Optimal Hedging with Futures Contracts: The Case for Fixed-Income Portfolios

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INTRODUCTION

This study applies the principles of continuous time finance to optimal hedging of portfolios of fixed-income securities. It makes use of specific properties of interest rate–sensitive securities which are not generally available for other commodities or futures contracts. The contribution of this article to the hedging and speculation literature is a methodological one. The geometry of the opportunity set available to the participant in the market for interest rate–sensitive securities is explored when interest rates follow a diffusion process with drift. The results are used to calculate the optimal hedging strategy for hedging agents with finite risk aversion.

Standard hedging methods include the minimum variance approach, regression models, the yield sensitivity, and the duration techniques.[1] Because of their intuitive appeal and obvious computational advantage, variance minimizations (or the equivalent price regression technique) have become standard hedging methods used by practitioners to determine desired interest rate futures positions. However, recognizing that most often the economic agents are utility maximizers rather than risk minimizers, several authors such as Black (1976), Anderson and Danthine (1980, 1981), Johnson (1981), and Nelson and Collins (1985), take a risk–return portfolio approach. Howard and D'Antonio (1984) develop a general measure of hedging effectiveness in a risk–return framework. The applicability of the hedging effectiveness measure is conditioned by knowledge of the opportunity set created by the spot–futures portfolio. Little work has been conducted to date regarding the risk–return tradeoffs available to the hedger of interest rate risk.

This article builds upon previous literature by exploring the computational implications of the properties in continuous time of securities sensitive to interest rates. In particular, the parameters which affect the instantaneous geometry of the opportunity set are examined. It concentrates on the specifics of spot and futures instruments in the

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[1]Important studies advancing methods accepted by practitioners include Johnson (1960); Stein (1961); Makin (1978); Ederington (1979); McEnally and Rice (1979); Gay, Kolb, and Chiang (1983); Hilliard (1984); and Hilliard and Jordan (1989).

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markets for fixed-income securities. As a tradeoff, the additional constraints provide a richer environment, which allows for stronger results than those obtainable in the framework of the general portfolio theory. A *risk characteristic equation* describing the efficient set when interest rates follow a correlated diffusion process is derived and used to calculate the optimal interest rate futures hedge as a function of the agent's subjective rate of substitution.

Interest rate futures contracts are often viewed as risky instruments due to their ability to create highly leveraged positions. At the same time they make good hedging instruments by allowing the holder of a position in the spot bond market to take offsetting positions in the futures markets and shift undesired interest rate risk from the hedger to the speculator. Hedging and speculation are most often easy to differentiate. A hedger is a participant in the futures market who maintains a vested interest in the underlying spot commodity; whereas, a speculator does not have an interest in the spot instrument, but hopes to capitalize on mispricing of contracts. The two types of market participation are harder to differentiate in the interest rate futures markets than in the commodity futures markets. The cash margin maintained by the speculator with interest rate futures is a position in spot fixed-income instruments which qualifies him as a hedger. On the other hand, the hedger who deviates from the risk minimization position by underhedging or overhedging is speculating. In the interest rate futures market, both hedger and speculator maintain positions in both spot and futures instruments; and, although semantically distinct, the separation between hedging and speculation is fuzzy. Transition between the two types of market participation occurs gradually as the agent changes the degree of leverage by varying the number of contracts relative to the spot position.

Risk/expected-return tradeoffs are of interest when controlling risk and return. This article examines the risk-return opportunity set and describes the implications for hedging portfolios of fixed-income securities by investors with finite and infinite risk aversion. Sufficient limiting conditions are found for the minimum volatility portfolio to be optimal. The results apply both to interest rate forward and futures contracts.

The following section presents the economic environment, the model, and notation used in this study. The relationship between volatility and instantaneous portfolio returns when interest rates follow diffusion processes with drift is described. The result is used to determine the interest rate futures position which achieves a certain target volatility of returns. The next section describes the geometry of the opportunity set and is followed by a section that presents the implications for calculating the optimal ratio for hedging and speculation with interest rate futures.

THE MODEL

Assumptions and Notation

Consider a portfolio consisting of fixed-income securities and interest rate forward or futures contracts. Futures are preferred in the exposition for being more liquid and more widely used. Denote by n the number of possible maturities. (A continuous term structure of interest rates can be represented by choosing n sufficiently large.) For ease of notation, assume that only one futures contract is included. The futures contract is indexed by $n + 1$. The results can be generalized for multiple futures with no conceptual difficulty. The equilibrium value of the portfolio

$V(r_1, r_2, \ldots, r_n, r_{n+1} = r_F, t)$ is a function of the applicable interest rates, the bond holdings, and the contractual interest rate implied by the futures (forward) contract. The change in value of the portfolio V is determined by the change of rates, which are assumed to follow continuous time correlated diffusion processes of the form:

$$dr_i = f_i dt + \sigma_i dz^i, \qquad i = 1, \ldots, n + 1 \tag{1}$$

where dz^i is the variation of a standard Wiener process (thus a normal random variable with mean zero and distribution dt); f_i and σ_i represent the drift; and, respectively, the volatility factors of the rate diffusion processes and the covariances of the underlying standard Wiener processes are of the form $\operatorname{cov}(dz^i, dz^j) = \rho_{ij} dt$, and describe the co-movement of rates of different maturities and contract rate, r_F. An important issue for the bond hedger is the choice of hedging contracts. Most often the instrument to be hedged is not deliverable under any contract. Cross hedging is the only alternative. The basis (difference between the futures and spot price) is an important source of uncertainty which must be controlled also. Hedging with interest rate futures is trading interest rate risk for basis risk; and, therefore, the choice of the appropriate hedging contract is important.

Continuous-time hedging strategies create state-contingent payoffs, where the state variables are represented by spot and futures prices [e.g., Breeden (1984) or Ho (1984)]. Adler and Detemple (1988) find the optimal hedge when, in addition to a single cash position, investors can hold freely shortable traded assets also. In that case, if commodity futures and cash prices are less than perfectly correlated, the only implementable strategy is the minimum variance strategy, which they show, does not necessarily maximize expected utility.

The price of a bond, P_i, the applicable interest rate, r_i, and the bond's promised cash flow rate, c_i, are related at time t through the relationship:

$$P_i(t) = \int_t^\infty c_i(s) e^{-r_i(s-t)} ds$$

where $c_i(s)$ represents the cash flow rate at time s.[2] Alternatively, the contractual rate implied by the futures contract is the solution to the implicit equation in r_F:

$$F_{tT} = \int_T^\infty c_F(s) e^{-r_F(s-T)} ds$$

where F_{tT} is the price of the futures contract at time t, with a settlement date T; $c_F(s)$ represents the cash flow of the cheapest-to-deliver from settlement date on. The price/interest rate relationship allows one to use implied rates as state variables which provides more robust estimators than prices alone.

Using the stochastic calculus rules for variance and covariance, one can easily verify that:

$$\operatorname{Var}(dr_i) = \sigma_i^2 dt, \qquad i = 1, \ldots, n + 1$$

$$\operatorname{Cov}(dr_i, dr_j) = \rho_{ij} \sigma_i \sigma_j dt, \qquad i, j = 1, \ldots, n + 1$$

[2]The discrete time payment can be treated conveniently also in this context, by writing the cash flow as a weighted function of the Dirac's density function centered around the time of payments, namely:

$$c(s) = \sum_{k=1}^{\infty} \operatorname{cash}(t_k) \delta(s - t_k)$$

where δ is the generalized distribution function with $\delta(0) = \infty$, $\delta(s) = 0$ for $s \neq 0$, and $\int_{-\infty}^{\infty} \delta(s)\, ds = 1$.

The coefficients, $\rho_{i,n+1} = \rho_{iF}$, describe the co-movement of the implied yields and the contractual yield implied by the futures contract as described by their correlation.

Prevailing interest rates must be compatible with feasible term structures of interest rates, and positive rate correlations must exist. The question as to what primary factors span the elements of the variance–covariance matrix of rates is interesting in itself and warrants separate investigation. Protection against particular shapes and types of changes in the term structure of interest rates is provided by immunization strategies. A portfolio is said to be immunized against a certain type of interest rate change if the value at the end of the holding period is guaranteed to be at least as large as a target value, should interest rate changes against which immunization is provided occur. (The target value is defined as the final portfolio value if interest rates do not change.)[3] While immunization techniques are effective in providing protection against a specific type of interest rate changes, the payoff is unpredictable when other types of term structures occur. The relative advantage of using historical data for the variance–covariance matrix is that estimates are compatible with revealed shapes of term structures and with the occurrence probability of changes. The present model is not a term structure model, but it is consistent with prevailing term structures of interest rates. The pragmatic advantage of historical estimation is that no feasible term structure is ruled out by a priori model considerations.

The Risk Characteristic Equation

Consider a portfolio, $V(t)$, formed of n bonds with prices $P_i(t)$, $i = 1, \ldots, n$, with distinct maturities, and a position in the futures contract F. Denote by h_t the futures position at time t; ($h_t > 0$ implies a long position and $h_t < 0$ describes a short position). The cash value of the futures position is zero at the time the contract is bought or sold, and the total value of the portfolio is equal to the value of the spot instruments. As interest rates change, gains and losses are incurred both in the spot and futures markets, and the change in value of the portfolio can be calculated using Ito's lemma for the function $V(r_1, \ldots, r_n, r_F, t)$:

$$dV = \left(\frac{\partial V}{\partial t} + \sum_{i=1}^{n+1} \frac{\partial V}{\partial r_i} f_i + \frac{1}{2} \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} \frac{\partial^2 V}{\partial r_i \partial r_j} \sigma_i \sigma_j \rho_{ij} \right) dt + \sum_{i=1}^{n+1} \frac{\partial V}{\partial r_i} \sigma_i dz^i \quad (2)$$

Taking the partial derivative involved in (2), one finds that:

$$dV = \left[\sum_{i=1}^{n} (r_i P_i - c_i) - \sum_{i=1}^{n} D_i P_i f_i - h D_F F f_F + \frac{1}{2} \sum_{i=1}^{n} H_i P_i \sigma_i^2 + \frac{1}{2} h H_F F \sigma_F^2 \right] dt + \sum_{i=1}^{n} D_i P_i \sigma_i dz^i + h D_F F f_F \sigma_F dz^F \quad (3)$$

where D_i and H_i represent the first and second derivatives of the bond price P_i. The first derivative is:

$$D_i(t) = \frac{1}{P_i(t)} \int_t^{\infty} (s - t) e^{-r_i(s-t)} c_i(s) ds$$

[3]Immunization techniques have been advanced for parallel term structure shifts by Fisher and Weil (1971), for multiplicative interest rate shocks by Bierwag (1977), and for term structure shifts with bounded concavity change by Fong and Vasicek (1984).

which also represents the Macaulay duration of the bond defined as average time to repayment weighted by the present value of the cash flows. Cooper (1977) proposed an alternative duration measure defined as the weighted average of the squared time to repayment, which he showed to measure the sensitivity of bonds to changes in the shape of term structure of interest rates. The definition of the Cooper duration for bonds in present notation is:

$$H_i(t) = \frac{1}{P_i(t)} \int_t^\infty (s - t)^2 e^{-r_i(s-t)} c_i(s) ds$$

and coincides with the second-order derivative of the price with respect to interest rates. (For the discrete time case, the weighted averages involved in the above expressions are calculated with summations instead of integral.) For futures contracts, both types of duration can be extended in a natural way as the duration of the underlying representative deliverable instrument.

Using eq. (3), it follows that the instantaneous return of the portfolio over an infinitesimal time period dt is:

$$R_P(dt) = \frac{\sum_{i=1}^{n} c_i dt + dV}{V} = E_P dt + \tilde{\eta}$$

where

$$E_P = \frac{1}{V}\left[\sum_{i=1}^{n} r_i P_i - \sum_{i=1}^{n} D_i P_i f_i - hD_F F f_F + \frac{1}{2}\sum_{i=1}^{n} H_i P_i \sigma_i^2 + \frac{1}{2} hH_F F \sigma_F^2\right] \tag{4}$$

and the random component of the return is:

$$\tilde{\eta} = \frac{1}{V}\left[\sum_{i=1}^{n} D_i P_i \sigma_i dz^i + hD_F F \sigma_F dz^F\right]$$

The variance $\sigma_P^2 = \mathrm{Var}(\tilde{\eta})$ of the portfolio return is:

$$\sigma_P^2 = \frac{1}{V^2}\left[\sum_{i=1}^{n}\sum_{j=1}^{n} D_i P_i D_j P_j \sigma_i \sigma_j \operatorname{cov}(dz^i, dz^j) + 2hD_F F \sigma_F \sum_{i=1}^{n} D_i P_i \sigma_i \operatorname{cov}(dz^i, dz^F) + h^2 D_F^2 F^2 \sigma_F^2\right] \tag{5}$$

and, thus, the instantaneous return can be written as:

$$R_P(dt) = E_P(dt) + \sigma_P dw$$

where dw is a drawing from a normal distribution of mean zero and variance dt.

Making the notation:

$$a = D_F F \sigma_F, \qquad b = \sum_{i=1}^{n} D_i P_i \rho_{iF}, \qquad c = \sum_{i=1}^{n}\sum_{j=1}^{n} D_i P_i D_j P_j \sigma_i \sigma_j \rho_{ij} \tag{6}$$

in (5), the following equation is obtained, which is called the *risk characteristic equation* because it describes the relation between the futures position and portfolio risk.

$$a^2h^2 + 2hab + c - \sigma_P^2 V^2 = 0 \tag{7}$$

Equation (4) describes the portfolio expected instantaneous return as a function of the futures position and the drift of interest rates. In an environment of increasing rates, the long position in the spot market decreases the expected portfolio return. The decrease is proportional to the magnitude of the trend f_i and the bond elasticity D_i. The contribution to the change in value is preceded by a minus sign denoting that an increase in interest rates is accompanied by a price decrease. The contribution of the futures position depends on the futures contract price F, the number of futures assumed, and the contract's sensitivity to interest rates as measured by its duration D_F. Since, for hedging a long cash position, one usually assumes a short position in the futures market ($h < 0$), an increase in interest rates ($f_F > 0$) increases the expected return, which partially offsets losses incurred in the spot position.

The volatility of instantaneous return is described in eq. (5). It depends on the volatility σ_i, the duration of the instruments involved, and on the futures position. The drift terms, f_i, cancel out in the expression of the volatility of unexpected changes in return. The futures position which minimizes the instantaneous volatility of returns can be found by taking the derivative of σ_P^2 with respect to h in eq. (5) and setting it equal to zero. If follows that:

$$h_{\text{minvol}} = -\frac{\sum_{i=1}^{n} D_i P_i \sigma_i \rho_{iF}}{D_F F \sigma_F} = -\frac{b}{a} \tag{8}$$

The minimum volatility hedge position depends on the durations, D_i, the holdings, P_i, and the rate correlations, ρ_{iF}. These provide a measure of hedging effectiveness of the futures contract with respect to each of the maturities comprising the portfolio. The minimum volatility position is maturity-additive; that is, the minimum volatility futures position of a portfolio is equal to the summation of the minimum-volatility positions for the underlying maturities.

The Futures Position for Target Volatilities

The risk characteristic equation (7) describes the relationship between the volatility of returns and the interest rate futures position. One can determine the futures positions which achieve a certain target volatility σ_{target} by solving for h:

$$h_{1,2} = \frac{-b + \text{sgn}\sqrt{\Delta}}{a} \tag{9}$$

where $\Delta = \sigma_{\text{target}}^2 V^2 + b^2 - c$ and the variable sgn can be plus or minus one. Any target volatility for which the discriminant Δ is strictly positive can be reached by two possible futures positions, one for sgn $= +1$ and one with sgn $= -1$. A risk position different from the minimum risk position can be achieved either by overhedging or by underhedging. From eqs. (8) and (9) it follows that futures contract positions for the target level of risk σ_{target}^2 can be rewritten as a function of the minimum volatility position as:

$$h_{1,2} = h_{\text{minvol}} + \text{sgn}\,\frac{\sqrt{\Delta}}{D_F F \sigma_F} \tag{10}$$

The minimum attainable volatility is for $\Delta = 0$, in which case the minimum volatility value is:

$$\sigma^2_{\text{minimum}} = \frac{1}{V^2}(c - b^2) \tag{11}$$

To find which of the two solutions in eq. (9) is mean–variance efficient, further analysis of the opportunity set is necessary. The risk–return tradeoffs available to the investor in bonds and interest rate futures is described in the next section.

THE GEOMETRY OF THE OPPORTUNITY SET

Theorem (risk-return tradeoff). The opportunity rate of substitution between portfolio return and portfolio volatility as the number of futures is varied around the position of length h, is

$$\text{MRS}^E_\sigma = \text{sgn}\frac{\sigma_P V}{\sqrt{\Delta}}\, d \tag{12}$$

where

$$d = \frac{1/2\, H_F F \sigma_F^2 - D_F F f_F}{D_F F \sigma_F} \tag{13}$$

Proof. The marginal rate of substitution between risk and return when h is varied is:

$$\frac{\partial E}{\partial \sigma_P} = \frac{\partial E_P}{\partial h}\frac{\partial h}{\partial \sigma_P} \tag{14}$$

Derivating E_P with respect to h in eq. (4),

$$\frac{\partial E_P}{\partial h} = \frac{1}{V}\left[\frac{1}{2} H_F F \sigma_F^2 - D_F F f_F\right] \tag{15}$$

Alternatively, taking the derivative of h with respect to σ_P in eq. (10),

$$\frac{\partial h}{\partial \sigma_P} = \text{sgn}\frac{\sigma_P V^2}{D_F F \sigma_F \sqrt{\Delta}} \tag{16}$$

Substituting (15) and (16) into the right-hand side of eq. (14) concludes the proof.

Corollary. The contour of the opportunity set available to an agent hedging with interest rate futures has the following two properties:

$$\text{(i)} \quad \left.\frac{\partial E_P}{\partial \sigma_P}\right|_{\sigma_{\text{minvol}}} = +\infty$$

$$\text{(ii)} \quad \lim_{\sigma \to \infty} \text{MRS}^E_\sigma = \text{sgn}\, d$$

Now it is possible to establish the value of the variable sgn. For the position to be on the efficient set, the rate of substitution must be positive (increase return by accepting additional risk), and thus sgn and d are of the same sign. The condition to move along the efficient set by varying the futures position is:

$$\text{sgn} \times \left[\frac{1}{2}\frac{H_F \sigma_F}{D_F} - \frac{f_F}{\sigma_F}\right] > 0 \tag{17}$$

If futures rates are expected in increase, ($f_F > 0.5 H_F \sigma_F^2 / D_F \geq 0$), futures prices are expected to decrease, and thus the futures position which achieves the target volatility and is on the efficient set is obtained by overhedging (sgn = −1) in eq. (10). If futures rates are expected to decrease, ($f_F < 0 \leq 0.5 H_F \sigma_F^2 / D_F$), the optimal futures position is obtained by underhedging relative to the minimum volatility hedge ratio.

OPTIMAL HEDGING FOR FINITE RISK AVERSION

Like investors, hedgers are utility maximizers and optimal positions must consider tradeoffs of risk and returns. Speculators and hedgers alike, maximize expected return on a risk-adjusted basis. Cecchetti, Cumby and Figlewski (1988) maximize expected utility. Briys, Crouhy, and Schlesinger (1989) find the optimal hedge position under intertemporally dependent preferences in a utility maximization framework. Utility maximization is attractive in a theoretical sense, and its use is contingent upon the availability of the utility function. The optimality condition is reached when the risk–return objectives of the hedge are met.

While hedging by sole proprietorship or closely held partnerships is motivated by utility maximization, the question as to why do institutions hedge is interesting also.[4] The literature explains observed institutional hedging by ownership structure [Stoll (1979)], taxes, bankruptcy costs, and contracting costs [Smith and Stulz (1985)], and by the role of management compensation contracts [Stulz (1984)]. In the present model, the marginal rate of substitution between risk (as measured by return volatility) and expected return are exogenous. This allows applicability for the institutional hedger who, due to the ownership structure of the firm, does not have a utility function. However, while outside of the scope of this article, it is possible to derive the marginal rate of substitution λ between risk and return as part of the policy of a corporation with multiple owners with individual utility functions.[5] Such an equilibrium model is both interesting and possible in a world with managerial risk aversion and imperfect markets where hedging can have a value-maximizing effect as described by Smith and Stulz (1985).

Based on the results derived in the previous sections it is possible to control risk and return along the efficient frontier. This can be accomplished by deviating from the minimum volatility position by adding or subtracting futures depending on the value of sgn. It is desirable to increase the expected return at the expense of volatility as long as the opportunity rate of substitution between return and risk is greater than the agent's marginal rate of substitution. The optimality condition is reached when the subjective and the opportunity rate of substitution between risk and return are equal. Denote by λ the marginal rate of substitution between risk and return for the hedging agent. The *optimality condition,* $\text{MRS}_\sigma^E = \lambda$, is:

$$\text{sgn}\,\frac{\sigma_{\text{target}} V}{\sqrt{\Delta}}\, d = \lambda$$

[4]In a world of perfect capital markets in which perfect and cost-free diversification is possible, all MRS' must be equal, in which case the problem is not interesting as hedging becomes undesirable. Baron (1976) explains hedging as a byproduct of market imperfections by showing that in a world of perfect capital markets, where shares in the production or storage process are freely traded, there is no need to hedge.

[5]For a related model see Feder, Just, and Schmitz (1980), who explore the effect of expected utility maximization on institutional production in the presence of futures markets and price uncertainty.

To determine the optimal target volatility, square both sides of the above equation, explicate Δ as defined in eq. (9), and solve for σ_{optimal}:

$$\sigma^2_{\text{optimal}} = \frac{\lambda^2(c - b^2)}{V^2(\lambda^2 - d^2)}$$

substituting it in Δ of eq. (9), one finds the discriminant for the optimal volatility level to be:

$$\Delta_{\text{optimal}} = \frac{d^2(c - b^2)}{\lambda^2 - d^2}$$

Using eq. (10), it follows that the *optimal hedge position* for the investor with risk aversion λ is:

$$h_{\text{optimal}} = \frac{-b}{a} + (\text{sgn})\frac{d}{D_F F \sigma_F}\sqrt{\frac{c - b^2}{\lambda^2 - d^2}} \tag{18}$$

where a, b, and c are defined in (6); d is defined in eq. (13); and sgn $= +1$ if $f_F \leq 0.5H_F\sigma_F^2/D_F$; and -1 if $f_F \geq 0.5H_F\sigma_F^2/D_F$. The numerator, $c - b^2$, of the fraction under the squared root is positive (since, according to (11), it is equal to $\sigma^2_{\text{minvol}}V^2$), and thus, the expression under the squared root in eq. (18) is positive for $\lambda \geq |d|$.

An optimal hedge ratio exists for any risk-aversion factor greater than the asymptotical value of the opportunity rate of substitution between expectation and standard deviation of instantaneous returns. The decomposition of the optimal hedge in eq. (18) should be interpreted independently from the existence of a hedging agent from whom the minimum volatility is optimal. In fact, as known from the seminal work of Arrow (1964), on decision under uncertainty, no economic agent invests solely in the minimum variance portfolio, and the minimum volatility position is thus a limiting optimal position when risk aversion increases unboundedly.

The optimal futures position described in eq. (18) explicates the optimal futures position for a hedging agent participating in the fixed-income securities markets. The optimal position depends on the sensitivity to interest rate risk as described by the Macaulay and Cooper durations of the underlying instruments.[6]

The length of the speculative component depends on the investor's risk aversion. For infinite risk aversion ($\lambda = +\infty$), the speculative component is zero and the minimum volatility position is optimal. For λ, finite but large ($\infty > \lambda \gg |d|$), the speculative component has a value different from zero, and the adjustment factor can be on either side of the market (long or short), depending on the value of the variable sgn. The absolute value of the speculative position depends on the trend and on risk aversion. When the risk aversion is relatively small ($\lambda = |d| + \epsilon$) the speculative component is significant and can exceed the minimum volatility position. In that situation, if the adjustment factor is on the other side of the market than the

[6]For example, consider the hedging with T-bond futures on May 31, 1983 of the T-bond issued in June 1960 with a coupon of 8.785% maturing in May 1986. The involved futures contract expired in September 1983. Monthly data covering the period February 1982 through May 1983 from the Center for Research in Securities Prices (CRSP) government bond file produces the following historical estimates: $\sigma_B^2 = 0.5464 * 10^{-4}$, $\sigma_F^2 = 0.2494 * 10^{-4}$, and $\rho_{BF} = 0.889$. The duration of the bond is $D_B = 2.73$ years and the price, B, is 94.25% of the face value. The futures contract duration is $D_F = 13.5$ years. Using an Auto-Regressive Conditional Heteroskedastic (ARCH) model of interest rates, the trend factor f_F is estimated to be 0.015 on the date of the hedge. The invoice price implied by the futures price is $F = 74{,}062.50$.

minimum volatility position, the hedger actually behaves as a speculator. If the expected interest rate trend, f_F, is significantly different from zero, and the risk aversion parameter is such that the denominator in the adjustment factor is small, it is possible for the sign of the adjustment factor to override that of the hedging component and the resulting futures position to be on the same side of the market as the spot position.

CONCLUSIONS

The model presented in this article explicitly accounts for interest rate risk, basis risk, the dynamics of portfolio sensitivity to interest changes, and intertemporal investor's attitude toward risk. Assuming interest rates to follow a diffusion process with drift, the instantaneous geometry of the risk–return opportunity set for the case of a portfolio of fixed-income securities hedged with interest rate futures is examined. Sufficient conditions for optimality of the minimum volatility portfolio are found. The model has a methodological importance in that it explicates the parameter calculation for the optimal position in continuous time hedging with interest rate futures. The method specifically accounts for the sensitivity of the portfolio to interest rate changes as measured by the Macaulay and Cooper durations and for the agent's risk aversion.

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