Hedging Risk on Futures Contracts under Stochastic Interest Rates

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INTRODUCTION

Several authors have argued that arbitrage will induce an equality relationship between forward and futures prices if interest rates are nonstochastic [Cox, Ingersoll, and Ross (1981); Richard and Sundaresan (1981); Jarrow and Oldfield (1981); and French (1982)]. Levy (1989) has argued that if only interest rates are nonstochastic, forward and futures prices are equal. This article addresses the question of the relative pricing of multiperiod futures and forward contracts under conditions of stochastic interest rates. The distribution of the terminal value of hedging the daily mark-to-market settlements for a general n-period futures contract is explicitly derived. In addition, the effects of fixed and variable transactions costs on the pricing relationship between forward and futures prices are evaluated.

The results show that the variance of the terminal value of the futures mark-to-market hedge is not zero if interest rates are stochastic; and the strategy cited in Levy (1989), of minimizing the interest rate forecast error by (validly) restricting the forecasting requirements to only one day ahead, is negated by the requirement of repeating the strategy $(n - 1)$ times over the contract's life. Because of the effective compounding of one-day interest rate forecasting errors over the repeated daily hedging of the cumulative mark-to-market, the resulting mark-to-market hedge risk is an increasing function of the number of days remaining in the life of the futures contract. Thus, the hedger effectively cannot escape the uncertainty of all interest rates over the hedge period by restricting the uncertainty to one-day-ahead interest rates. Finally, it is found that transactions costs have a nontrivial effect on the pricing of futures relative to forward contracts and that there is no probable offsetting parallel cost to forwards.

BASIC MODEL WITH CERTAINTY

To assure, at time 0, the future price of an asset at time t, one could enter into a forward contract; with futures contracts, however, one cannot achieve a perfect hedge because of the daily cash settlement associated with them. To achieve a perfect hedge with futures, it is sufficient to assume that at any day i, the next day's yield on a bond maturing on the delivery date of the futures contract, can be predicted with certainty [Levy (1989)]. This implies that the futures price and the forward price must be the same due to the

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law of one price. Assume:

F_t = futures price on day t;

B_t = price, on day t, of a pure discount bond maturing on day n

(this implies that $B_t = \dfrac{1}{(1 + r_t)^{n-t}}$, r_t being a spot rate);

d_t = number of futures contracts bought on day t;

FVW_t = value of W_t on day n, where

$W_t = (F_t - F_{t-1}) * d_{t-1}$ = gain or loss in the futures position.

To lock in the futures price available on day t: buy d_t futures contracts deliverable on day n; on $(t + 1)$ close the open positions; reinvest the proceeds until day n; and buy d_{t+1} futures contracts. Repeating this strategy, the cash flow that will accumulate on day n is given by:

$$TV_n = \sum_{t=1}^{n} d_{t-1}(F_t - F_{t-1}) * (1 + r_t)^{n-t}, \quad t \leq n \tag{1}$$

This equation could be rearranged to yield:

$$TV_n = F_0[-d_0(1 + r_1)^{n-1}] + F_1[d_0(1 + r_1)^{n-1} - d_1(1 + r_2)^{n-2}] + \ldots + F_n(d_{n-1}) \tag{2}$$

A perfect hedge, with futures contracts, can be achieved if the cash flow on day n is function of F_0 only. This requires:

$$d_t = \frac{1}{(1 + r_{t+1})^{n-t-1}}, \quad t = 0, \ldots, (n - 1)$$

This implies that if one knows on day i, with certainty, the next day's yield (r_{i+1}), the number of futures contracts to be bought on day i can be determined, and a perfect hedge will be possible. If this strategy is applied, the cash flow on day n will be reduced to $(S_n - F_0)$, S_n being the cash price on day n. But the perfect hedge in the forward contract will yield a cash flow of $(S_n - G_0)$, G_0 being the forward price at time 0. As a result, the futures price and forward price are the same $(F_0 = G_0)$.

THE MODEL WITH UNCERTAINTY

Consider now the introduction of uncertainty in the forecast of the next day's yield. Mathematically, it is assumed

$$r_t = \hat{r}_t + \epsilon_t$$

where:

r_t = spot rate on day t;
$\hat{r}_t$ = forecast, on day $(t - 1)$, of spot rate on day t; and
ϵ_t = error term (assumed normally distributed with a mean of zero and standard deviation of σ).

By following the same strategy described above and choosing

$$d_{t-1} \quad \text{as} \quad (1/(1 + \hat{r}_t))^{n-t}$$

Equation (1) will be:

$$TV_n = \left(\sum_{t=1}^{n} (F_t - F_{t-1})(1 + \hat{r}_t + \epsilon_t)^{n-t}\right) / (1 + \hat{r}_t)^{n-t} \tag{3}$$

or

$$TV_n = \sum_{t=1}^{n} (F_t - F_{t-1})(1 + e_t)^{n-t} \tag{4}$$

where $e_t = \epsilon_t/(1 + \hat{r}_t)$.

For a finite n, an infinitesimal e_t, $(1 + e_t)^{n-t}$, could be approximated with $(1 + (n - t)e_t)$; (this approximation is made for simplicity; it could be seen from $\lim_{e_t \to 0} ((1 + e_t)^{n-t}/(1 + (n - t)e_t)) \to 1$ from above.

As a result, eq. (4) could be written as:

$$TV_n = \sum_{t=1}^{n} (F_t - F_{t-1})(1 + (n - t)e_t) \tag{5}$$

The expected value of the cash flow, TV_n, on day n, is:

$$E(TV_n) = F_n - F_0 = S_n - F_0$$

(which is the same value found above for the cash flow at time n).

The variance of the cash flow, TV_n, on day n is:

$$\mathrm{Var}(TV_n) = E\left[\sum_{t=1}^{n} (F_t - F_{t-1})(1 + (n - t)e_t) - (S_n - F_0)\right]^2 \tag{6}$$

If one assumes $dF_t = F_t - F_{t-1} = \mu + f_t$, where μ is a constant drift and f_t an error term (assumed to be normally distributed with a mean of zero and standard deviation of σ_f), eq. (6) becomes:

$$\begin{aligned}\mathrm{Var}(TV_n) &= E\left[\sum_{t=1}^{n} (\mu + f_t)(1 + (n - t)e_t) - (S_n - F_0)\right]^2 \\ &= E\left[\sum_{t=1}^{n} (\mu + f_t)(n - t)e_t\right]^2\end{aligned} \tag{7}$$

If e_t and f_t are uncorrelated, eq. (7) becomes:

$$\begin{aligned}\mathrm{Var}(TV_n) &= E\left[\sum_{t=1}^{n} \mu(n - t)e_t\right]^2 \\ &= \mu^2\sigma_e^2 \sum_{t=1}^{n-1} t^2 = \mu^2\sigma_e^2(n - 1)(n)(2n - 1)/6\end{aligned} \tag{8}$$

Equation (8) shows that the standard deviation of the terminal value, $\sigma(TV_n)$, increases at an increasing rate with the number of days (n) remaining in the life of the futures contract.

For illustration, this equation is applied for 30 periods (days). A drift μ of 1, and a standard deviation of the error term of interest rates of 0.1% (for the $(n - t)$ period, $t = 1, \ldots, n - 1$) are assumed. For each day, 64 iterations are used. Table I shows the variations of $\sigma(TV_n)$ with the number of days remaining in the life of the futures contracts.

Table I
VARIATIONS OF THE TERMINAL VALUE (TV_n) WITH THE NUMBER OF DAYS REMAINING IN THE LIFE OF THE FUTURES CONTRACT
(μ = 1; σ_e = 0.001; NUMBER OF DAILY ITERATIONS = 64)

n	$\sigma(TV_n)$
1	0.0000
2	0.0010
3	0.0022
4	0.0037
5	0.0055
6	0.0074
7	0.0095
8	0.0118
9	0.0143
10	0.0169
11	0.0196
12	0.0225
13	0.0255
14	0.0286
15	0.0319
16	0.0352
17	0.0387
18	0.0422
19	0.0459
20	0.0497
21	0.0536
22	0.0575
23	0.0616
24	0.0658
25	0.0700
26	0.0743
27	0.0787
28	0.0832
29	0.0878
30	0.0925

EFFECT OF TRANSACTION COSTS

Even if the hedging strategy were riskless under stochastic interest rates, the effect of fixed and/or variable transactions costs required to implement the futures hedging would be (n)-times that of implementing the forward hedging strategy. Adding terms to reflect transactions costs in (1) yields:

$$TV_n = \sum_{t=1}^{n} d_{t-1}[(F_t - F_{t-1})(1 - \alpha) - T]/B_t$$

where:

α = variable transaction cost as a percent of the dollar value of the position taken in the futures.

T = fixed transaction cost for each time a position is taken in futures.

[A perfect hedge is invoked by setting $d_{t-1} = B_t/(1 - \alpha)$].

Assuming that the transactions cost for the forward (G) is tc, the pricing relationship between current forward and futures values is:

$$F_0 + n[T/(1 - \alpha)] = G_0 + tc$$

which means that $F_0 = G_0$ only if

$$tc = \frac{n \cdot T}{(1 - \alpha)}$$

Thus, the cost of implementing the futures hedging strategy is, in general, different from that of the forward strategy. And even if there were no unresolvable uncertainty in the futures hedging strategy, transactions costs would induce a potentially significant pricing difference to the disadvantage of futures hedging in proportion to the length of the hedging period (n).

CONCLUSION

The results show that the uncertainty surrounding the hedger's future wealth, due to uncertain next-day interest rates, increases at an increasing rate with the number of days n of the hedging period. This results from the combined effects of each day t's forecasting error being compounded for $(n - t)$ periods and for having to face that compounding error $(n - 1)$ times over the n-period life of the futures contract. Since this study assumes away any covariance effects between f_t and e_t, the true potential risk in using futures contracts for this kind of hedging is potentially significantly understated. In addition, daily exposure to transactions costs further reduce the attractiveness of futures contracts, relative to forward contracts, for this kind of hedging.

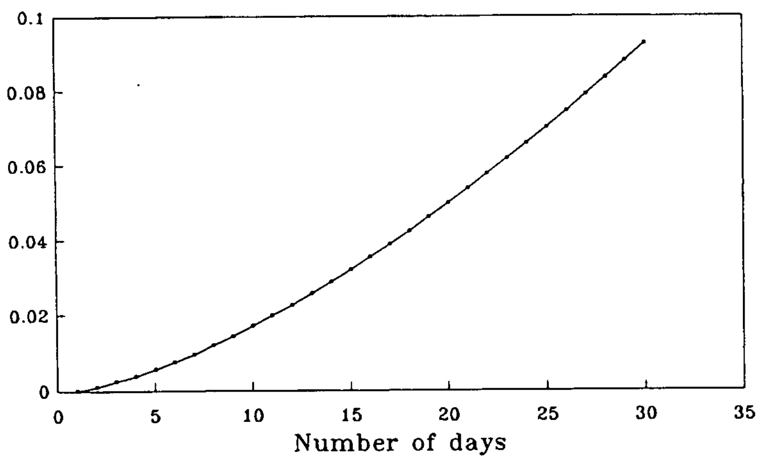

Figure 1

Variations of TV_n with number of days remaining in the life of the futures.

Bibliography

Cox, J., Ingersoll, J., and Ross, S. (1981): "The Relation Between Forward Prices and Futures Prices," *Journal of Financial Economics,* 9:321–346.

French, K. R. (1983): "A Comparison of Forward and Futures Prices," *Journal of Financial Economics,* 12:311–342.

Jarrow, R., and Oldfield, G. (1981): "Forward Contracts and Futures Contracts," *Journal of Financial Economics,* 9:373–382.

Levy, A. (1989): "A Note on the Relationship between Forward and Futures Contracts," *The Journal of Futures Markets,* 9:171–173.

Richard, S. F., and Sundaresan, M. (1981): "A Continuous Time Equilibrium Model of Forward Prices and Futures Prices in a Multigood Economy," *Journal of Financial Economics,* 9:347–371.