Averaging and Deferred Payment Yield Agreements

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INTRODUCTION

The financial markets continually introduce contracts that facilitate efficient risk sharing by investors. Recent introductions include a family of contracts where terminal payoffs are linked to arithmetic averages of prices collected over finite periods of time rather than to end-of-period prices. Examples of such contracts traded in over-the-counter markets include averaging options on commodities, lookback contracts on the average, and a host of interest rate mortgages, swaps, and interest rate caps with averaging features. In addition, averaging contracts exist in organized exchanges. For example, the 30-day interest rate futures contract that trades on the Chicago Board of Trade has its terminal price linked to the arithmetic average daily federal funds rate computed over a 30-day period. With an open interest of over 5000 contracts and a notional principal of $5 million each, these account for over $25 billion in the exchange. Contracts, with terminal values based on average prices, are less susceptible to price manipulation than contracts having payouts linked to terminal prices alone. Such contracts are therefore especially advantageous when the underlying asset is illiquid.[1]

The literature on averaging and deferred payment contracts has mushroomed over the last few years.[2] In all models, however, interest rates are assumed constant in spite of the fact that the majority of averaging contracts encountered in practice are related to interest rate products. This article provides a valuation model for futures and forward contracts having terminal payouts linked to averages of yields. The 30-day interest rate futures contract is used to illustrate the valuation procedure. A special case of an averaging contract arises when the terminal payout is linked to a price or yield determined at an earlier point in time. Examples of such contracts, called deferred payment contracts, arise in many interest rate swaps where the size of the variable payment to be exchanged for a

[1]Average based contracts have other advantages. For example, Ritchken, Sankarasubramanian, and Vijh (1990) show how options on the average can be used to precisely cap risks arising from daily price fluctuations facing traders who transact continuously in the underlying commodity. For further examples of averaging contracts, see Kemna and Vorst (1990).

[2]See Kemna and Vorst (1990), Ritchken, Sankarasubramanian, and Vijh (1989) and Turnbull and Wakeman (1991).

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The Journal of Futures Markets, Vol. 13, No. 1, 23–41 (1993)

CCC 0270-7314/93/010023-19

fixed payment is known one period prior to the payment date. Explicit valuation models for these contracts are provided.

This article is organized as follows: The next section highlights the terms of the 30-day interest rate futures contract and places the valuation issue of this contract in a broader perspective of valuing interest rate sensitive averaging contracts. Since these contracts adjust to new information that affects forward rates along the relevant range of the yield curve, it is necessary to specify how the set of forward rates jointly evolve over time before establishing viable models. Section 3 reviews an arbitrage free pricing approach for interest rate claims where forward rates are generated by a single factor model with all rates initialized to their currently observable values. The main results are presented in section 4 where models for pricing futures and forward averaging contracts are developed. Section 5 focuses on the 30-day interest rate futures contract. The theoretical futures price is compared to the price of a forward contract, identical to the futures contract but without the marking to market feature. This comparison is made because many traders use the *forward* 30-day yield-to-maturity observed in the term structure as the basis for establishing a theoretical price for the 30-day interest rate futures contract.[3] The goal is to measure the deviation between the theoretical yield implied by the futures contract and the observed forward yield. It is found that these deviations are small for the short-term 30-day interest rate futures contract. However, the deviations are significant for longer term contracts. Hence, the common strategy practiced by traders may not induce large errors for short term contracts, but may cause problems for longer term contracts.[4] In addition, for other over-the-counter forward rate agreements, where the averaging is based on a set of longer yields to maturity, or when the time between averaging points is large, the impact of these adjustments may also be large and should not be ignored.

2. THE 30-DAY INTEREST RATE FUTURES CONTRACT

The 30-day interest rate futures contract that trades on the Chicago Board of Trade is a cash settled contract, where the terminal price is linked to the arithmetic average of the daily federal funds effective rates calculated and reported over each of 30 consecutive days by the Federal Reserve Bank of New York. A unique feature of this contract is the way in which the cash settlement price is established. Specifically, the final average is a simple arithmetic average of the daily federal funds rates collected over the delivery month. However, cash settlement can occur anytime in the delivery month. The futures settlement price is calculated as 100 minus the monthly average of the implied average federal funds rate. Since the notional principal is 5 million dollars, each basis point is worth \$41.67 ($\frac{1}{100}$ of 1% of 5 million dollars on a 30-day basis). Notice that no delivery options are embedded in this contract.

During the delivery period, the contract rate reflects a weighted average of average rates-to-date and term rates through the rest of the month. As the end of the month draws near, the contribution of the average rate-to-date dominates the calculation and hence the volatility of the contract price diminishes. Prior to the expiration month, the contract tracks the one month forward rate for the delivery month. However, this relationship is not precise for two reasons. First, the implied forward rate is based on a compounded or geometric average, whereas the rate implied by the contract is based on an uncompounded, or arithmetic average. Second, the implied rate from a futures contract

[3]Refer to the Chicago Board of Trade publications on the 30-day interest rate futures contract.

[4]This could be particularly important since the Chicago Board of Trade has recently introduced longer term contracts.

will differ from the forward rate because futures contracts are not equivalent to forward contracts. Before providing a valuation model for this contract, recently developed pricing methodology for valuing interest rate claims is reviewed.

3. ARBITRAGE-FREE PRICING OF INTEREST RATE CLAIMS

Since the price of averaging futures and forward contracts adjust to new information that affects forward rates over the relevant range along the yield curve, to model the prices it is necessary to specify how the set of forward rates jointly evolve stochastically over time. Toward this goal, let the instantaneous forward rate for date T viewed from date t be $f(t,T)$, where

$$f(t,T) = \frac{\partial}{\partial T} \ln [P(t,T)] \tag{1}$$

with $P(t,T)$ representing the price at time t, of a default-free discount bond maturing at time T. Further, let the annualized forward yield to maturity at time 0, for the time period $[t,T]$ be denoted by $y_0(t,T)$ where

$$y_0(t,T) = -\frac{1}{T-t} \ln \left[\frac{P(0,T)}{P(0,t)} \right] \tag{2}$$

The evolution of the forward rate is given by

$$df(t,T) = \mu_f(t,T)dt + \sigma_f(t,T)dw(t) \quad \text{for} \quad T > t \tag{3}$$

with the forward rate function initialized to its current observable value. Here $\mu_f(t,T)$ and $\sigma_f(t,T)$ are the drift and volatility structures and $dw(t)$ is the standard Weiner increment.

The instantaneous return on a discount bond of maturity T, when forward rates are given by eq. (3) is of the form[5]

$$\frac{dP(t,T)}{P(t,T)} = \mu_p(t,T)\,dt - \sigma_p(t,T)dw(t)$$

In this single factor model, the instantaneous drift and volatility structure of bond returns must be related to the drift and volatility structure of forward rates. To see this relationship more clearly, note that from eq. (1)

$$\begin{aligned} df(t,T) &= -\frac{\partial}{\partial T} d(\ln [P(t,T)]) \\ &= -\frac{\partial}{\partial T} \left(\left[\mu_p(t,T) - \frac{1}{2}\sigma_p^2(t,T) \right] dt + \sigma_p(t,T)dw(t) \right) \end{aligned} \tag{4}$$

Hence,

$$\begin{aligned} \sigma_f(t,T) &= \frac{\partial}{\partial T} \sigma_p(t,T) \\ \sigma_p(t,T) &= \int_t^T \sigma_f(t,s)ds \end{aligned} \tag{5}$$

[5]The negative coefficient of the stochastic component reflects the inverse relationship between interest rates and bond prices.

and

$$\mu_f(t,T) = -\frac{\partial}{\partial T}\mu_p(t,T) + \frac{\partial}{\partial T}\frac{\sigma_p^2(t,T)}{2}$$

from which it follows that

$$\frac{\partial}{\partial T}\mu_p(t,T) = -\mu_f(t,T) + \sigma_p(t,T)\sigma_f(t,T) \tag{6}$$

Further, the usual dynamic no-arbitrage condition is given by

$$\mu_p(t,T) - r(t) = \lambda(t)\sigma_p(t,T) \tag{7}$$

where $\lambda(t)$ is the market price of risk and $r(t) = f(t,t)$ is the instantaneous spot interest rate. Differentiating eq. (7) with respect to T yields

$$\begin{aligned}\frac{\partial}{\partial T}\mu_p(t,T) &= \lambda(t)\left[\frac{\partial}{\partial T}\sigma_p(t,T)\right] \\ &= \lambda(t)\sigma_f(t,T)\end{aligned} \tag{8}$$

Substituting (7) into (6) one obtains

$$\mu_f(t,T) = -\lambda(t)\sigma_f(t,T) + \sigma_p(t,T)\sigma_f(t,T) \tag{9}$$

Hence, the evolution of the forward rates can be represented by

$$df(t,T) = \sigma_f(t,T)[\sigma_p(t,T) - \lambda(t)]dt + \sigma_f(t,T)dw(t) \tag{9a}$$

This equation relates the drift component of forward rates to the volatility structure of forward rates and the market price of risk.

Now consider the pricing of an interest rate futures contract. Specifically, let $H(0,s)$ be the time 0 price of a futures contract with settlement date s. Let $g(s) = H(s,s)$ be the settlement value of the interest rate contract at time s. The futures price will fluctuate with all forward rates and with time.[6] Suppressing arguments, using subscripts to denote derivatives, and applying Ito's lemma yields

$$dH = \mu_H dt + \sigma_H dw(t)$$

where

$$\begin{aligned}\mu_H &= \mu_f(t,s)H_f + H_t + \frac{1}{2}\sigma_f^2(t,s)H_{ff} \\ \sigma_H &= \sigma_f(t,s)H_f\end{aligned}$$

Now consider a portfolio of n_1 futures contracts with settlement date T_1 and n_2 discount bonds with maturity date T_2. If n_1 and n_2 are selected such that the portfolio is riskless, it must earn the riskless rate of return to preclude arbitrage opportunities. This yields

$$n_1\mu_H + n_2\mu_p(t,T_2)P(t,T_2) = r(t)n_2P(t,T_2)$$

$$n_1\sigma_H - n_2\sigma_p(t,T_2)P(t,T_2) = 0$$

In conjunction with eq. (7) this leads to

$$\frac{\mu_H}{\sigma_H} = -\frac{\mu_p(t,T_2) - r(t)}{\sigma_p(t,T_2)} = -\lambda(t)$$

[6]In a single factor model, all forward rates are instantaneously perfectly correlated. Hence, any single forward rate, including the spot rate could be used as the underlying factor. The forward rate for the time increment at date $s, f(t,s)$, is chosen to be the state variable.

Substituting for μ_H and $\sigma_H$yields

$$\mu_f(t,s)H_f + H_t + \frac{1}{2}\sigma_f^2(t,s)H_{ff} + \lambda(t)\sigma_f(t,s)H_f = 0$$

Now substituting eq. (9) for $\mu_f(t,s)$ into the above equation yields the following partial differential equation that all futures prices must satisfy to preclude arbitrage opportunities

$$\sigma_f(t,s)\sigma_p(t,s)H_f + H_t + \frac{1}{2}\sigma_f^2(t,s)H_{ff} = 0$$

Note that the market price of risk has dropped out of this equation. The solution of this partial differential equation subject to the boundary condition of $H(s,s) = g(s)$ can be expressed as

$$H(0,s) = \overline{\mathbf{E}}_0\{H(s,s)\} \tag{10a}$$

where the expectation is taken under the *risk-neutralized process*

$$df(t,s) = -\sigma_p(t,s)\sigma_f(t,s)dt + \sigma_f(t,s)dw(t) \tag{10b}$$

The forward price of an otherwise identical contract can be expressed as[7]

$$G(0,s) = \tilde{\mathbf{E}}_0\{g(s)\} \tag{10c}$$

where the expectation is taken under the *forward risk-adjusted process*

$$df(t,s) = \sigma_f(t,s)dw(t) \tag{10d}$$

Equations (5)–(10) allow contingent claim models to be developed for a variety of interest rate sensitive contracts. In the Black–Scholes model, options are priced relative to an observed stock price and conditional on a volatility structure. Similarly, in this approach, interest rate claims are priced relative to an observed term structure and conditional on a volatility structure for forward rates. This approach is particularly attractive for pricing claims such as the 30-day interest rate futures contract, where the prices are set relative to an existing, observable set of forward rates.

An alternative to the above arbitrage approach is to adopt an equilibrium model such as Cox Ingersoll, and Ross (1985) or Longstaff and Schwartz (1990). While such equilibrium models have many advantages, if the goal is to price claims contingent on observable yields, the parameters of these models must be carefully chosen such that the set of forward rates produced by the model closely duplicate the observed set of forward rates. If a perfect matching of theoretical and observed prices is not attained, then these models yield prices that admit arbitrage opportunities with respect to observed prices. In contrast, the arbitrage based approach initializes prices to their observed values and is therefore adopted here.

Of course, like the original Black–Scholes model, this *no-arbitrage term-structure-constrained* methodology has some disadvantages. While a market structure for risk is not required, a volatility structure is required. In the original Black–Scholes model, a constant volatility is assumed. Here, a volatility structure for the entire set of instantaneous forward rates is required. While the volatility structure could in principle be stochastic, for simplicity a deterministic structure is used. with this assumption, analytical solutions for a variety of claims can be computed. A volatility structure that is used by Heath, Jarrow, and Morton (1990) and Vasicek(1977) is the exponentially dampened volatility

[7]See Appendix 1 for a proof of this result.

structure, where

$$\sigma_f(t,T) = \sigma e^{-\kappa(T-t)} \quad \text{for} \quad T > t \quad \text{and} \quad \kappa \geq 0$$

This function captures the notion that the volatility of forward rates declines with time. If $\kappa = 0$, than all forward rates have the same constant volatility and the model reduces to the continuous time limit of the Ho–Lee (1986) model. The parameters σ and κ can be estimated by inverting option prices. Under this structure, from eq. (5), the instantaneous volatility of the bond is computed as

$$\sigma_p(t,T) = \frac{\sigma}{\kappa}\left[1 - e^{-\kappa(T-t)}\right]$$

Notice that as $t \to T, \sigma_p(t,T)$ converges to zero. This follows because the bond converges to its face value at maturity. A common criticism of the constant volatility structure used by Ho and Lee is that it allows for negative interest rates.[8] However, if the parameters κ and σ are carefully curtailed, then the probability of having negative rates can be made arbitrarily small. Notice too that since the drift term is unspecified, it could be designed to exhibit very strong mean reversion especially when rates fall below some threshold. This further mitigates the problem.

For this volatility structure the price of a bond at time t equals the time 0 forward price of the same bond, $F_0(t,T)$, multiplied by an adjustment factor. Specifically,

$$P(t,T) = F_0(t,T)e^{\alpha(t,T)-\beta(t,T)r(t)} \tag{11}$$

where

$$F_0(t,T) = \frac{P(0,T)}{P(0,t)} \tag{11a}$$

$$\alpha(t,T) = \beta(t,T)f(0,t) - \frac{1}{2}\beta^2(t,T)\phi^2(t) \tag{11b}$$

$$\beta(t,T) = \frac{1}{\kappa}\left[1 - e^{-\kappa(T-t)}\right] \tag{11c}$$

$$\phi^2(t) = \frac{\sigma^2}{2\kappa}\left[1 - e^{-2\kappa t}\right] \tag{11d}$$

and $r(t) = f(t,t)$ is the spot rate at time t. A proof of this result is provided in Appendix 2. Note that as $t \to T, P(t,T)$ converges to its face value of unity.

Given the exponentially dampened volatility structure, the distribution of $f(t,T)$ at time t viewed from time 0, under the risk-neutralized process described in eq. (6) is normal with mean and variance given by

$$\overline{\mathbf{E}}_0\{f(t,T)\} = f(0,T) + \frac{\sigma^2}{2\kappa^2}\left[2e^{-\kappa(T-t)} - 2e^{-\kappa T} - e^{-2\kappa(T-t)} + e^{-2\kappa T}\right] \tag{12a}$$

$$\overline{\mathrm{Var}}_0\{f(t,T)\} = \frac{\sigma^2}{2\kappa}\left[1 - e^{-2\kappa(T-t)}\right] \tag{12b}$$

4. DEFERRED PAYMENT AND AVERAGE YIELD FORWARD AND FUTURES CONTRACTS

Consider a claim (forward or futures) where the terminal price at date s, say, is determined by the sequence of k yields to maturity that occurred at times $t_1 < t_2 < \cdots < t_k$, with

[8]See Rabinovitch (1989) and Ritchken and Boenewan (1990) for a discussion on this.

$t_k \leq s$. Let $y_{t_i}(t_i, s_i)$ be the yield to maturity at date t_i for the period $[t_i, s_i]$. Here $s_i \geq t_i$ for $i = 1, \cdots, k$. Let the accrued weighted average at time s be

$$A(t_1, t_k) = \sum_{i=1}^{k} \omega_i y_{t_i}(t_i, s_i) \tag{13}$$

The value of claims at time $t_0 = 0$ on the above average is of interest here. The 30-day interest rate futures contract belongs to this class. Here $k = 30$, and t_1 through t_k correspond to successive days in the deliverable month, with the final settlement date, $s = t_k$. The terminal value is based on the average $A(t_1, t_k)$where $s_i - t_i$ equals one day and ω_i equals one-thirtieth.[9]

Before determining the value of averaging contracts, simple deferred payment contracts are examined where $k = 1$ and $s \geq t_1$. Consider an agreement to exchange, at time s, the yield $y_{t_1}(t_1, s_1)$ for a constant, $g(0, s)$. If this constant $g(0, s)$ is chosen so as to make the fair value of the agreement equal to zero, then $g(0, s)$ may be interpreted as the *implied forward yield.* Further, if the contract is marked to market continuously, then denote by $h(0, s)$, the fair value of the constant. $h(0, s)$ can be interpreted as the *implied futures yield.*

Proposition 1

(i) The implied forward yield, $g(0, s)$, on the yield, $y_{t_1}(t_1, s_1)$, is given by

$$g(0, s) = y_0(t_1, s_1) + \gamma_0(t_1, s_1, s) \tag{14}$$

where

$$\gamma_0(t_1, s_1, s) = \frac{\sigma^2}{4\kappa^3(s_1 - t_1)}\left(1 - e^{-\kappa(s_1 - t_1)}\right)$$
$$\left(1 - e^{-2\kappa t_1}\right)\left(2e^{-\kappa(s - t_1)} - e^{-\kappa(s_1 - t_1)} - 1\right)$$

(ii) The implied futures yield, $h(0, s)$, on the yield, $y_{t_1}(t_1, s_1)$ is given by

$$h(0, s) = g(0, s) + \chi_0(t_1, s_1, s) \tag{15}$$

where

$$\chi_{0(t_1, s_1, s)} = \frac{\sigma^2}{2\kappa^3}\left[\frac{1 - e^{-\kappa(s_1 - t_1)}}{s_1 - t_1}\right] e^{\kappa(s - t_1)}$$
$$\left[2e^{-\kappa(s - t_1)} - 2e^{-\kappa s} - e^{-2\kappa(s - t_1)} + e^{-2\kappa s}\right]$$

Proof. See Appendix 3.

Equation (15) states that the implied forward yield will be set below the implied futures yield with the difference, $\chi_0(t_1, s_1, s) \geq 0$, reflecting the cost of marking to market.[10]

Notice that when $t_1 = s$, the equations simplify to cash settlement contracts on yields with no payments deferred. Even in this case, the fair implied forward yield, $g(0, s)$ is not set equal to the observed forward yield, $y_0(s, s_1)$. The reason they are not equal stems from the fact that yields are not traded securities. $\gamma_0(t_1, s_1, s)$ therefore reflects an adjustment that is necessary to make to the forward yield account for the fact that yields are not directly tradable.

[9]Actually, $\omega_i = 1/30$ for Mondays through Sunday, but the weekend rate is taken equal to Friday's rate.
[10]If the contracts are expressed in price form, then $G(0, s) = 100[1 - g(0, s)], H(0, s) = 100[1 - h(0, s)]$, and $G(0, s) - H(0, s) \geq 0$ reflects the marking to market costs.

Proposition 1 indicates that the implied forward and futures yields may not equal the observed forward yields to maturity. Proposition 2 below characterizes the sign of the deviation between the implied yields and the observed forward yield.

Proposition 2

(i) The implied forward yield of a deferred forward contract can be higher or lower than the observed forward yield to maturity depending on the relationship between the deferment interval, $s - t_1$, and the maturity of the yield, $s_1 - t_1$, upon which the payment is based. Specifically

$$\begin{aligned} &g(0,s) > y_0(t_1,s_1) \quad \text{if} \quad s - t_1 < \frac{1}{\kappa}\left(\ln(2) - \ln\left[1 + e^{-\kappa(s_1-t_1)}\right]\right) \\ &g(0,s) \leq y_0(t_1,s_1) \quad \text{otherwise} \end{aligned} \tag{16}$$

(ii) The implied futures yield of a deferred futures contract is always bounded below by the observed yield to maturity. That is

$$h(0,s) \geq y_0(t_1,s_1)$$

Proof. See Appendix 3.

Proposition 1 is of interest in its own right. First, it provides models for valuing forward rate and deferred forward rate agreements. Such contracts arise in a variety of forms in over-the-counter markets. Notice that for the special case where there is no deferment, i.e., $t_1 = s$, $\gamma_0(s, s_1, s) > 0$ and fair implied yields will exceed observed forward yields. Hence, for forward rate agreements with no deferment, the implied yield exceeds the observed forward yield. Second, averaging forward and futures contracts can be viewed as portfolios of deferred forward rate agreements. Hence, Proposition 1 also forms the basis for valuing more complex contracts. This is explored next.

Assume the period $[0,s]$ is partitioned into k time increments with $0 = t_0 < t_1 < \cdots < t_k = s$. Let $\bar{g}(0,s)$ be the fair implied forward yield at time 0 for a contract based on the accrued average, $A(t_1,t_k)$. Further, let $\bar{h}(0,s)$ be the fair implied futures yield at time 0 for a contract based on $A(t_1,t_k)$. Proposition 3 below prices these contracts.

Proposition 3

(i) A forward rate agreement on an arithmetic average is equivalent to a portfolio of deferred forward rate agreements on the appropriate yields. The implied forward yield on the average is

$$\bar{g}(0,s) = \sum_{i=1}^{k} \omega_i y_0(t_1,s_1) + \bar{\gamma} \tag{17}$$

where

$$\bar{\gamma} = \sum_{i=1}^{k} \omega_i \gamma_i$$

$$\gamma_i = \frac{\sigma^2}{4\kappa^3(s_1 - t_1)}\left(1 - e^{-\kappa(s_i-t_i)}\right)\left(1 - e^{-2\kappa t_i}\right)\left(2e^{-\kappa(s-t_i)} - e^{-\kappa(s_i-t_i)} - 1\right)$$

(ii) A futures contract on the arithmetic average is equivalent to a portfolio of deferred futures contracts on the appropriate yields. The implied futures yield on the average is

$$\bar{h}(0,s) = \bar{g}(0,s) + \bar{\chi} \tag{18}$$

where

$$\bar{\chi} = \sum_{i=1}^{k} \omega_i \chi_i$$

$$\chi_i = \frac{\sigma^2}{2\kappa^3(s_i - t_i)}(1 - e^{-\kappa(s_i - t_i)})e^{\kappa(s-t_i)} \left[2e^{-\kappa(s-t_i)} - 2e^{-\kappa s} - e^{-2\kappa(s-t_i)} + e^{-2\kappa s}\right]$$

Proof. See Appendix 3.

Equation (18) together with (17) shows that the 30-day futures rate, $\bar{h}(0,s)$, can be expressed as

$$\bar{h}(0,s) = \sum_{i=1}^{k} \omega_i y_0(t_i, s_i) + \bar{\gamma} + \bar{\chi} \tag{19}$$

That is, the futures price equals the weighted average of the relevant daily forward rates, together with two adjustments. The first reflects the fact that the arithmetic averages of yields are not tradable, the second reflects a total marking to market cost. Notice, too, that when the weights ω_i are all equal and when $s_i = t_{i+1}$ with $s = t_{k+1}$, then[11]

$$\sum_{i=1}^{k} \omega_i y_0(t_i, s_i) = \frac{1}{k} \sum_{i=1}^{k} y_0(t_i, s_i)$$

$$= \frac{1}{k} y_0(t_1, t_{k+1}) \tag{20}$$

In this case, $\bar{\gamma}$ and $\bar{\chi}$ represent corrections that must be made to the forward yield to maturity to obtain the fair implied yields.

5. PRICING THE 30-DAY INTEREST RATE FUTURES CONTRACT

The 30-day interest rate futures contract is used to illustrate the valuation procedure. The literature published by the Chicago Board of Trade on this contract claims that, prior to the settlement month, the futures price closely tracks the appropriate forward rate over the 30 days. This is a true statement, provided the adjustment terms in the above pricing equations are small. In light of Proposition 2, approximating the futures price using the observed forward rate causes a bias in one direction. This section investigates the potential of this bias.

Figures 1 and 2 show the dollar magnitudes of $\bar{\gamma}$ and $\bar{\chi}$ respectively, for a wide range of parameter values on the volatility structure. Observe from Figure 1 that the deviation between the observed forward yield and the implied forward yield is for the most part less than \$20. The marking to market costs in Figure 2, on the other hand, are somewhat larger in magnitude and often exceed \$50. However, the approximate error caused by ignoring these two terms is of a magnitude less than two basis points. Therefore, the yield implied by the 30-day interest rate futures contract should closely track the observed forward yield. Nevertheless, this approximation worsens as the time to expiration increases. To illustrate this, consider a 30-day interest rate futures contract with

[11]This corresponds to the 30-day interest rate futures contract.

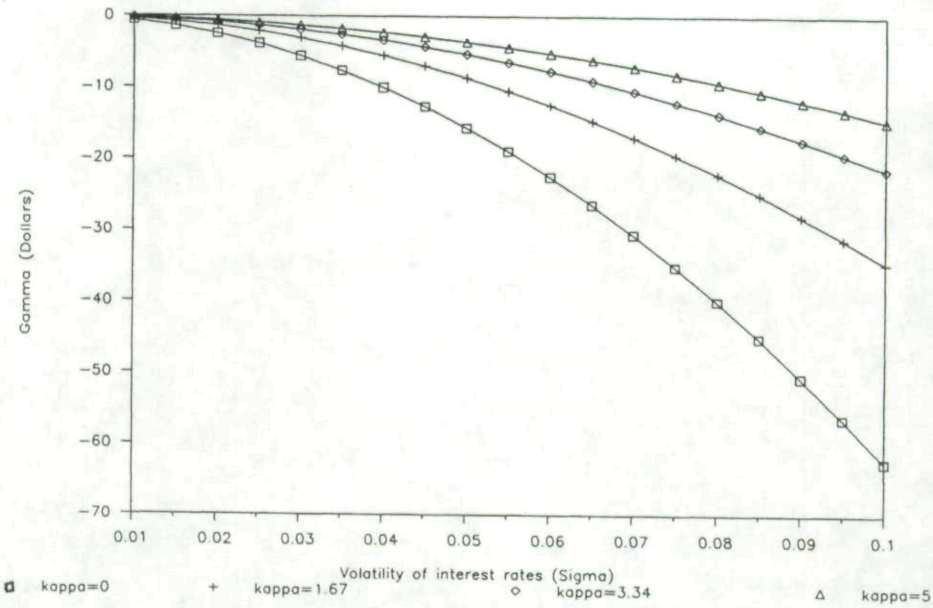

Figure 1
The 30-day interest rate contract.

24 months to expiration.[12] With $\sigma = 3\%$ and $\kappa = 0$, the error is $718, which represents more than 17 basis points. Clearly, as the maturity of the contract expands, so does the error in the approximation. This is illustrated below in Figure 3.

6. PRICING AVERAGING FORWARD RATE CONTRACTS

For an averaging yield forward contract, the deviation from the observed forward yield is given by $\bar{\gamma}$. The magnitude of this deviation depends critically on the maturities of the yields used, $s_i - t_i$, and on the intervals between consecutive averaging points, $t_{i+1} - t_i$. In general, these cannot be ignored. To illustrate this, consider a three-month forward contract ($t_1 - t_0$ =three months), where the terminal payout is based on the daily average of the six-month yields computed over ten consecutive days.[13] Let $\sigma = 0.03$ and let $\kappa = 0$. The notional principal is $5 million. For this case, the value of $\bar{\gamma}$ is $5725. Clearly, this of a magnitude that should not be ignored.

Figure 4 shows that $\bar{\gamma}$ is sensitive to the maturity of the yields comprising the average. Figure 4 shows that $\bar{\gamma}$ decreases as the interval between averaging points expands. Figure 5 also illustrates Proposition 2, namely that the zero value for $\bar{\gamma}$ occurs at a point which is independent of σ [see eq. (16)]. Figures 4 and 5 highlight the fact that the magnitude of the deviation between implied and observed yields varies dramatically according to the terms of the contract.

CONCLUSION

This article is concerned with the valuation of path dependent contracts that have payouts linked to the arithmetic mean of yields measured at finite points in time. Such contracts

[12] The Chicago Board of Trade has permission to trade 30-day contracts using as many as 24 consecutive calendar months.

[13] That is, $k = 10$ with $t_{i+1} - t_i = 1$ day.

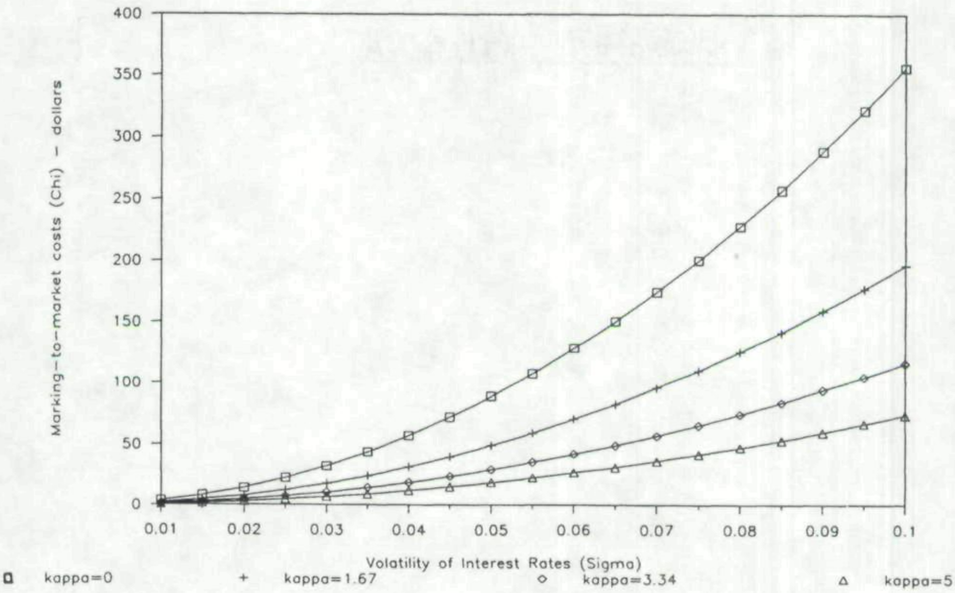

Figure 2
The 30-day interest rate contract.

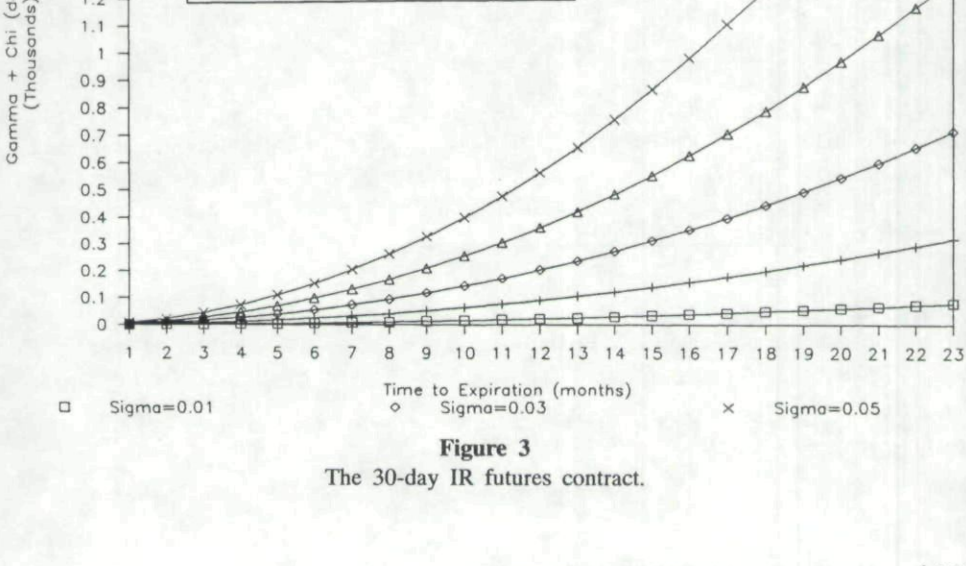

Figure 3
The 30-day IR futures contract.

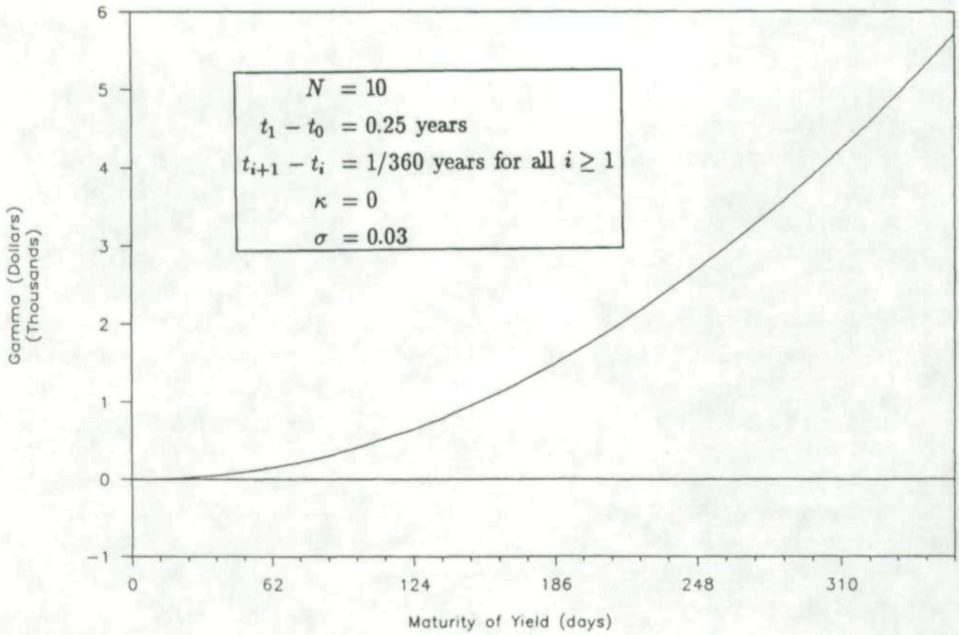

Figure 4
The impact of the yield maturity.

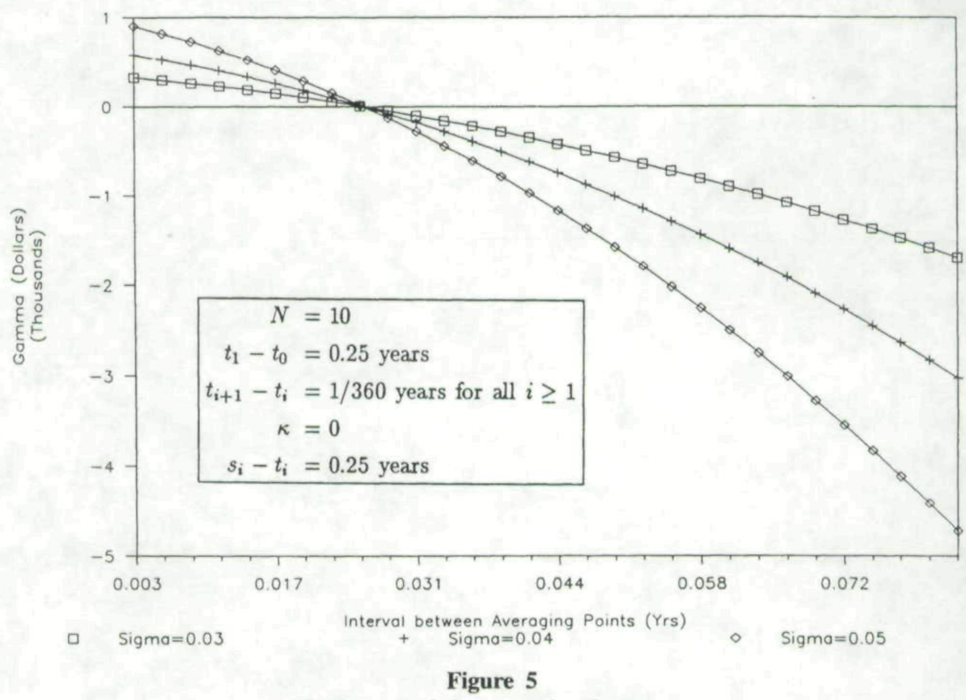

Figure 5
The impact of the averaging frequency.

have become increasingly common in recent years and include as a special case, the deferred payment contract. Their valuation is made complex because of path dependence issues and because the average yield is not a traded security. This article investigates how forward the futures contracts on averages can be computed. It is shown that the implied fair yields of such contracts can be obtained by appropriately adjusting observed forward yields to maturity. The magnitude of the adjustments are shown to depend critically on the type of averaging contract. The fair implied yield of a short-term 30-day interest rate futures contract is shown to be closely linked to the forward yield to maturity over the delivery month. However, when the time to delivery increases, the approximation weakens. Further, it is shown that in the over-the-counter markets, the fair implied yields can deviate significantly from their observed forward yields. In general, the magnitude of the deviations from the forward yields depends critically on the maturity of the yields averaged and the time points between yield measurements.

This article develops analytical pricing expressions for forward and futures contracts based on the average for the case where the volatility structure of forward rates is deterministic. A similar arbitrage-free model can be established when the volatility structure is stochastic and, in particular, when the volatility structure is linked to the level of the short rate. However, analytical solutions are not readily available for this more general framework.

Appendix 1

THE FORWARD RISK-ADJUSTED PROCESS

Let $G(0, s)$ be the forward price, at time 0, of a zero-copon claim with settlement date s, whose value at expiry, $g(s)$ could depend on the entire term structure at that time. Hence,

$$G(0, s) = \frac{g(0)}{P(0, s)} \tag{A1.1}$$

where $g(0)$ is the value of the underlying claim at time 0. Further, let

$$\frac{dg}{g} = \mu_g[f(t, T), t]\,dt + \sigma_g[f(t, T), t]\,dw(t)$$

Using Ito's division rule for the right hand side of (A1.1), and suppressing the arguments leads to

$$\begin{aligned}\frac{dG}{G} &= [(\mu_g - \mu_p) + \sigma_p(\sigma_g + \sigma_p)]\,dt + (\sigma_g + \sigma_p)\,dw \\ &= \mu_G\,dt + \sigma_G\,dw(t)\end{aligned} \tag{A1.2}$$

Using Ito's lemma directly for the left hand side of eq. (A1.1) one obtains

$$\frac{dG}{G} = \left[\frac{\frac{1}{2}\sigma_f^2(t, T)G_{ff} + \mu_f(t, T)G_f + G_t}{G}\right]dt + \left[\frac{\sigma_f(t, T)G_f}{G}\right]dw(t)$$

Now consider a portfolio of n_1 units of the claims valued at $g(t)$ and n_2 discount bonds with maturity date T. If n_1 and n_2 are selected such that the portfolio is riskless, it must earn the riskless rate of return to preclude arbitrage opportunities. This yields

$$n_1\mu_G + n_2\mu_p(t, T)P(t, T) = r(t)(n_2 P(t, T) + n_1 g(t))$$

$$n_1\sigma_G - n_2\sigma_p(t, T)P(t, T) = 0$$

In conjunction with eq. (7) this leads to

$$\frac{\mu_G - r(t)}{\sigma_G} = -\frac{\mu_p(t,T_2) - r(t)}{\sigma_p(t,T_2)} = -\lambda(t) \tag{A1.3}$$

Substituting for μ_H, σ_H from (A1.2), and using (A1.3) yields

$$(\mu_f(t,T) + \lambda(t)\sigma_f(t,T) - \sigma_f(t,T)\sigma_p(t,T))G_f + G_t + \frac{1}{2}\sigma_f^2(t,T)G_{ff} = 0$$

Now substituting eq. (9) for $\mu_f(t,s)$ into the above equation yields the following partial differential equation that all forward prices must satisfy to preclude arbitrage opportunities

$$\frac{1}{2}\sigma_f^2(t,T)G_{ff} + G_t = 0$$

with the terminal condition

$$G(s,s) = g(s)$$

The solution to this partial differential equation is given by

$$G(0,s) = \tilde{E}_0\{g(s)\}$$

where the expectation is taken under the *forward risk-adjusted process*

$$df(t,T) = \sigma_f(t,T)\,dw(t)$$

Notice that by setting $g(s) = P(s,T)$ the expected future bond price is given by the current forward price under this process. Further, the distribution of $r(s)$ under the forward risk-adjusted process is normal with its mean equal to the forward rate, $f(0,s)$, and variance equal to $\phi^2(s)$.

Appendix 2

THE BOND PRICING EQUATION

Integrating eq. (9a), one obtains

$$\begin{aligned} f(t,T) - f(0,T) &= \int_0^t \sigma_f(s,T)[\sigma_p(s,T) - \lambda(s)]\,ds + \int_0^t \sigma_f(s,T)\,dw(s) \\ &\equiv R(0,t;T) \end{aligned} \tag{A2.1}$$

Here, $R(0,t;T)$ is defined as the difference between the forward rate at date t and that at the original date.

Now assume that the volatility structure, $\sigma_f(s,T)$ can be expressed as[14]

$$\sigma_f(s,T) = h(s,T)k(t,T)$$

where $h(s,T)$ could depend on information at time s such as the current level of interest rates. On the other hand, $k(t,T)$ is a deterministic function with $k(t,t) = 1$. With this

[14]Clearly the exponentially dampened function belongs to this class. Here $h(s,t) = \sigma e^{-\kappa(t-s)}$ and $k(t,T) = e^{-\kappa(T-t)}$.

restriction, eq. (A2.1) can be rearranged as

$$\frac{R(0,t;T)}{k(t,T)} - \int_0^t h(s,t)\sigma_p(s,T)\,ds = -\int_0^t h(s,t)\lambda(s)\,ds + \int_0^y h(s,t)\,dw(s) \quad \text{(A2.2)}$$

Since the right hand side of eq. (A2.2) is independent of T, take $T = t$ to obtain

$$\frac{R(0,t;T)}{k(t,T)} - \int_0^t h(s,t)\sigma_p(s,T)\,ds = \frac{R(0,t;t)}{k(t,t)} - \int_0^t h(s,t)\sigma_p(s,t)\,ds$$

which simplifies to

$$\begin{aligned}
R(0,t;T) &= k(t,T)\left\{R(0,t;t) - \int_0^t h(s,t)[\sigma_p(s,t) - \sigma_p(s,T)]\,ds\right\} \\
&= k(t,T)\left\{R(0,t;t) + \int_0^t h(s,t)\int_t^T \sigma_f(s,x)\,dx\,ds\right\} \\
&= \frac{\sigma_f(t,T)}{\sigma_f(t,t)}\left\{R(0,t;t) + \int_0^t h^2(s,t)\,ds\int_t^T k(t,x)\,dx\right\} \\
&= \frac{\sigma_f(t,T)}{\sigma_f(t,t)}\left\{R(0,t;t) + \int_0^t \sigma_f^2(s,t)\,ds\int_t^T \frac{\sigma_f(t,x)}{\sigma_f(t,t)}\,dx\right\} \\
&= \sigma_f(t,T)\frac{R(0,t;t)}{\sigma_f(t,t)} + \frac{\sigma_f(t,T)}{\sigma_f^2(t,t)}\sigma_p(t,T)\phi^2(0,t;t)
\end{aligned}$$

where

$$\phi^2(0,t;t) = \int_0^t \sigma_f^2(s,t)\,ds$$

Equivalently,

$$f(t,T) = f(0,T) + \frac{\sigma_f(t,T)(f(t,t) - f(0,t))}{\sigma_f(t,t)} + \frac{\sigma_f(t,T)}{\sigma_f^2(t,t)}\sigma_p(t,T)\phi^2(0,t;t)$$

This leads to

$$f(t,T) - f(0,T) = \frac{\partial}{\partial T}\left[\sigma_p(t,T)\frac{R(0,t;t)}{\sigma_f(t,t)} + \frac{\phi^2(0,t;t)}{2\sigma_f^2(t,t)}\sigma_p^2(t,T)\right]$$

Further, from eq. (1)

$$P(t,T) = e^{-\int_t^T f(0,s)\,ds - \int_t^T \frac{\partial}{\partial x}\left[\sigma_p(t,x)\frac{R(0,t;t)}{\sigma(t)} + \frac{\phi^2(0,t;t)}{2\sigma_f^2(t,t)}\sigma_p^2(t,x)\right]dx}$$

which upon simplification yields the result. For more detail, please see Jamshidian (1987) and Ritchken and Sankarasubramanian (1991).

Appendix 3

PROOFS OF THE PROPOSITIONS

The following lemma is useful for the propositions.

Lemma

The stochastic interest rate at time t, $r(t)$, is related to forward rates that prevailed at an earlier time 0, and to an arbitrary stochastic forward rate at time t, $f(t,T)$. In particular, for $T > t$,

$$r(t) = e^{\kappa(T-t)}[f(t,T) - f(0,T)] + f(0,t) - \frac{\sigma^2}{2\kappa^2}[1 - e^{-\kappa(T-t)}][1 - e^{-2\kappa t}] \quad \text{(A3.0)}$$

Proof. From eq. (1)

$$f(t,T) = -\frac{\partial}{\partial T}\ln[P(t,T)] \quad \text{(A3.1)}$$

Take the derivative of $P(t,T)$ in eq. (11) with respect to T to obtain

$$f(t,T) = -\frac{\partial}{\partial T}\ln[P(0,T)] - \frac{\partial}{\partial T}\left[\beta(t,T)f(0,t) - \frac{1}{2}\beta^2(t,T)\phi^2(t)\right] + \frac{\partial}{\partial T}[\beta(t,T)]r(t) \quad \text{(A3.2)}$$

Now differentiate eq. (11c) with respect to T to obtain

$$\frac{\partial}{\partial T}\beta(t,T) = e^{-\kappa(T-t)} \quad \text{(A3.3)}$$

Further, from eq. (A3.1)

$$f(0,T) = -\frac{\partial}{\partial T}\ln[P(0,T)] \quad \text{(A3.4)}$$

Substitute eqs. (A3.3) and (A3.4) in eq. (A3.2) to obtain

$$f(t,T) = f(0,T) + e^{-\kappa(T-t)}(r(t) - f(0,t) + \beta(t,T)\phi^2(t)) \quad \text{(A3.5)}$$

Rearrange eq. (A3.5) above and substitute for $\beta(t,T)$ and $\phi^2(t)$ from eqs. (11c) and (11d) to complete the proof. □

Now the two propositions can be proved.

Proof of Proposition 1

(i) From eq. (10), under the forward risk-adjusted process, for a claim that has a settlement date s,

$$f(t_1,s) = f(0,s) + \int_0^{t_1} \sigma_f(x,s)\,dw(x) \quad \text{(A3.6)}$$

Substitute eq. (A3.6) in eq. (A3.0) with $t = t_1$ and $T = s$ to obtain

$$r(t_1) = e^{\kappa(s-t_1)}\int_0^{t_1} \sigma_f(x,s)\,dw(x) + f(0,t_1) - \frac{\sigma^2}{2\kappa^2}[1 - e^{-\kappa(s-t_1)}][1 - e^{-2\kappa t_1}] \quad \text{(A3.7)}$$

From eq. (2),

$$y_{t_1}(t_1, s_1) = -\frac{1}{s_1 - t_1} \ln [P(t_1, s_1)] \qquad \text{(A3.8)}$$

Substitute for $P(t_1, s_1)$ using eq. (11) and further substitute for $r(t_1)$ using eq. (A3.7) to yield

$$\begin{aligned} y_{t_1}(t_1, s_1)(s_1 - t_1) = &- \ln [P(0, s_1)] + \ln [P(0, t_1)] + \frac{1}{2}\beta^2(t_1, s_1)\phi^2(t_1) \\ &+ \beta(t_1, s_1)\left\{ e^{\kappa(s-t_1)} \int_0^{t_1} \sigma_f(x, s)\, dw(x) \right. \\ &\left. - \frac{\sigma^2}{2\kappa^2}[1 - e^{-\kappa(s-t_1)}][1 - e^{-2\kappa t_1}] \right\} \end{aligned} \qquad \text{(A3.9)}$$

From eq. (10c) it is known that the implied forward yield, $g(0, s)$ on $y_{t_1}(t_1, s_1)$ is the expectation of the yield $y_{t_1}(t_1, s_1)$ under the forward risk-adjusted process. This is obtained by taking the expectation of eq. (A3.9). This yields

$$\begin{aligned} g(0, s)(s_1 - t_1) = &- \ln [P(0, s_1)] + \ln [P(0, t_1)] + \frac{1}{2}\beta^2(t_1, s_1)\phi^2(t_1) \\ &- \beta(t_1, s_1)\left\{ \frac{\sigma^2}{2\kappa^2}[1 - e^{-\kappa(s-t_1)}][1 - e^{-2\kappa t_1}] \right\} \end{aligned} \qquad \text{(A3.10)}$$

Substitute for $\beta(t_1, s_1)$ and $\phi^2(t_1)$ using eqs. (11c) and (11d) and rearrange terms to obtain the result.

(ii) To obtain the implied futures yield, one must take the expectation under the *risk-neutralized process* detailed in eq. (10a). Under this process, eq. (A3.6) must be modified to yield

$$f(t_1, s) = f(0, s) + \int_0^{t_1} \sigma_f(x, s)\sigma_p(x, s)\, dx + \int_0^{t_1} \sigma_f(x, s)\, dw(x) \qquad \text{(A3.11)}$$

Substitute from the exponentially dampened volatility structure and evaluate the first integral to obtain

$$\begin{aligned} f(t_1, s) = &f(0, s) + \frac{\sigma^2}{2\kappa^2}[2e^{-\kappa(s-t_1)} - 2e^{-\kappa s} - e^{-2\kappa(s-t_1)} + e^{-2\kappa s}] \\ &+ \int_0^{t_1} \sigma_f(x, s)\, dw(x) \end{aligned} \qquad \text{(A3.12)}$$

Using (A3.12) instead of (A3.6) in the proof of part (i) yields the desired result.□

Proof of Proposition 2

(i) From eq. (14), $g(0, s)$ exceeds $y_0(t_1, s_1)$ provided $\gamma_0(t_1, s_1, s)$ is positive. Using the definition of $\gamma_0(t_1, s_1, s)$ and rearranging terms leads to the result.

(ii) From eqs. (14) and (15),

$$\gamma_0(t_1, s_1, s) + \chi_0(t_1, s_1, s) = \frac{\sigma^2(1 - e^{-\kappa(s_1-t_1)})}{4\kappa^3(s_1 - t_1)}$$

$$\cdot \left[(1 - e^{-2\kappa t_1})(2e^{-\kappa(s-t_1)} - e^{-\kappa(s_1-t_1)} - 1) + 2e^{\kappa(s-t_1)}(2e^{-\kappa(s-t_1)} - 2e^{-\kappa s} - e^{-2\kappa(s-t_1)} + e^{-2\kappa s}) \right]$$

This simplifies to

$$\gamma_0(t_1, s_1, s) + \chi_0(t_1, s_1, s) = \frac{\sigma^2}{4\kappa^3(s_1 - t_1)}(1 - e^{-\kappa(s_1-t_1)}) \cdot (1 - e^{-\kappa t_1})[3 - e^{-\kappa t_1} + e^{-\kappa(s_1+t_1)}]$$

The above term is positive. □

Proof of Proposition 3

(i) First consider the distribution of $\sum_{i=1}^{k} \omega_i y_{t_i}(t_i, s_i)$ under the forward risk-adjusted process. From eq. (A3.9) one obtains

$$\begin{aligned} y_{t_i}(t_i, s_i)(s_i - t_i) = &- \ln[P(0, s_i)] + \ln[P(0, t_i)] + \frac{1}{2}\beta^2(t_i, s_i)\phi^2(t_i) \\ &+ \beta(t_i, s_i)\left\{ e^{\kappa(s-t_i)} \int_0^{t_i} \sigma_f(x, s)\, dw(x) \right. \\ &\left. - \frac{\sigma^2}{2\kappa^2}[1 - e^{-\kappa(s-t)}][1 - e^{-2\kappa t}] \right\} \end{aligned} \tag{A3.13}$$

Divide through by $(s_i - t_i)$ and use eq. (1a) to obtain

$$\begin{aligned} y_{t_i}(t_i, s_i) = y_0(t_i, s_i) &+ \frac{1}{2(s_i - t_i)}\beta^2(t_i, s_i)\phi^2(t_i) \\ &+ \frac{\beta(t_i, s_i)}{s_i - t_i}\left\{ e^{\kappa(s-t_i)} \int_0^{t_i} \sigma_f(x, s)\, dw(x) \right. \\ &\left. - \frac{\sigma^2}{2\kappa^2}[1 - e^{-\kappa(s-t_i)}][1 - e^{-2\kappa t_i}] \right\} \end{aligned} \tag{A3.14}$$

Multiply by ω_i, sum across all i and take expectations to obtain the result.

(ii) The proof follows in a similar manner to (i) above, but this time eq. (A3.13) is replaced by its analog under the risk-neutralized process as in the proof of Proposition 1. □

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