Arbitrage Free Pricing of Interest Rate Futures and Forward Contracts

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1. INTRODUCTION

The tremendous growth in the exchange markets for financial futures over the last decade has spurred a lot of academic research on the pricing of such contracts. Many recent articles are concerned with the empirical and theoretical differences between futures and forward prices arising from the periodic marking to market that takes place in the futures market. In addition to its intrinsic interest, the quantitative difference between futures and forward prices is important for several applications where it is often necessary to model futures prices as if they are forward prices to maintain tractability. A recent example of this is Boyle's (1989) analysis of the quality and timing options embedded in Treasury-bond futures contracts. The analysis of Cox, Ingersoll, and Ross (CIR) (1981) indicates that the size of the difference is an increasing function of the magnitude of the covariance between the underlying asset and a stochastic interest rate. Still, there does not seem to be general agreement on how significant these price differences are. It appears, however, that if significant differences between forward and futures prices exist, the most likely place to find them is in interest rate based contracts.

This article contains two analytical innovations: it derives a closed form continuous time pricing formula for discretely marked to market futures contracts, and it prices a futures contract whose final settlement is a nonlinear function of the underlying asset price.[1] Applying the resulting formulas, it is subsequently demonstrated how the price differential between futures and forwards depends on the contract specifications and the volatility of interest rates. Moreover, the effect of different frequencies of marking to market is analyzed. Particular emphasis is put on assessing the error made by previous researchers who, for analytical tractability, assume futures contracts to be marked-to-market continuously.

Section 2 contains background material on futures and forward pricing. Section 3 contains a brief introduction to the term structure model and valuation procedure. The main theoretical results are presented in section 4, which is followed by a discussion of how one can compute hedge portfolios in the current setting. This is of independent

This article is based on the first chapter of the author's doctoral dissertation at the University of California at Berkeley. Thanks are due to the dissertation committee members: Gerard Gennotte, Hayne Leland, Pravin Varaiya, and especially, David Modest. Funding from the Norwegian Council for Research in Social Science and the Humanities is gratefully acknowledged.

[1]This is the futures contract on the Eurodollar time deposit rate that is traded at the Chicago Mercantile exchange.

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The Journal of Futures Markets, Vol. 13, No. 1, 77–91 (1993)

CCC 0270-7314/93/010077-15

interest, but it is not required for the pricing of futures and forward contracts. Numerical values for the theoretical difference between futures and forward contracts are provided in section 6.

2. BACKGROUND

Beginning with Black (1976) theoretical research has treated futures contracts as inherently different from forward contracts. Black's model used a constant interest rate, however, and therefore found that the two contracts carry the same price. CIR (1981), Richard and Sundaresan (1981), and Jarrow and Oldfield (1981) all pointed out that when interest rates are stochastic, futures prices, in general, differ from forward prices. Their research suggests that when interest rates are high and if the price of the underlying asset is positively correlated with the spot interest rate, the holder of a futures contract tends to receive a positive cash flow from the marking to market process. Conversely, when interest rates are low, the holder pays out cash from the marking to market process. Thus, when the underlying asset is positively correlated with the spot interest rate, rational investors favor the marking to market process over total settlement at maturity, as in the forward contract. Conversely, the maturity payoff from a forward contract is favored over the cash flow from a futures contract when the price of the underlying asset is negatively correlated with interest rates. Thus, in equilibrium, the price differential between futures and forwards should have the same sign as the correlation between the price of the underlying instrument and the spot interest rate.

In a recent article, Jarrow (1988) derived forward, futures, and options prices for a general setting that includes the model presented in this article as a special case. However, like the research cited above, his model features futures contracts as marked to market continuously, rather than at the end of every trading day, as is the case in organized futures markets. Ho (1985) developed a discrete time bond futures pricing model that (necessarily) involves discrete marking to market; but that takes place at the end of each of his "time periods," the length of which is arbitrary and not directly related to the trading day. A common feature of all these models is that the futures contracts are written on a tradeable asset, which implies that the corresponding forward price can be found from a simple static hedging argument. To appreciate why that is not the only relevant case, one can point to the fact that futures contracts on the three-month Eurodollar time deposit rate has a final settlement that is linear in the reciprocal of the price of the Eurodollar time deposit. In this case, an exact static hedge is no longer feasible for the corresponding forward contract, and most of the previously developed valuation techniques will break down. This necessitates the use of dynamic hedging arguments to solve for both the forward and futures price.

There is limited empirical evidence on the magnitude of the difference between forward and futures prices due to marking to market of the futures contracts. Although French (1983) found some statistically significant empirical differences in metal markets, other studies by Capozza and Cornell (1979); Cornell and Reinganum (1981); Elton, Gruber, and Rentzler (1984); and Park and Chen (1985) indicate that the differences for contracts on Treasury-bills and foreign currencies are low and of little economic significance. These kinds of studies generally suffer from a lack of parallel futures and forward markets where the important institutional features, including tax treatment, are equal.

The approach of this study is comparable to that of Rendleman and Carabini (1979) and Gibbons and Ramaswamy (1986). They both use parameter estimates obtained for the CIR interest rate model together with closed form pricing formulas to compute model-based price differences between short term interest rate forward and futures contracts.

Rendleman and Carabini conclude that the difference between forward and futures prices should be relatively trivial; whereas Gibbons and Ramaswamy demonstrate the possibility of large differences for certain parameter vectors. It is believed that the difference between their conclusions is due primarily to the difficulty of getting good parameter estimates for the CIR model. This study develops a model that depends on a single volatility parameter that can be accurately estimated from contemporaneous interest rate options prices, allowing sharper predictions about the valuation effect of marking to market.

3. THE TERM STRUCTURE MODEL

This section provides a brief introduction to a special case of the interest rate model developed by Heath, Jarrow, and Morton (HJM) (1992). The model takes the initial term structure of default free interest rates as given. It thereby avoids the initial term structure mismatch problem inherent in models that posit a functional form relating bond prices to a small set of state variables [e.g., the model of CIR (1985)]. The standard contingent claims pricing assumptions of a perfect continuous market is employed throughout the study, excluding transaction costs, taxes, etc.

The stochastic dynamics of the model are driven by a single Brownian Motion, $\{W(t), t \in [0, t^*]\}$, where t^* is a fixed but arbitrary time horizon. The quadruplet $(\Omega, F, \{\mathcal{F}_t\}, Q)$ denotes the underlying filtered probability space, where $\{\mathcal{F}_t\}$ is taken to be the filtration generated by $W()$, augmented to contain the Q-null sets in F[2]. Let $P(t,T)$ denote the time t price of a default free discount bond paying \$1 at time T. Then the forward rates, $f(t,s)$, are implicitly defined through the relationship:

$$P(t,T) = e^{-\int_t^T f(t,s)ds} \tag{1}$$

The "risk free return," i.e., the return on an instantaneously maturing bond, is given by $f(t,t)$ and is normally written as $r(t)$.

The whole term structure of forward rates is assumed to evolve as given by the stochastic differential equation, which holds simultaneously for each T[3]:

$$df(t,T) = [-\sigma\phi(t) + \sigma^2(T-t)]dt + \sigma dW(t) \tag{2}$$

Here, σ is a constant and $\phi(t)$ is a stochastic process that is predictable and bounded, but otherwise unrestricted. $\phi(t)$ can be thought of as the randomly time varying market risk premium for interest rate risk at time t. There will then be no arbitrage opportunities in this economy if and only if there exists a probability measure Q^*, equivalent to Q,[4] with the property that the expected instantaneous return on any discount bond is equal to the risk free rate. Thus, defining[5]:

$$B(t) = e^{\int_0^t r(s)ds} \tag{3}$$

and letting E^* denote expectation with respect to Q^*, it follows that:

$$P(t,T) = B(t)E^*\left[\frac{P(s,T)}{B(s)} \;\middle|\; \mathcal{F}_t\right] \qquad \forall\; s,t,T\; s.t. 0 \le t \le s \le T \le t^* \tag{4}$$

[2]The original probability measure is here denoted by Q, rather than the more customary P, since P will be reserved as notation for bond prices.

[3]It should be stressed that the only assumption going into the model at this point is constant volatility. The form of the drift term is then necessary for no arbitrage opportunities between bonds to exist, as demonstrated by HJM.

[4]Two probability measures, μ and ν, defined on the same measurable space, are said to be equivalent (or mutually absolutely continuous) if $\mu(A) = 0 \Leftrightarrow \nu(A) = 0$ for all measurable sets A.

[5]$B(t)$ can be visualized as the value at time t of \$1 left to grow in a money market account since time 0.

Thus, the process $\{P(t,T)/B(t)\}$ is a Q^*-martingale for all T, when restricted to $t \leq T$. The measure Q^* is unique and given by the Radon–Nikodym derivative:

$$\frac{dQ^*}{dQ} = e^{\int_0^{t^*} \phi(s)dW(s) - \frac{1}{2}\int_0^{t^*} [\phi(s)]^2 ds} \tag{5}$$

Defining the process:

$$W^*(t) = W(t) - \int_0^t \phi(s)ds \tag{6}$$

it follows from Girsanov's theorem that $W^*(t)$ is a standard Brownian Motion under Q^*. Using this transformation, and given the initial term structure, eq. (2) can be integrated to get (7) which can be used along with (1) to get (8):

$$f(t,T) = f(0,T) + \sigma^2 t\left(T - \frac{t}{2}\right) + \sigma W^*(t) \tag{7}$$

$$P(t,T) = e^{-\int_t^T f(0,s)ds - \frac{\sigma^2}{2} tT(T-t) - \sigma(T-t)W^*(t)} \tag{8}$$

The general pricing operator for any interest rate contingent claim can then be represented by an expectation under the equivalent martingale measure, Q^*. Thus, a claim with a time T random cash flow $G(T)$, where $G(T)$ is $\mathcal{F}_T$-measurable and integrable *w.r.t.* Q^*, will have a time t no-arbitrage value of:

$$G(t) = B(t)E^*\left[\frac{G(T)}{B(T)} \,\middle|\, \mathcal{F}_t\right] \tag{9}$$

This generalizes to the valuation of contingent claims that have multiple cash flows by the linearity of the expectation operator.

The model described above is closely related to the discrete time model of Ho and Lee (1986), as shown in HJM (1990). An inherent problem is that the assumed dynamics allows interest rates to go negative with a strictly positive probability, even though this probability may be made arbitrarily small by a judicious choice of the risk premium $\phi()$ as a function of past and current interest rates.

4. PRICING FORMULAS

This section derives exact pricing formulas for forward and futures contracts when the underlying index ("asset") is linear in either a pure discount bond (Proposition 2) or in the reciprocal of the price of a discount bond (Propositions 1 and 3). Before turning to the new results, the following standard result is noted for purposes of completeness and comparison. Its proof follows from a simple static hedging argument: the no-arbitrage forward price at time t, $G(t,\tau,T)$, for a forward contract maturing at time τ with a discount bond maturing at time T as the underlying instrument is given by:

$$G(t,\tau,T) = \frac{P(t,T)}{P(t,\tau)} \tag{10}$$

Proposition 1 concerns the corresponding forward price when the forward contract has a payoff at maturity that is the difference between the reciprocal of the bond price and the forward price. In this case, there does not exist a static perfect hedging strategy for the contract, since the payoff is a nonlinear function of the bond price. In the following, let G denote a forward price and H_k denote a futures price for a contract that is marked to market at the end of every k^{-1} time period. An i superscript indicates that the underlying index is the inverse of a discount bond price.

Proposition 1

The no-arbitrage forward price at time t, $G^i(t,\tau,T)$, for a forward contract maturing at time τ, with a payoff at maturity to the long side equal to $P(\tau,T)^{-1} - G^i(t,\tau,T)$, is given by:

$$
\begin{aligned}
G^i(t,\tau,T) &= e^{\int_\tau^T f(t,s)ds+\sigma^2(\tau-t)(T-\tau)^2} \\
&= \frac{P(t,\tau)}{P(t,T)}\, e^{(\tau-t)(T-\tau)^2}
\end{aligned}
\tag{11}
$$

Proof. The forward price is by definition set so as to render the present value of the cash flow at maturity equal to zero. Thus, using the valuation operator given in (9):

$$
B(t)E^*\left[\frac{P(\tau,T)^{-1} - G^i(t,\tau,T)}{B(\tau)} \,\middle|\, \mathcal{F}_t\right] = 0 \tag{12}
$$

A slight reorganization, using the fact that the forward price at time t must be observable at time t, gives:

$$
G^i(t,\tau,T) = \frac{E^*\left[\dfrac{B(t)}{P(\tau,T)B(\tau)} \,\middle|\, \mathcal{F}_t\right]}{E^*\left[\dfrac{B(t)}{B(\tau)} \,\middle|\, \mathcal{F}_t\right]} \tag{13}
$$

Using a conditioning argument from HJM (1992) the Appendix shows that the numerator in (13) is equal to:

$$
E^*\left[\frac{B(t)}{P(t,T)B(\tau)} \,\middle|\, \mathcal{F}_t\right] = \frac{P(t,\tau)^2}{P(t,T)}\, e^{\sigma^2(\tau-t)(T-\tau)^2} \tag{14}
$$

Since (4) implies that the denominator in (13) is equal to $P(t,\tau)$, the desired result now follows by simplification. ∎

Proposition 2

The no-arbitrage futures price at time t, $H_k(t,\tau,T)$, for a futures contract maturing at time τ, written on a discount bond that matures at time T, and that is marked to market every k^{-1} periods is given by:

$$
\begin{aligned}
H_k(t,\tau,T) &= \frac{P(t,T)}{P(t,\tau)}\, e^{-\frac{\sigma^2}{2}\left[(\tau-t)^2(T-\tau)-\frac{(\tau-t)(T-\tau)}{k}\right]} \\
&= G(t,\tau,T)e^{-\frac{\sigma^2}{2}\left[(\tau-t)^2(T-\tau)-\frac{(\tau-t)(T-\tau)}{k}\right]}
\end{aligned}
\tag{15}
$$

Remark. Note that for daily marking to market one will have $k \approx 250$.[6] The following simple lemma is used in the proof of Proposition 2, and is generally useful for manipulating models of this type.

[6]This is based on the number of trading days in a year, averaging out weekends and holidays. The proposition and proof could, in principle, easily be modified to account for a different length of time between two consecutive markings to market when a weekend lies in between, but doing so would add complexity to the formulas without a reasonable corresponding gain in accuracy. This is particularly true when considered in light of the maintained assumption of continuously open markets and a constant volatility process.

Lemma. When the time scale of the model is reduced by a factor of k, the per period interest rate volatility parameter, σ, must be reduced by a factor of $k^{3/2}$.

Proof of Lemma. Consider the forward interest rate process given by:

$$f(t,T) = f(0,T) + \sigma^2 t\left(T - \frac{t}{2}\right) + \sigma W^*(t) \tag{16}$$

Now, reduce the time scale by a factor of k and let s denote the volatility parameter and f' the forward rates on the new scale, and let $Y()$ be a process equal to $W^*()$ in distribution. Then, the term structure dynamics can be expressed as:

$$f'(kt,kT) = f'(0,kT) + s^2 kt\left(kT - \frac{kt}{2}\right) + sY(kt) \tag{17}$$

From the definition of the forward rates it is clear that $f(t,T) \approx kf'(kt,kT)$, where $\approx$ refers to equality in distribution. Applying this once gives:

$$f'(kt,kT) = \frac{1}{k}f(0,T) + s^2k^2t\left(T - \frac{t}{2}\right) + sY(kt) \tag{18}$$

and, once again, to yield:

$$\frac{1}{k}f(t,T) = \frac{1}{k}\left[f(0,T) + s^2k^3t\left(T - \frac{t}{2}\right) + skY(kt)\right] \tag{19}$$

which is identical to eq. (7) if and only if $s^2 = \sigma^2 k^{-3}$. ∎

Proof of Proposition 2. The lemma is used to change the time scale in such a way that the marking to market takes place at the end of every time period, measured on the new scale. The proof proceeds by induction. The induction hypothesis holds that the futures price n periods before maturity is given by:

$$H(\tau k - n, \tau k, Tk) = \frac{P(\tau k - n, Tk)}{P(\tau k - n, \tau k)}\, e^{A_n \sigma^2 k^{-3}} \tag{20}$$

where:

$$A_{n+1} = A_n - (kT - k\tau)n \tag{21}$$

$$A_0 = 0 \tag{22}$$

The induction hypothesis is trivially satisfied at maturity for $n = 0$. To check that it holds for an arbitrary n, consider the general valuation relationship that relates the futures price at time $\tau k - n$ to the futures price at time $\tau k - n + 1$:

$$0 = B(\tau k - n)E^*\left[\frac{H(\tau k - n + 1, \tau k, Tk) - H(\tau k - n, \tau k, Tk)}{B(\tau k - n + 1)} \;\middle|\; \mathcal{F}_{\tau k - n}\right] \tag{23}$$

After substitution of the hypothesized value of $H(\tau k - n + 1, \tau k, Tk)$ the computations to verify the induction hypothesis are identical to those in the proof of Proposition 1. To complete the proof one can solve the difference equation (21) for A_n subject to the boundary condition (22) to get:

$$A_n = \frac{-n(n-1)(kT - k\tau)}{2} \tag{24}$$

Changing back to the original time scale, it follows that at time t there are exactly $k(\tau - t)$ markings to market left until maturity, so the correct adjustment factor is found by setting n equal to $k(\tau - t)$ and substituting (24) into (20). ∎

Proposition 3

The no-arbitrage futures price at time t, $H_k^i(t,\tau,T)$, for a futures contract that corresponds to the forward contract valued in Proposition 1 and that is marked to market at the end of every k^{-1} period, is given by:

$$H_k^i(t,\tau,T) = \frac{P(t,\tau)}{P(t,T)} e^{\sigma^2\left[(\tau-t)(T-\tau)^2+\frac{1}{2}(\tau-t)^2(T-\tau)-\frac{1}{2}\frac{1}{k}(\tau-t)(T-\tau)\right]}$$
$$= G^i(t,\tau,T)e^{\frac{\sigma^2}{2}\left[(\tau-t)^2(T-\tau)-\frac{1}{k}(\tau-t)(T-\tau)\right]} \tag{25}$$

Proof. The proof is nearly identical to the proof of Proposition 2 and will, therefore, be only sketched. The induction hypothesis is now given by:

$$H^i(\tau k - n, \tau k, Tk) = \frac{P(\tau k - n, \tau k)}{P(\tau k - n, Tk)} e^{A_n \sigma^2 k^{-3}} \tag{26}$$

where:

$$A_{n+1} = A_n + (kT - k\tau)^2 + (kT - k\tau)n \tag{27}$$
$$A_0 = 0 \tag{28}$$

Again, the hypothesis trivially holds at maturity, and can be verified for general n by computation. The solution to (27)–(28) is given by:

$$A_n = n(kT - k\tau)^2 + \frac{n(n-1)(kT - k\tau)}{2} \tag{29}$$

Reversing the time change by substituting in $n = k(\tau - t)$, simplifying the resulting expression, and combining it with (26) gives the first line of (25). The second line follows from Proposition 1. ∎

For both futures pricing models one can note that setting $k = (\tau - t)^{-1}$, corresponding to a single marking to market at the contract's expiration, appropriately results in equality between futures and forward prices. The same result applies if either interest rates or the bond price at maturity of the futures contract are deterministic, i.e., if either $\sigma = 0$ or $T = \tau$. Furthermore, by comparing (15) to (25) it becomes clear that the difference between the futures and forward prices are of identical magnitude and opposite sign in the two cases. The reason for this is fairly intuitive: using Ito's lemma it can be shown that the stochastic parts of $dP(t,T)/P(t,T)$ and $d[1/P(t,T)]/[1/P(t,T)]$ have the same variance and have perfect negative and positive correlation with $dr(t)$, respectively. The result could then be conjectured from the analysis of CIR (1981), based on the argument that the difference between forward and futures prices depends only on the covariance between the changes in the "underlying price" and the change in interest rates. This would merely be a conjecture, however, since CIR's analysis rests on the "underlying price" being the price of a tradeable asset. The proofs above serve to formally justify such a conjecture.

Letting k go to infinity in (15) and (25) results in futures prices for contracts that are continuously marked to market, and setting it at a value around 250 represents daily marking to market. This allows for an evaluation of the quantitative importance of marking to market in actual markets, as well as of the errors made in previous studies that employed a continuous marking to market approximation. Before proceeding to report numerical values based on the formulas developed in this section, it is shown how to use these results to form hedge portfolios for the instruments involved here.

5. HEDGE PORTFOLIOS

One of the main practical differences between applying the equivalent martingale measure valuation method and the Black–Scholes (1973)/Merton (1973) method of contingent claims pricing is that, in the latter, one computes the hedge portfolios as a part of the pricing process. Since this is not required in the equivalent martingale measure approach, it has to be done separately if one actually wants to hedge. However, as is shown below, it is a simple exercise once the pricing formulas for the contingent claims in question have been obtained. For expositional ease, it is demonstrated below how to hedge a newly issued forward contract on the reciprocal of a pure discount bond price, of the form valued in Proposition 1. The derivation generalizes readily to futures, options, and other interest rate dependent contingent claims, as long as the pricing formula for the instrument can be differentiated with respect to the contemporaneous value of $W^*()$. It is shown how to hedge the forward contract with a position in an instantaneously maturing bond (or, equivalently, a money market account) and a bond of arbitrary maturity.

Assume that one enters into a fairly priced (according to Proposition 1) forward contract at time t and that the goal is to hedge the resulting uncertainty for the instant from t to $t + dt$ by taking a position in the two bonds maturing at time $t + dt$ and s, where $s > t + dt$, but otherwise arbitrary. If the forward position is offset at time $t + dt$, and the resulting future gain (or loss) is liquidated, the resulting cash flow is:

$$\begin{aligned} C &= [G^i(t + dt, \tau, T) - G^i(t, \tau, T)]P(t + dt, \tau) \\ &= dG^i(t, \tau, T)[P(t, \tau) + dP(t, \tau)] \end{aligned} \tag{30}$$

Writing $G^i(t, \tau, T)$ as a function of $W^*()$ and t, $g[W^*(t), t]$, a straightforward application of Ito's lemma yields:

$$\begin{aligned} C &= g_{W^*(t)} P(t, \tau) dW^*(t) \\ &= \sigma(T - \tau)G(t, \tau, T)P(t, \tau)dW^*(t) \end{aligned} \tag{31}$$

Letting $X(s)$ denote the number of dollars a hedge portfolio would hold in a bond of maturity s, financed by $-X(s)$ dollars in the instantaneously maturing bond, it can be shown in the same manner that the value of the hedge portfolio at time $t + dt$ will be:

$$h = -\sigma(s - t)X(s)dW^*(t) \tag{32}$$

Finally, setting:

$$X(s) = \frac{T - \tau}{s - t} G(t, \tau, T)P(t, \tau) \tag{33}$$

it can be verified by inspection that $C + h = 0$ with probability 1 under the martingale measure Q^*, and by the equivalence of Q and Q^* it follows that the portfolio has a time $t + dt$ value that is (almost) surely 0. Thus, the goal of hedging the forward price risk is accomplished.

6. NUMERICAL EXAMPLES

To apply the results of section 3 to evaluate the differences between forward and futures prices, one needs an estimate of the interest rate volatility parameter, σ. Such estimates are taken from Flesaker (1992), where this model is applied to the pricing of Eurodollar futures options. In an empirical study using daily data on futures and option prices from 1985 to 1988, option implied interest rate volatilities (estimates of σ) are found to be in the range of 1–2%.

Since the type of instruments dealt with in this article are conventionally discussed in terms of yields rather than prices, the futures and forward prices derived in section 4 are transformed to yields. The notation defined below is used, corresponding as closely as possible to the previous usage, i.e., G refers to forwards, H to futures, and an i superscript relates to the case where the contracts are written on the reciprocal of a pure discount bond price. For instance, the first variable defined, $R_G(t,\tau,T)$, is the continuously compounded yield implied by the theoretical discount bond forward price at time t, when the forward contract matures at time τ, and the underlying bond matures at time T.

$$R_G(t,\tau,T) = \frac{-\ln[G(t,\tau,T)]}{T-\tau} \tag{34}$$

$$R_G^i(t,\tau,T) = \frac{\ln[G^i(t,\tau,T)]}{T-\tau} = R_G(t,\tau,T) + \sigma^2(\tau-t)(T-\tau) \tag{35}$$

$$\begin{aligned} R_{H,k}(t,\tau,T) &= \frac{-\ln[H_k(t,\tau,T)]}{T-\tau} \\ &= R_G(t,\tau,T) + \frac{\sigma^2}{2}(\tau-t)^2 - \frac{\sigma^2}{2}\,\frac{1}{k}(\tau-t) \end{aligned} \tag{36}$$

$$\begin{aligned} R_{H,k}^i(t,\tau,T) &= \frac{\ln[H_k^i(t,\tau,T)]}{T-\tau} \\ &= R_G(t,\tau,T) + \sigma^2(\tau-t)(T-\tau) + \frac{\sigma^2}{2}(\tau-t)^2 - \frac{\sigma^2}{2}\,\frac{1}{k}(\tau-t) \\ &= R_G^i(t,\tau,T) + \frac{\sigma^2}{2}(\tau-t)^2 - \frac{\sigma^2}{2}\,\frac{1}{k}(\tau-t) \end{aligned} \tag{37}$$

Addressing the primary issue first, define $D(\tau,k,\sigma)$ as the difference between the futures yield and the forward yield for contracts with τ years to maturity, where the futures contract is marked to market every k^{-1} years, and where the interest rate volatility is given by σ. That is, for any t and for $T > \tau > 0$:

$$\begin{aligned} D(\tau,k,\sigma) &= R_{H,k}(t,t+\tau,t+T) - R_G(t,t+\tau,t+T) \\ &= R_{H,k}^i(t,t+\tau,t+T) - R_G^i(t,t+\tau,t+T) \\ &= \frac{\sigma^2\tau^2}{2} - \frac{\sigma^2\tau}{2k} \end{aligned} \tag{38}$$

As previously discussed, the magnitude of the difference between the futures and forward price is equal for the contracts based on a discount bond and on the reciprocal of a discount bond price. When compared on a yield basis, the difference is independent of the maturity of the underlying discount bond. It then follows that these yield differences will apply equally to futures and forwards on coupon bonds.[7] Setting $k = 250$ one can now evaluate the magnitude of the effect of daily marking to market on the price of interest rate futures contracts (Table I). The entries in Table I are $D(\tau, 250, \sigma)$, measured in basis points, for selected values of τ and σ. For example, if $\sigma = 2\%$, the effect on the yield implicit in the price of a one-year interest rate futures contract from daily marking to market is two basis points. Generally, the difference between futures and forward yields is quite small for contract maturities of less than one year. However, the

[7]However, existing U.S. coupon bond futures contracts are loaded with delivery options that obscure the basic relationship between futures and forwards explored here.

difference is much different for contract maturities of two and three years. Even for a moderate σ value of 1.5%, one finds nontrivial yield differences of 4.5 and 10.1 basis points, respectively.

To interpret the economic significance of the numbers in Table I, it should be pointed out that futures contracts on short term interest rates are settled in whole basis points, so differences below a basis point are of the same order of magnitude as institutional round-off errors. For example, the three-month Treasury-bill and Eurodollar futures prices both change by $25 for every basis point change in the yield, regardless of the interest rate level. On the other hand, since the duration of coupon bonds change with the level of interest rates, and since the volatility of a bond price increases with its duration, a change in the yield has a stronger effect on coupon bond futures prices when interest rates are lower. For the Treasury-bond and note futures contracts, the value of a basis point yield change will be in the range of $25 to $150 for typical interest rate levels. Thus, a difference of 4.5 basis points (corresponding to $\sigma = 1.5\%$ and $\tau = 2$ years) translates into differences of $112–725, depending on contract type and interest rate level. To put this in perspective, the margin required to open an interest rate futures contract will normally range from $1200 to $4000. Therefore, the economic significance of marking interest rate futures contracts to the market on a daily basis is fairly trivial for futures contracts with maturities well below a year, but is of significance when the time to maturity is longer. In particular, if the underlying instrument is a long term bond and the interest rate level is low, the dollar value of the effect can, according to this model, be substantial.

As mentioned in the Introduction, futures contracts are sometimes treated as if they are forward contracts when valuing futures options or options embedded in the futures contracts. One such example is approximating the Eurodollar futures price with the forward price of a three-month discount bond.[8] In this case, two separate simplifications are made: the terminal cash-flow is linearized and the effect of the marking to market is ignored. The resulting approximation error can be found from the second line of (37) by setting $T = \tau + (3/12)$. Numerical values for this error are presented in Table II,

Table I
FUTURES VS. FORWARD YIELD DIFFERENCES WITH DAILY MARKING TO MARKET

	σ			
Maturity	**1.0%**	**1.5%**	**2.0%**	**2.5%**
1 month	0.0	0.0	0.0	0.0
2 months	0.0	0.0	0.1	0.1
3 months	0.0	0.1	0.1	0.2
6 months	0.1	0.3	0.5	0.8
9 months	0.3	0.6	1.1	1.7
1 year	0.5	1.1	2.0	3.1
2 years	2.0	4.5	8.0	12.5
3 years	4.5	10.1	18.0	28.1

Entries represent the theoretical difference between interest rate futures and forward prices due to the daily marking to market of the futures contracts, measured in annualized basis points for the equivalent yields. For example, the difference between futures and forward yields for contracts with one year to maturity when the interest rate volatility parameter, σ, is equal to 2% is 2.0 basis points.

the format of which is identical to Table I. The error is the sum of the futures/forward yield difference from Table I and the "nonlinearity effect" on the forward yield. One can see that the nonlinearity effect for short maturities is stronger in Table II than in Table I. For example, for a three-month maturity with $\sigma = 2\%$, the difference between the futures and forward yields is only 0.1 basis points in Table I; whereas, the total error in Table II is 0.4 basis points. Still, the total error is rather insignificant for maturities up to six months. As the maturities lengthen the relative effect of the marking to market starts to increase, so the total errors in these cases are fairly similar to the corresponding differences in Table I.

Finally, the effect of varying the frequency of marking to market (k) is investigated. Of particular interest is whether one, for valuation purposes, can safely set $k = \infty$ and thus assume continuous marking to market. Referring back to eqs. (36) and (37), it is clear that continuous marking to market eliminates the last term on the right hand side of these equations. Therefore, the effect of discrete marking to market can be measured directly by the contribution for this term for finite values of k. To get a reasonable upper bound on this contribution for real world futures contracts, one can set $\sigma = 2.5\%$, $k = 250$, and $\tau - t = 4$ years. With these extreme values, the effect comes to a mere 5/100 of a basis point. Thus, even admitting that the effect of discreteness in the marking to market process depends on the specific model used, one can feel relatively certain that previous researchers did not commit any significant error in assuming continuous marking to market.

From a theoretical perspective, however, as k varies from $(\tau - t)^{-1}$ to ∞, the resulting yield effect is strictly increasing and highly concave in k. This is illustrated in Table III, which contains the effect on the futures yield from marking to market $k(\tau - t)$ times before maturity, as a proportion of the maximum possible effect, which arises from marking to market continuously. The entries in Table III are found as $[R_{H,k} - R_G]/[R_{H,\infty} - R_G]$ for k equal to $(\tau - t)^{-1}$, $2(\tau - t)^{-1}$, and so on. Note that

Table II
APPROXIMATION ERROR MADE BY MODELING A EURODOLLAR FUTURES CONTRACT AS A DISCOUNT BOND FORWARD CONTRACT

	σ			
Maturity	**1.0%**	**1.5%**	**2.0%**	**2.5%**
1 month	0.0	0.1	0.1	0.2
2 months	0.1	0.1	0.2	0.3
3 months	0.1	0.2	0.4	0.6
6 months	0.2	0.6	1.0	1.6
9 months	0.5	1.1	1.9	2.9
1 year	0.7	1.7	3.0	4.7
2 years	2.5	5.6	10.0	15.6
3 years	5.2	11.8	21.0	32.8

Entries represent the error, measured in annualized basis points, made by pricing a Eurodollar futures contracts as if it were a forward contract on a three-month bond discount bond rather than a futures contract on the reciprocal of a three-month bond. This error is found as the sum of the effects of ignoring the daily marking to market and of linearizing the nonlinear payoff function for the Eurodollar contract. For example, the error is 3.0 basis points if $\sigma = 2\%$ and the contract has one year to maturity.

[8] This appears to be what Ho (1985) is doing, although it is never explicitly discussed.

this variable is independent of time to maturity, the maturity of the underlying bond, and the value of σ. It is apparent from the table that marking to market only twice during the life of the contract (including the final settlement) captures a full half of the maximum potential difference between the futures and forward yield. If a one-year futures contract is marked to market once a month it has more than 90% of the valuation effect of continuous marking to market. Finally, daily marking to market of a one-year contract has a yield effect that is 99.9% of hat of continuous marking to market, and so it is as good as continuous from all practical points of view.

7. CONCLUSIONS

There are three major conclusions from this study. First, for contract maturities up to around six months, interest rate futures prices are not significantly different from the corresponding forward prices. Second, for longer maturities, the magnitude of the difference may be quite significant and increases as the maturity and the level of interest rate volatility increases. When measured on a dollar basis, the difference will also increase with the duration of the underlying instrument. Third, through explicit calculations it is shown that the difference in the pricing effects of daily versus continuous marking to market of interest rate futures contracts is trivial.

There is reason to believe that the effects of marking to market are stronger for interest rate futures than for other futures contracts, since the asset values underlying these other contracts are likely to have lower covariance with the interest rate process. Thus, based on the results in this article, one can feel reasonably comfortable pricing any short term futures contract as if it were a forward contract.

Table III
RELATIVE EFFECT OF DISCRETE MARKING TO MARKET OF INTEREST RATE FUTURES AS A FUNCTION OF FREQUENCY

Number of Times	Relative Effect
1	0.0%
2	50.0%
3	66.7%
4	75.0%
5	80.0%
6	83.3%
10	90.0%
12	91.7%
24	95.8%
52	98.1%
100	99.0%
250	99.6%
∞	100.0%

Entries represent the percentage of the difference between forward prices and continuously market to market futures prices captured by discretely marking the futures contracts N times over its remaining lifetime. For example, if an interest rate futures contract is marked to market 24 times before it matures, the difference between its yield and the yield implied by a forward price differ by an amount equal to 95.8% of the difference between the forward yield and the yield of a continuously marked to market futures contract.

The model applied in this study is rather stringent in its assumptions, especially regarding the nature of the interest rate process. In return for giving up some generality, closed form solutions are derived for all the valuation problems considered, which facilitate the numerical analysis. It remains to be seen whether this model and these conclusions carry over to a more general interest rate process. In part based on other work by the author [Flesaker (1991)], it is conjectured that the conclusions regarding short maturity futures contracts are quite robust, but that a more realistic term structure model featuring multiple factors and/or nonconstant volatility across maturities will tend to modify the strong marking to market effects found here for long maturity contracts.

Appendix

To evaluate the numerator on the right hand side of eq. (13) each of the component terms is first re-expressed by using eqs. (1)–(8) and simplification:

$$\begin{aligned}\frac{1}{P(\tau,T)} &= e^{\int_\tau^T f(\tau,\nu)d\nu} \\ &= e^{\int_\tau^T f(\tau,\nu)d\nu + \int_\tau^T \int_t^\tau \sigma^2(T-s)ds\,d\nu + \int_\tau^T \int_t^\tau \sigma dW^*(s)d\nu} \\ &= \frac{P(t,\tau)}{P(t,T)} e^{\frac{\sigma^2}{2}(T-\tau)(\tau-t)(T-t)+\sigma(T-\tau)[W^*(\tau)-W^*(t)]}\end{aligned} \tag{A-1}$$

$$\begin{aligned}\frac{B(t)}{B(\tau)} &= e^{-\int_t^\tau r(s)ds} \\ &= e^{-\int_t^\tau f(t,s)ds - \int_t^\tau \int_t^s \sigma^2(s-\nu)d\nu\,ds - \int_t^\tau \int_t^s \sigma dW^*(\nu)ds} \\ &= P(t,\tau)e^{\frac{\sigma^2}{2}\left[t\tau(\tau-t)-\frac{1}{3}(\tau^3-t^3)\right]-\sigma\int_t^\tau [W^*(s)-W^*(t)]ds}\end{aligned} \tag{A-2}$$

Taking advantage of the Markov property of Brownian Motion, one can define a new Q^*-Brownian Motion $W'(s)$, as:

$$W'(s) = W^*(s) - W^*(t) \tag{A-3}$$

Combining the stochastic parts of (A-1) and (A-2) with this definition, one needs to evaluate the expectation of the following expression under the equivalent martingale measure, Q^*:

$$e^{\sigma(T-\tau)W'(\tau-t)-\sigma\int_0^{\tau-t} W'(s)ds}$$

This expectation can be evaluated by first conditioning on the value of $W'(\tau - t)$. The process defined by $\{W'(s)|W'(\tau - t) = u\}$ is a Brownian bridge under the equivalent martingale measure, and it follows that:

$$\left[\int_0^{\tau-t} (W'(s)|W'(\tau - t) = u)ds\right] \sim N\left(\frac{u(\tau - t)}{2}, \frac{(\tau - t)^3}{12} + \frac{(\tau - t)^2u^2}{4}\right)$$

Using the moment generating function of the normal distribution it then follows that:

$$E^*\left[e^{-\sigma\int_0^{\tau-t} W'(s)ds} \;\middle|\; W'(\tau - t) = u\right] = e^{\frac{\sigma^2}{24}(\tau-t)^3-\frac{\sigma}{2}u(\tau-t)} \tag{A-4}$$

The conditional expectation in (A-4) can now be integrated over the distribution of $W'(\tau - t)$, which is given by $N[0,(\tau - t)]$, and subsequently simplified to give:

$$
\begin{aligned}
&E^*\left[e^{\sigma(T-\tau)W'(\tau-t)-\sigma\int_0^{\tau-t}W'(s)ds}\right]\\
&\quad=\int_{-\infty}^{\infty}\left(e^{\sigma(T-\tau)u+\frac{\sigma^2}{24}(\tau-t)^3-\frac{\sigma}{2}u(\tau-t)}\right)\frac{1}{\sqrt{2\pi(\tau-t)}}\,e^{\frac{-u^2}{2(\tau-t)}}du\\
&\quad=e^{\frac{\sigma^2}{24}(\tau-t)^3+\frac{\sigma^2}{8}(\tau-t)^3-\frac{\sigma^2}{2}(\tau-t)^2(T-\tau)+\frac{\sigma^2}{2}(\tau-t)(T-\tau)^2}
\end{aligned}
\tag{A-5}
$$

The proof is now completed by combining the expectation of the stochastic part given in eq. (A-5) with the deterministic parts of (A-1) and (A-2) and algebraic simplification.

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