-2^a	−1.51	-2^a	−2.27	−3	−3.02	-4^a	−3.72	-4^a	−4.47	−5	−5.07	-6^a	−5.80
CD	0	−0.47	−1	−2.13	−4	−3.20	−4	−4.27	−5	−5.80	−8	−6.95	−9	−9.30	−10	−10.63
BP	-2^a	−0.91	−2	−1.86	-2^a	−2.72	−4	−3.63	−4	−4.48	−5	−5.38	-7^a	−6.32	−7	−7.23
DM	0	−0.74	−1	−1.07	−3	−2.47	−3	−3.29	−5	−4.86	−6	−5.83	−6	−6.14	−7	−7.02
BP	−1	−0.92	-1^a	−1.85	−3	−2.78	−4	−3.71	-4^a	−4.63	−6	−5.55	-7^a	−6.48	-8^a	−7.40
JY	0	−0.56	−2	−1.69	−3	−2.97	−4	−3.96	−5	−4.48	−5	−5.38	−6	−6.29	−7	−7.19
BP	−1	−0.95	−2	−1.89	-2^a	−2.86	-5^a	−3.81	−5	−4.76	−6	−5.71	−7	−6.66	-7^a	−7.61
SF	−1	−0.81	−2	−1.90	−3	−2.58	−3	−3.44	−4	−4.31	−5	−5.17	−6	−6.03	−7	−6.89
CD	0	−0.16	-2^a	−1.46	-3^a	−2.07	−3	−2.75	-3^a	−3.75	−4	−4.51	−6	−6.40	−7	−7.32
DM	−1	−0.75	−1	−1.03	−2	−2.38	−3	−3.17	−5	−4.65	−6	−5.59	−6	−5.80	−7	−6.63
CD	0	−0.23	−1	−1.40	−2	−2.08	−3	−2.78	−4	−3.87	−5	−4.64	−6	−6.36	−7	−7.27
JY	−1	−0.57	−2	−1.65	−3	−2.91	−4	−3.88	−4	−4.37	−5	−5.24	−6	−6.09	−7	−6.96
CD	0	−0.14	−1	−1.31	-1^a	−2.00	-4^a	−2.67	-5^a	−3.76	-5^a	−4.51	−6	−6.33	−7	−7.24
SF	−1	−0.83	−2	−1.90	−3	−2.58	−3	−3.43	−4	−4.28	−5	−5.14	−6	−5.94	−7	−6.79
DM	0^a	−0.72	−1	−1.08	-3^a	−2.39	−3	−3.19	-4^a	−4.63	-5^a	−5.56	−6	−5.90	-6^a	−6.74
JY	−1	−0.63	−2	−1.78	−3	−3.16	−4	−4.21	−5	−4.86	−6	−5.83	−7	−6.77	−8	−7.74
DM	-2^a	−1.19	-1^a	−1.95	-5^a	−3.92	-7^a	−5.23	−7	−7.45	-8^a	−8.94	-8^a	−9.62	-10^a	−10.99
SF	0	−0.54	−2	−1.39	−1	−1.69	−1	−2.26	−3	−2.69	−4	−3.23	−5	−3.89	−5	−4.45
JY	0^a	−0.62	−2	−1.80	−3	−3.12	-5^a	−4.16	−5	−4.75	−6	−5.70	−7	−6.66	−8	−7.61
SF	−1	−0.82	−2	−1.91	−3	−2.61	−3	−3.48	−4	−4.36	−5	−5.23	−6	−6.10	−7	−6.97

Note: C = Choice constrained by integer number of futures contracts.
U = Unconstrained choice.
[a] = Constrained choice is not the same as the rounded unconstrained choice.

nearest integer) optimal futures position under the unconstrained choice approach. This indicates that using the unconstrained choice model to solve for the optimal futures position and implementing the hedge with rounded values of the solutions for the number of futures contracts, would lead to suboptimal decisions for a substantial proportion of cases. Assuming that the hedged portfolio is terminated on various days in Periods I and II, the optimal futures position under the constrained choice and unconstrained choice approach are then used to calculate the variance of the profit of the hedged portfolio in Period I (termed the ex post test) and in Period II (termed the ex ante tests). An F test is used to determine if the variance of the hedged portfolio profit under the constrained choice is significantly higher than the variance of the hedged portfolio profit under the unconstrained choice. Since there are 96 data points in each of Period I and Period II, the cut off value of the F statistic is given by $F_{95,95} \simeq 1.69$. Table II shows the standard deviation of the hedged portfolio profit under the constrained choice and unconstrained choice for the ex post and ex ante tests, and the associated F statistics for the spot positions corresponding to size 1 and parameters estimated for the March, 1986 contract, for various currency portfolios. It may be observed that for this size of the spot position, the risk of the hedged portfolio is significantly higher for the constrained choice than for the unconstrained choice for all the combinations, except for the ex post test of the Deutschmark, Swiss franc combination.

Table III compares the relative hedging performance of the constrained choice and the unconstrained choice approaches with particular emphasis on the influence of the size of the spot positions. The results of Table III shows that for the ex post tests, there are only two types of outcomes—either the constrained choice approach performs significantly worse than the unconstrained choice approach in reducing risk or there is no significant difference in risk of the hedged portfolio under the two approaches. In the ex ante tests, where the optimal hedge positions are calculated with the parameters estimated from Period I data, while the hedge is assumed to be terminated in various days in Period II (out-of-sample data), it is shown that there are cases where the constrained choice provides hedge portfolios with significantly lower risk than the unconstrained choice. The number of cases where this occurs is not large. This result is due to the parameter instability previously described and the effect of being unable to foresee the future perfectly. However, in this case, there are three possible types of outcomes.

The null hypothesis that the outcomes are equally likely is tested. The probability of each type of outcome according to the null hypothesis is $p_i = 1/k$ where $k =$ the number of possible types of outcomes. The expected number of each outcome of type i is given by $E(n_i) = np_i$, where $n =$ total number of outcomes and $n_i =$ number of outcomes of type i, where i varies from 1 to k. A $\chi^2_{(k-1)}$ test statistic is used to test the null hypothesis, where

$$\chi^2_{k-1} = \sum_{i=1}^{k} \frac{(n_i - np_i)^2}{np_i}$$

[See Mendenhall and Sheaffer (1973)].

Table III shows the actual number of each outcome, proportion (%) of each outcome and the results of the χ^2 test. The χ^2 test determines whether a particular type of outcome is favored. Considering the ex post tests, it is concluded that agents with spot positions of sizes less than 2× size of each futures contract would be clearly disadvantaged because of having to apply the constrained choice approach. If, however, the spot position is of sizes greater than 3× the size of each futures contract, the agent is reasonably certain of not being disadvantaged because of having to apply the constrained choice approach.

Table II

STANDARD DEVIATION OF HEDGED PORTFOLIO PROFIT UNDER THE CONSTRAINED CHOICE AND UNCONSTRAINED CHOICE FOR PARAMETERS ESTIMATED FROM THE MARCH 1986 CONTRACT

		Ex Post Tests			Ex Ante Tests		
		Standard Deviation of Hedged Portfolio Profit under ($)			Standard Deviation of Hedged Portfolio Profit under ($)		
Size of Spot Position	Currency Combination	Unconstrained Choice (Vu)1/2	Constrained Choice (Vc)1/2	$F = \frac{Vc}{Vu}$	Unconstrained Choice (Vu)1/2	Constrained Choice (Vc)1/2	$F = \frac{Vc}{Vu}$
Size 1	BP, CD	137.07	231.26	2.85[a]	152.85	319.65	4.37[a]
	BP, DM	188.11	670.40	12.70[a]	205.30	1845.61	80.81[a]
	BP, JY	139.19	855.53	37.78[a]	150.39	1497.95	99.21[a]
	BP, SF	255.00	684.10	7.20[a]	266.95	372.14	1.94[a]
	CD, DM	102.57	511.58	24.88[a]	207.41	557.22	7.22[a]
	CD, JY	63.81	1122.18	309.28[a]	84.66	1603.14	358.58[a]
	CD, SF	207.80	516.85	6.19[a]	245.21	351.02	2.05[a]
	DM, JY	119.84	761.98	40.43[a]	204.20	842.08	17.01[a]
	DM, SF	252.80	308.61	1.49	318.24	839.77	6.96[a]
	JY, SF	229.61	1183.54	26.57[a]	240.81	1871.07	60.37[a]

Note: [a] Significant at the 95% confidence level.

Table III

EFFECT OF THE SIZE OF THE SPOT POSITION UPON THE RELATIVE HEDGING PERFORMANCE OF THE CONSTRAINED CHOICE AND UNCONSTRAINED CHOICE

	Ex Post Tests			Ex Ante Tests			
	Number of Combinations/Proportion %			Number of Combinations/Proportion %			
Size of Spot Position	Hedged Portfolio had Significantly Higher Risk under Constrained Choice	No significant Difference in Hedged Portfolio Risk under Constrained and Unconstrained Choice	χ_1^2	Hedged Portfolio had Significantly Higher Risk Under Constrained Choice	No Significant Difference in Hedged Portfolio Risk Under Constrained and Unconstrained Choice	Hedged Portfolio had Significantly Lower Risk Under Constrained Choice	χ_1^2
1	110	10		92	19	9	
	91.67	8.33	83.33[a]	76.67	15.83	7.50	102.65[a]
2	75	45		45	50	25	
	62.50	37.50	7.5[a]	37.50	41.67	20.83	8.75[a]
3	51	69		52	57	11	
	42.50	57.50	2.7	43.33	47.50	9.17	31.85[a]
4	35	85		41	67	12	
	29.17	70.83	20.83[a]	34.17	55.83	10.00	37.85[a]
5	23	97		29	81	10	
	19.17	80.83	45.63	24.17	67.50	8.33	67.55[a]
6	14	106		28	82	10	
	11.67	88.33	70.53[a]	23.33	68.33	8.33	70.20[a]
7	7	113		21	83	16	
	5.83	94.17	93.63[a]	17.50	69.17	13.33	69.65[a]
8	3	117		18	94	8	
	2.50	97.50	108.30[a]	15.00	78.33	6.67	110.60[a]

Note: $\chi_{k-1}^2 = \Sigma_i[(n_i - np_i)^2/np_i]$ where: n_i = actual number of outcomes of type i, n = total number of outcomes, p = probability of each possible outcome = $1/k$, and k = number of possible outcomes.

[a]Significant at the 95% confidence level.

Considering the ex ante tests, it is concluded that agents with spot positions of sizes less than 1× size of each futures contract would be clearly disadvantaged because of having to apply the constrained choice approach. If, however, the spot positions are of sizes greater than 3× size of each futures contract, the agent is reasonably certain of not being disadvantaged because of having to apply the constrained choice approach.

Table IV compares the relative hedging performance of the constrained choice and the *rounded unconstrained* choice approaches with particular emphasis on the influence of the size of the spot position. For the ex post tests, there are only two types of outcomes—either the constrained choice approach performs significantly better than the rounded unconstrained choice approach in reducing risk or there is no significant difference between the risk of the hedged portfolio under the two approaches. The number of outcomes and proportion of outcomes are presented in the table. Again, the null hypothesis that the different types of outcomes are equally likely is tested using the χ_1^2 test statistic. The χ^2 test statistic indicates that the two outcomes are not equally likely. There is a greater chance that the two approaches will give similar results as indicated by the proportion of outcomes. It is found that in size category 1, 35% of the constrained choice cases perform better than the rounded unconstrained choice cases. This superiority of the constrained choice approach falls off as the size of the spot position is increased. The χ^2 statistic indicates that the two outcomes are not equally likely. Its more likely that the two approaches are equally effective. The agent is reasonably certain that the two approaches will lead to the same result especially in the higher size categories.

Regarding the ex ante tests, the three possible types of outcomes are that the constrained choice approach has significantly lower risk, the two approaches are equally effective, and that the constrained choice approach has significantly higher risk. The χ_2^2 test statistic indicates that the three types of outcomes are not equally likely for all size categories. The proportion of cases indicates that it is more likely that the two approaches will lead to similar hedging results. At size category 1, the proportion of cases in which the constrained choice approach performed better is 19.17% and seems to fall as the size of spot positions increases.

It should be noted that the ex ante tests, in size category 1, and the proportion of cases in which the constrained choice approach performs worse than the rounded unconstrained choice approach is 10% and this diminishes as the size of the spot positions increases. The possibility in the ex ante test of finding that, in a few cases, the rounded unconstrained choice approach performs better than the constrained choice approach is due to parameter instability and the impossibility of foreseeing the future with 100% accuracy.

CONCLUSION

This article investigates the effect of the standardized size specifications of futures contracts upon hedging performance. It extends the optimal hedging model with the constraint of indivisible choices, and proposes solutions to the resulting nonlinear *integer* optimizing problem. The model is applied to hedging of foreign exchange risk with financial futures contracts. The results show both ex post and ex ante that if the hedging agent's spot positions are of small sizes, the agent suffers the disadvantage of significantly reduced hedging effectiveness due to the fixed sizes of futures contracts. This disadvantage disappears if the spot positions to be hedged are sufficiently large. The performance of this model is compared to the common current practice of rounding to the nearest integer. The results show that, ex post, this model performs better than or as well as the common practice. This performance depends on the size of the spot positions. As the agent's spot positions increase, for a greater proportion of the cases, the model

Table IV

EFFECT OF THE SIZE OF THE SPOT POSITION UPON THE RELATIVE HEDGING PERFORMANCE OF THE CONSTRAINED CHOICE AND ROUNDED UNCONSTRAINED CHOICE

	Ex Post Tests			Ex Ante Tests			
	Number of Combinations/Proportion %			Number of Combinations/Proportion %			
Size of Spot Position	Hedged Portfolio had Significantly Lower Risk under Constrained Choice	No Significant Difference in Hedged Portfolio Risk under Constrained and Rounded Unconstrained Choice	χ_1^2	Hedged Portfolio had Significantly Lower Risk under Constrained Choice	No Significant Difference in Hedged Portfolio Risk under Constrained and Rounded Unconstrained Choice	Hedged Portfolio had Significantly Higher Risk under Constrained Choice	χ_2^2
1	42	78		23	85	12	
	35.00	65.00	10.80[a]	19.17	70.83	10.00	77.45[a]
2	16	104		9	102	9	
	13.33	86.67	64.53[a]	7.50	85.00	7.50	144.15[a]
3	20	100		10	86	24	
	16.67	83.33	53.33[a]	8.33	71.67	20.00	81.80[a]
4	18	102		14	92	14	
	15.00	85.00	58.80[a]	11.67	76.67	11.67	101.40[a]
5	11	109		14	92	14	
	9.17	90.83	80.03[a]	11.67	76.67	11.67	101.40[a]
6	10	110		9	98	13	
	8.33	91.67	83.33[a]	7.50	81.67	10.83	126.35[a]
7	4	113		10	97	13	
	3.33	9.67	104.53[a]	8.33	80.83[a]	10.83	121.95[a]
8	5	115		16	88	16	
	4.17	93.83	100.83[a]	13.33	73.33[a]	13.33	86.40[a]

[a]Significant at the 95% confidence level.

performs as well as current practice. Parameter instability reduces the effectiveness of the model in the ex ante test. However, as the agent's spot position increases, there are a larger proportion of cases where the performance of this model and current practice are the same.

The above findings hold on an overall basis. The above implies that large hedgers are not disadvantaged by fixed sizes of futures contracts and could use the common practice of rounding with little loss, but this is not valid if differences in hedging effectiveness are measured at the margin. It then becomes necessary for all hedgers to pay attention to and recognize the fixed sizes of futures contracts in hedging practice.

One way of doing this is to use the model proposed in this study to hedge in futures markets. An alternative would be to use the unconstrained choice model with futures prices to determine the hedge, use the futures markets for the rounded integer position, and hedge the residual (positive or negative) in the forward markets. The second approach assumes that futures and forward prices are indistinguishable. While most of the empirical research on foreign exchange has provided evidence consistent with this, it has to be remembered that published forward price quotes available for research are confined to maturities of 30, 60, and 90 days which may weaken the grounds for substitutability of futures and forward prices.

The findings of this article should be useful for research on the factors responsible for a futures contract's success or failure and to the designers of new futures contracts.

Appendix I

The constrained choice problem could be generalized by the following: Minimize

$$\sum_{i=1}^{n} a_i x_i^2 + \sum_{i=1}^{n} b_i x_i + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} x_i x_j \tag{A.1}$$

with respect to x_i, $i = 1 \ldots n$ subject to the constraint $x_i \gtreqless 0$ and x_i = integer for all i.

This nonlinear integer optimization problem may either be solved by (1) Method I. Transform the problem into a 0–1 linear programming problem or by (2) Method II. Use the concept of penalty functions. These methods are explained below and are adapted from Rao (1984), pp. 557–568.

Method I

Let $x_i = x_{i1} - x_{i2}$ for all i, where x_{i1} and x_{i2} both > 0 for all i. Let u_{ij} be the upper limit of x_{ij}, for $i = 1 \ldots n$, and $j = 1, 2$. Then

$$x_{ij} = \sum_{k=0}^{N_{ij}} 2^k y_{ij}^{(k)} \tag{A.2}$$

where $y_{ij}^{(0)}, y_{ij}^{(1)}, y_{ij}^{(2)} \ldots$ denote binary numbers which can take on a value of either 0 or 1 and N_{ij} is the smallest integer so that

$$\left(\frac{u_{ij} + 1}{2}\right) \leq 2^{N_{ij}}$$

Substitute (A.2) into (A.1). Note that after substitution there will be squared terms $[y_{ij}^{(k)}]^2$ in the objective function which could be replaced by $y_{ij}^{(k)}$. There will be cross product terms $y_{ij}^{(k)} y_{rs}^{(t)}$ which are to be dealt with before the problem can be transformed into a 0–1 linear programming problem. Let

$$y_p = y_{ij}^{(k)} y_{rs}^{(t)}$$

and write

$$y_{ij}^{(k)} + y_{rs}^{(t)} - y_p \leq 1 \tag{A.3}$$

$$-y_{ij}^{(k)} - y_{rs}^{(t)} + 2y_p \leq 0 \tag{A.4}$$

$$y_p = 0 \quad \text{or} \quad 1 \tag{A.5}$$

Equation (A.3) and (A.5) will ensure that $y_p = 1$ when $y_{ij}^{(k)}$ and $y_{rs}^{(t)}$ are both 1. Equation (A.4) will ensure that if $y_{ij}^{(k)} = y_{rs}^{(t)} = 0$ or if one of $y_{ij}^{(k)}$ and $y_{rs}^{(t)}$ but not the other equals 0, the $y_p = 0$.

Adding eq. (A.3) through (A.5) as constraints to the problem and substituting for cross terms $y_{ij}^{(k)} y_{rs}^{(t)}$ in the objective function by y_p, the problem is transformed into a 0–1 linear programming problem.

Note that if the original problem has two decision variables and the upper limit for each x_{ij} is four, the number of constraints added to the problem would be 132 constraints, and the number of 0–1 variables to solve for would be 78 (12 binary variables $y_{ij}^{(k)}$ plus 66 binary variables y_p).

Method II

The problem may be formulated as follows: Minimize

$$f(X) = \sum_{i=1}^{n} a_i x_i^2 + \sum_{i=1}^{n} b_i x_i + \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} x_i x_i$$

subject to the constraints $g_j(X) \geq 0$, for $j = 1, \ldots m$ where $g_j(X)$ is given by

$$\begin{aligned} -x_i + U_i &\geq 0 \\ x_i - L_i &\geq 0 \end{aligned} \qquad \text{for all } i = 1, \ldots n$$

where U_i and L_i are upper and lower limits of x_i and $x_i \gtreqless 0$ and integer, and X is the vector of x_i's.

This method uses a generalized interior penalty function approach to define the following transformed problem. Minimize

$$\phi_k(X, r_k, s_k) = f(X) + r_k \sum_{j=1}^{m} \frac{1}{g_j(X)} + s_k Q_k(X)$$

The second term ensures that once the minimization is started from a feasible point, the point remains in the feasible region always. The third term is a penalty term which gives a penalty whenever some of the variables x_i take on other than integer values. We use

$$Q_k(X) = \sum_{i=1}^{n} \left\{ 4\left(\frac{x_i - y_i}{z_i - y_i}\right)\left(1 - \left(\frac{x_i - y_i}{z_i - y_i}\right)\right)\right\} \beta_k$$

where y_i and z_i are the two neighboring integer values for the value of x_i, and $y_i \leq x_i$ and $z_i \geq x_i$.

The ϕ_k function is minimized for a sequence of values of r_k and s_k such that for $k \to \infty$, the optimum integer solution to the basic problem is obtained.

The following algorithm is used.

Step 1. Let $k = 1$
Choose initial feasible values X_1 of x_i, $i = 1 \ldots n$
Determine r_1, s_1, and β_1 (For details see Rao).
Step 2. Formulate $\phi_k(X)$.
Step 3. Find the unconstrained minimum X_k^* of $\phi_k(X, r_k, s_k)$. We used a subroutine in the IMSL library, which uses a quasi-Newton method and a finite difference gradient.
Step 4. Check the stopping criterion.

$$\frac{f(X_k^*) - f(X_{k-1}^*)}{f(X_k^*)} \leq \epsilon$$

where $\epsilon = 10^{-3}$
If it is satisfied go to Step 6, otherwise go to Step 5.
Step 5. Set $X_k = X_k^*$. Update the values of r_{k+1}, s_{k+1}, and β_{k+1}. Go to Step 2.
Step 6. Check if X_k^* is an integer solution. If yes, take $X^* = X_k^*$ and stop. If not update the recovery cycle number i to $i + 1$ and go to Step 7.
Step 7. If this is the first step of the application of the recovery process, set $(\mathrm{r})_{\mathrm{new}_1} = r_{k-i-1}$ and $(\mathrm{s})_{\mathrm{new}_1} = 2s_{k+i-2}$. Execute Step 3. If this is the second step of the recovery process, set $(\mathrm{r})_{\mathrm{new}_2} = r_{k-i}$ and $(\mathrm{s})_{\mathrm{new}_2} = 2s_{k+i-1}$. Execute Step 3. Check the stopping criterion. If satisfied go to Step 6. If not, update the recovery cycle number i to $i + 1$, and go to Step 7.

Appendix II

When the hedged portfolio contains one spot commodity and futures contracts on only one commodity, the rounded solution to the unconstrained choice problem is the same as the solution to the constrained choice problem. The conditions to be fulfilled for the rounded solution to the unconstrained choice problem to equal the solution to the constrained choice problem are shown using two spot commodities and futures contracts on two commodities.

Futures Contracts on One Commodity

The variance of the profit on the hedged portfolio of a single commodity and the futures contract may be written as

$$V = N_1^2 X_1^2 \sigma_{F_1}^2 + 2N_1 X_1 C_1 \sigma_{S_1 F_1} + C_1^2 \sigma_{S_1}^2$$

where

N_1 = number of futures contracts

X_1 = size of each futures contract

$\sigma_{F_1}^2$ = variance of the return on the futures contract

C_1 = number of units of the spot commodity

$\sigma_{S_1 F_1}$ = covariance between the return on the spot commodity and futures contract

$\sigma_{S_1}^2$ = variance of the return on the spot commodity

The variance of the hedged portfolio may be written as

$$V = a_1 N_1^2 + a_2 N_1 + a_3$$

where

$$a_1 = X_1^2 \sigma_{F_1}^2$$
$$a_2 = 2X_1 C_1 \sigma_{S_1 F_1}$$
$$a_3 = C_1^2 \sigma_{S_1}^2$$

The unconstrained optimum $N_{1,0}$ is given by solving the following equation

$$\frac{\partial V}{\partial N_1} = 2a_1 N_1 + a_2 = 0 \tag{II.1}$$
$$N_{1,0} = -\frac{a_2}{2a_1}$$

$$\frac{\partial^2 V}{\partial N_1^2} = 2a_1 > 0 . \quad \therefore a_1 > 0 \tag{II.2}$$

Consider two integer values of N_1, above and below $N_{1,0}$, respectively, given by $N_{1,h}$ and $N_{1,l}$, such that $N_{1,l}$ is the rounded integer value of $N_{1,0.}$

$$N_{1,0} - N_{1,l} < N_{1,h} - N_{1,0} \tag{II.3}$$

It is proved below that

$$V_l < V_h$$

where

$$V_l = \text{variance of the hedged portfolio with } N_1 = N_{1,l}$$
$$V_h = \text{variance of the hedged portfolio with } N_1 = N_{1,h}$$
$$V_l - V_h = a_1 N_{1,l}^2 + a_2 N_{1,l} + a_3 - a_1 N_{1,h}^2 - a_2 N_{1,h} + a_3$$
$$= (N_{1,l} - N_{1,h})[a_1(N_{1,l} + N_{1,h}) + a_2]$$

From (II.3)

$$V_l - V_h = (N_{1,l} - N_{1,h})[2a_1 N_{1,0} + a_1 k_1 + a_2]$$

where K_1 is a positive constant $= (N_{1,h} + N_{1,l}) - 2N_{1,0}$
Using (II.1)

$$V_l - V_h = (N_{1,l} - N_{1,h})[a_1 k_1]$$

The first term on the right-hand side is negative and the second term is positive.

$$\therefore V_l < V_h$$

This process can be repeated by comparing $N_{1,l}$ with other integer values of $N_{1,h}$ satisfying (II.3) to show that the variance of the hedged portfolio at the rounded integer solution $N_{1,l}$ is the lowest of all integer solutions.

Futures Contracts on Two Commodities

The variance of the return on the hedged portfolio of two spot and futures commodities can be written as

$$V = a_1N_1^2 + a_2N_1 + a_3N_1N_2 + a_4N_2^2 + a_5N_2$$

$$N_1, N_2 = \text{number of futures contracts on commodity 1}$$

$$a_1 = X_1^2\sigma_{F_1}^2$$

$$a_4 = X_2^2\sigma_{F_2}^2$$

$$a_2 = 2X_1(C_1\sigma_{S_1,F_1} + C_2\sigma_{S_2,F_1})$$

$$a_5 = 2X_2(C_1\sigma_{S_1F_2} + C_2\sigma_{S_2F_2})$$

$$a_3 = 2X_1X_2\sigma_{F_1F_2}$$

Solving the unconstrained choice problem

$$\frac{\partial V}{\partial N_1} = 2a_1N_1 + a_2 + a_3N_2 = 0 \tag{II.4}$$

$$\frac{\partial V}{\partial N_2} = 2a_4N_2 + a_5 + a_3N_1 = 0 \tag{II.5}$$

$$\frac{\partial^2 V}{\partial N_2} = 2a_1 > 0$$

$$a_1 > 0$$

$$\frac{\partial^2 V}{\partial N_2^2} = 2a_4 > 0$$

$$a_4 > 0$$

a_2, a_3, and a_5 can be positive, negative, or zero.

Let $N_{1,0}$, $N_{2,0}$ represent the solution to the unconstrained choice problem. Let $N_{1,l}$ and $N_{1,h}$ represent integers below and above $N_{1,0}$, respectively. Let $N_{2,l}$ and $N_{2,h}$ represent integers below and above $N_{2,0}$, respectively. Let $N_{1,l}$, $N_{2,l}$ represent the rounded solution to the unconstrained choice problem. Then

$$N_{1,0} - N_{1,l} < N_{1,h} - N_{1,0} \tag{II.6}$$

$$N_{2,0} - N_{2,l} < N_{2,h} - N_{2,0} \tag{II.7}$$

Let $V_{i,j}$ represent the variance of the hedged portfolio with $N_1 = N_{1,i}$ and $N_2 = N_{2,j}$. Under certain conditions

$V_{l,l}$ could be greater than $V_{l,h}$, $V_{h,l}$ and $V_{h,h}$.

$$\begin{aligned}
V_{l,l} - V_{l,h} &= a_1N_{1,l}^2 + a_2N_{1,l} + a_3N_{1,l}N_{2,l} \\
&\quad + a_4N_{2,l}^2 + a_5N_{2,l} \\
&\quad - a_1N_{1,l}^2 - a_2N_{1,l} - a_3N_{1,l}N_{2,h} \\
&\quad - a_4N_{2,h}^2 - a_5N_{2,h} \\
&= (N_{2,l} - N_{2,h})[a_4(N_{2,l} + N_{2,h}) + a_5a_3N_{1,l}]
\end{aligned}$$

From (II.7)

$$2N_{2,0} < N_{2,l} + N_{2,h}$$
$$N_{2,l} + N_{2,h} = 2N_{2,0} + k_1$$

where

$$k_1 > 0$$
$$N_{1,l} = N_{1,0} - k_2$$

where

$$k_2 > 0$$
$$\therefore V_{l,l} - V_{l,h} = (N_{2,l} - N_{2,h})[2a_4N_{2,0} + a_5a_3N_{1,0} + a_4k_1 - a_3k_2]/2$$

From (II.5)

$$2a_4N_{2,0} + a_5 + a_3N_{1,0} = 0$$
$$V_{l,l} - V_{l,h} = (N_{2,l} - N_{2,h})[a_4k_1 - a_3k_2]/2 \tag{II.8}$$

The first term in brackets in the right-hand side of (II.8) is negative. The second term is positive if $a_4k_1 - a_3k_2 > 0$.

a_4, k_1, and k_2 are positive, a_3 could be positive or negative.

If a_3 is negative, then

$$V_{l,l} < V_{l,h}$$

If a_3 is positive, then

$$V_{l,l} < V_{l,h}$$

only if

$$\frac{a_4}{a_3} > \frac{k_2}{k_1}.$$

In a similar way, it can be shown that

$$V_{l,l} < V_{h,l}$$

only if

$$a_1k_3 - a_3k_4 > 0$$

where

$$2N_{1,0} = N_{1,l} + N_{1,h} + k_3$$
$$N_{2,l} = N_{2,0} - k_4$$

where

$$k_3, k_4 > 0$$

Comparing $V_{l,l}$ and $V_{h,h}$

$$\begin{aligned}
V_{l,l} - V_{h,h} &= a_1 N_{1,l}^2 + a_2 N_{1,l} + a_3 N_{1,l} N_{2,l} \\
&\quad + a_4 N_{2,l}^2 + a_5 N_{2,l} \\
&\quad - a_1 N_{1,h}^2 - a_2 N_{1,h} - a_3 N_{1,h} N_{2,h} \\
&\quad - a_4 N_{2,h}^2 - a_5 N_{2,h} \\
&= a_1\left(N_{1,l}^2 - N_{1,h}^2\right) \\
&\quad + a_2(N_{1,l} - N_{1,h}) \\
&\quad + a_3(N_{1,l}N_{2,l} - N_{1,h} - N_{2,h}) \\
&\quad + a_4\left(N_{2,l}^2 - N_{2,h}^2\right) \\
&\quad + a_5(N_{2,l} - N_{2,h})
\end{aligned}$$

It can be stated that the first term and the fourth term are negative. However if $a_2 < 0$, $a_3 < 0$ and $a_5 < 0$, the second, third, and fifth terms would be positive. This would make $V_{l,l} > V_{h,h}$.

The above comparisons could be applied to other integer values of N_1 and N_2 to make the same points. The above shows that if futures contracts on two commodities are used for hedging, the rounded solution to the unconstrained choice problem would under certain conditions (which are possible) not be the same as the solution to the constrained choice problem.

The above proof could be extended to a hedged portfolio of n commodities and futures contracts on n commodities.

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