A Note on the Effect of No-Arbitrage Conditions

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INTRODUCTION

This note argues that satisfaction of no-arbitrage conditions, as derived from the cost-of-carry model, tends to increase the correlation between spot and futures prices. To demonstrate this conjecture explicitly, spot and futures prices are assumed to be jointly lognormally distributed. Attention is then directed to the effects on the joint distribution and, subsequently, the correlation between the two random variables when the distribution is either truncated or censored in accordance with obtaining the no-arbitrage conditions. Via truncation, those observations that violate the no-arbitrage conditions are excluded; likewise with censorship.

This approach highlights a relationship between the arbitrage behavior (i.e., attaining the no-arbitrage condition) and hedging behavior in the context of portfolio selection. Furthermore, this approach shows that both types of behavior are mutually supportive.[1] As the analysis shows, imposition of the no-arbitrage conditions improves the correlation between spot and futures prices, which increases hedging effectiveness. Thus, the no-arbitrage conditions and the risk-minimizing approach to hedging can be viewed as an integrated system of analysis.

Let S and F denote spot and futures prices, respectively. They are assumed to be jointly lognormally distributed such that $E[\log F] = \mu_1$, $E[\log S] = \mu_2$, $\text{Var}[\log F] = \sigma_{11}$, $\text{Var}[\log S] = \sigma_{22}$, and $\text{Cov}[\log F, \log S] = \sigma_{12}$. As usual, $E(\cdot)$ is the expectation operator, $\text{Var}(\cdot)$ is the variance operator, and $\text{Cov}(\cdot,\cdot)$ is the covariance operator. From the moment generating function of a bivariate normal random vector [in this case $(\log F, \log S)$], it can be shown easily that

$$\text{Corr}^2(F,S) = (\exp(\sigma_{12}) - 1)^2/[(\exp(\sigma_{11}) - 1)(\exp(\sigma_{22}) - 1)] \tag{1}$$

where $\text{Corr}(F,S)$ denotes the correlation coefficient between F and S. Of course, eq. (1) provides a measure of the hedging effectiveness of the futures contract [Ederington (1979)].

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[1]When no-arbitrage conditions are lifted, riskless profits are available upon holding equal but opposite spot and futures positions. The problem analyzed in this article presumes that the no-arbitrage conditions are strictly adhered to. It is true that portfolio analysis is unnecessary when the no-arbitrage condition is violated, but the situation investigated here assumes the no-arbitrage condition is satisfied and investigates whether, consequently, the construct of the portfolio is necessarily altered.

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The no-arbitrage condition is now imposed: $F \leq \theta S$, where θ is a positive number. For example, it can be assumed that $\theta = \exp\{(r + u - y)(T - t)\}$ where r is the risk-free interest rate, u and y are the storage cost and convenience yield per unit of time as a proportion of the spot price, respectively, and $T - t$ is the time to maturity [Hull (1989)]. Note that in the original setting the probability the futures price exceeds θ times the spot price is nonnegligible; that is, $\text{Prob}(F > \theta S) > 0$. With the no-arbitrage condition, the new joint distribution must satisfy the constraint, $\text{Prob}(F > \theta S) = 0$. There are various methods of constraining a distribution. The most widely used methods are truncation and censoring.[2]

Using the truncation method, the new joint distribution is derived from the original by (i) setting a zero density for any combination of S and F satisfying $F > \theta S$; and (ii) adjusting the density at those combinations of S and F where $F < \theta S$ by a common factor to ensure its integration equals one. Let $g(F,S)$ denote the original density function and $h(F,S)$ the new joint density function. Mathematically, $h(F,S)$ satisfies: (1) $h(F,S) = 0$, if $F > \theta S$; (2) $h(F,S) = cg(F,S)$, otherwise. For the integration of $h(\cdot,\cdot)$ over the area $F \leq \theta S$ to equal one, c must be appropriately chosen. Thus, for any function of F and S, say $f(F,S)$, the expectation under the joint density $h(\cdot,\cdot)$ equals the conditional expectation under the joint density $g(\cdot,\cdot)$. That is, $E_h(f(F,S)) = E_g(f(F,S)|F \leq \theta S)$. Upon applying Theorem 1 of Lien (1985), one obtains the following equations (see Appendix I):

$$E_h(F) = a \cdot \exp(\mu_1 + \sigma_{11}/2)\Phi((p - \sigma_{11} + \sigma_{12})/\sigma) \tag{2a}$$

$$E_h(S) = a \cdot \exp(\mu_2 + \sigma_{22}/2)\Phi((p - \sigma_{12} + \sigma_{22})/\sigma) \tag{2b}$$

$$E_h(F^2) = a \cdot \exp(2\mu_1 + 2\sigma_{11})\Phi((p - 2\sigma_{11} + 2\sigma_{12})/\sigma) \tag{2c}$$

$$E_h(S^2) = a \cdot \exp(2\mu_2 + 2\sigma_{22})\Phi((p - 2\sigma_{12} + 2\sigma_{22})/\sigma) \tag{2d}$$

$$E_h(FS) = a \cdot \exp(\mu_1 + \mu_2 + (\sigma_{11} + 2\sigma_{12} + \sigma_{22})/2)\Phi((p - \sigma_{11} + \sigma_{22})/\sigma) \tag{2e}$$

where $\sigma^2 = \sigma_{11} + \sigma_{22} - 2\sigma_{12}$, $p = \log\theta - \mu_1 + \mu_2$, $a = 1/\Phi(p/\sigma)$ with $\Phi(\cdot)$ being the cumulative density function of the standard normal random variable. The coefficient between the two prices under the new joint density function, $\text{Corr}(F,S;h)$, can be calculated easily from the above equations.

To provide a numerical examination of the conjecture, it is assumed that $\mu_1 = \mu_2 = \mu$ and $\sigma_{11} = \sigma_{22} = 1$. That is, the two prices are identically distributed. This assumes that correlation coefficient $\text{Corr}(F,S;h)$ is independent of the value of μ. To select an appropriate value for θ, consider the simplest case where both storage cost and convenience yield are zero. Given $u = y = 0$, then $\theta = \exp(r(T - t))$. Suppose $r = \log(1.10)$ and that $T - t$ can take on two different values, 1 or 0.25 (1 year or 1 quarter). The results are summarized in Table I.

Clearly, the no-arbitrage condition improves the correlation between spot and futures prices; and, therefore, the hedging effectiveness of the futures contract. Note

[2]After the no-arbitrage condition is imposed, the support for the joint density function for spot and futures prices differ from the Cartesian product of the two supports for the corresponding marginal density function. Holland and Wong (1987) term this case "regional dependence." They show that regional dependence implies the two random variables are not stochastically independent. The result is qualitatively much less satisfactory than desired. However, based upon their definition, it is shown easily that the regional dependence between the two prices increases when the upper limit for the futures price becomes tighter in the "truncation" case. For the "censoring" case, the resulting joint density is not continuous, and the existence of the measure of regional dependence is itself dubious.

Table I
THE EFFECT OF NO-ARBITRAGE CONDITIONS

σ_{12}	$T - t$	Corr.$^2(F, S; g)$	Corr.$^2(F, S; h)$	Corr.$^2(F, S; k)$
0.0	1	0.0	0.1266	0.1435
0.0	0.25	0.0	0.1349	0.1570
0.1	1	0.0037	0.1815	0.1931
0.1	0.25	0.0037	0.1915	0.2084
0.2	1	0.0166	0.2441	0.2497
0.2	0.25	0.0166	0.2556	0.2669
0.3	1	0.0415	0.3139	0.3136
0.3	0.25	0.0415	0.3268	0.3324
0.4	1	0.0819	0.3905	0.3850
0.4	0.25	0.0819	0.4045	0.4052
0.5	1	0.1425	0.4738	0.4641
0.5	0.25	0.1425	0.4885	0.4854
0.6	1	0.2289	0.5636	0.5513
0.6	0.25	0.2289	0.5785	0.5732
0.7	1	0.3481	0.6600	0.6468
0.7	0.25	0.3481	0.6745	0.6687
0.8	1	0.5087	0.7631	0.7512
0.8	0.25	0.5087	0.7764	0.7718
0.9	1	0.7216	0.8736	0.8650
0.9	0.25	0.7216	0.8843	0.8820

that the improvement is greater when the original joint density displays zero or low correlation, i.e., when σ_{12} is small. For example, the no-arbitrage condition induces a correlation coefficient of 0.35–0.37, even though the two prices are independently distributed without the condition. For the moderate range of σ_{12}, correlation coefficients also increase dramatically. It appears, at least empirically, that the no-arbitrage condition itself induces a moderate correlation between spot and futures prices. This case occurs not infrequently when the futures contract allows for multiple deliverable grades. Finally, note that the improvement in correlation coefficients is greater as $T - t$ becomes smaller. This is not surprising since a smaller $T - t$ generates a smaller upper bound for the futures price.

An alternative method of creating the new joint density is by "censoring" the original density. Using this method, the probability for those combinations of F and S with $F > \theta S$ is removed and added to combinations or points $(\theta S, S)$. Consequently, the new joint density possesses mass points along the line $F = \theta S$. Let $k(F, S)$ denote the new joint density obtained via censoring. Thus, (1) $k(F, S) = g(F, S)$ if $F < \theta S$; (2) $k(\theta S, S) = \int_{\theta S}^{\infty} g(x, S)\,dx$; and (3) $k(F, S) = 0$, otherwise. To compute the correlation coefficient between F and S under the new joint density, Corr$(F, S; k)$, the following equations are required (see Appendix I):

$$E_k(S) = E_g(S) = \exp(\mu_2 + \sigma_{22}/2) \tag{3a}$$

$$E_k(S^2) = E_g(S^2) = \exp(2\mu_2 + 2\sigma_{22}) \tag{3b}$$

$$\begin{aligned} E_k(F) = {} & \exp(\mu_1 + \sigma_{11}/2)\Phi((p - \sigma_{11} + \sigma_{12})/\sigma) \\ & + \theta \exp(\mu_2 + \sigma_{22}/2)\Phi(-(p - \sigma_{12} + \sigma_{22})/\sigma) \end{aligned} \tag{3c}$$

$$E_k(F^2) = \exp(2\mu_1 + 2\sigma_{11})\Phi((p - 2\sigma_{11} + 2\sigma_{12})/\sigma) + \theta^2\exp(2\mu_2 + 2\sigma_{22})\Phi(-(p - 2\sigma_{12} + 2\sigma_{22})/\sigma) \tag{3d}$$

$$E_k(FS) = \exp(\mu_1 + \mu_2 + (\sigma_{11} + 2\sigma_{12} + \sigma_{22})/2)\Phi((p - \sigma_{11} + \sigma_{22})/\sigma) + \theta\exp(2\mu_2 + 2\sigma_{22})\Phi(-(p - 2\sigma_{12} + 2\sigma_{22})/\sigma) \tag{3e}$$

Using the same numerical values for σ_{11}, σ_{22}, θ, and $T - t$ as in the truncation case, the correlation coefficient $\mathrm{Corr}(F, S; k)$ is calculated. As shown in Table I, a similar conclusion is derived. Imposing a no-arbitrage condition leads to larger correlation coefficients and, therefore, greater hedging effectiveness. When σ_{12} is large, $\mathrm{Corr}(F, S; k)$ is smaller than $\mathrm{Corr}(F, S; h)$. That is, truncation increases more of the correlation coefficient than censoring. Nevertheless, as long as $T - t$ is small, both truncation and censoring induce a large increase in correlation coefficients. The implication of this result is clear, the no-arbitrage condition promotes the usefulness of portfolio analysis. It is also clear that the selection of an optimal portfolio may be greatly influenced by the existence of arbitrage behavior. It should be noted that only asymmetric arbitrage is considered. That is, $F \leq \theta S$. If the opposite arbitrage is possible then the no-arbitrage condition would establish a lower bound for the futures price as well; say $F > \phi S$ with $\phi < \theta$. This sort of symmetric arbitrage is discussed in Appendix II. Several numerical calculations are performed, and the results indicate that the existence of both upper and lower bounds dramatically increase the correlation between spot and futures prices (especially when the correlation is low a priori).[3] In general, high correlations between spot and futures prices are observed in full-carry markets.

The above conclusions are obtained with the assumption of a bivariate lognormal density. Consideration is given now to some distribution-free results. More specifically, the minimal correlation coefficient between two positive random variables S and F are characterized with unknown distribution function but obeying the no-arbitrage condition: $\theta_1 S \leq F \leq \theta_2 S$; $\theta_2 > \theta_1 \geq 0$. First, a new random variable Z is defined as follows:

$$Z = ((F/S) - \theta_1)/(\theta_2 - \theta_1) \tag{4}$$

Hence, Z is a bounded random variable with support $[0, 1]$. The neutrality concept from the geostatistics literature is used. See Darroch (1969) for an explanation of mean neutrality, Snow (1975) for quadrant neutrality, and Darroch and Ratcliff (1978) for median neutrality. Darroch and James (1974) and Aitchison (1981) provide some general discussions.

The main idea is that a minimal correlation coefficient is obtained if the futures price is independent of the spot price after the bounds induced by the latter are taken into account. In other words, the following neutrality is required: Z is stochastically independent of S. Given this property, one obtains: $E(F) = E[\theta_1 + (\theta_2 - \theta_1)Z]E(S)$; $E(F^2) = E[(\theta_1 + (\theta_2 - \theta_1)Z)^2]E(S^2)$; and $E(FS) = E[\theta_1 + (\theta_2 - \theta_1)Z]E(S^2)$. Moreover, after algebraic manipulation, one derives:

$$\mathrm{Corr}^2(F, S) = \{1 + \gamma(\theta_2 - \theta_1)^2\mathrm{Var}(Z)/(\theta_1 + (\theta_2 - \theta_1)E(Z))^2\}^{-1} \tag{5}$$

[3]Consider the case of symmetric bounds: $S/\theta \leq F \leq \theta S$. Let $\theta = \exp(r(T - t))$ with $r = \log(1.10)$ and $T - t = 1$ or 0.25. Both truncation and censoring produce a near-perfect correlation between F and S, regardless of the initial correlation coefficient. In fact, for a large $T - t$ (say, $T - t = 5$), and for uncorrelated spot and futures prices, the correlation coefficient becomes 0.91 (resp. 0.92) after truncation (resp. censoring).

where $\gamma = E(S^2)/\text{Var}(S)$. Note that Z is bounded in $[0, 1]$. Assume that Z has an absolutely continuous density function. Seaman, Young, and Marco (1986) show that, subject to the constraint $E(Z) = \mu$, the supremum of Var(Z) is attained at $\mu(1 - \mu)$. Thus,

$$\text{Corr}^2(F,S) \geq \{1 + \gamma(\theta_2 - \theta_1)^2\mu(1 - \mu)/(\theta_1 + (\theta_2 - \theta_1)\mu)^2\}^{-1} \tag{6}$$

which characterizes minimal correlation coefficient when γ and μ are both given. On the other hand, if μ is also a choice variable, the last term within the bracket achieves its maximum at $\mu = \theta_1/(\theta_1 + \theta_2)$, implying

$$\text{Corr}^2(F,S) \geq \{1 + \gamma(\theta_2 - \theta_1)^2(\theta_1 + \theta_2)/4\theta_1\theta_2\}^{-1} \tag{7}$$

Further characterizations of minimal correlation coefficient require assumptions on the random variable S, the spot price.

The above approach may be applied also to the additive no-arbitrage condition: $-a + S \leq F \leq b + S$ with a, $b > 0$. In this case, the required neutrality states that w is stochastically independent of S, where $w = (F - S + a)/(b + a)$. As a result, one obtains:

$$\text{Corr}^2(F,S) = \{1 + (b + a)^2\text{Var}(w)/\text{Var}(S)\}^{-1} \tag{8}$$

Provided that w has an absolutely continuous density function (note that w is bounded in $[0, 1]$), Seaman, Young, and Marco (1986) show that $\text{Var}(w) \leq 1/4$. Hence, the minimal correlation coefficient is characterized by:

$$\text{Corr}^2(F,S) \geq \{1 + (b + a)^2/4\text{Var}(S)\}^{-1} \tag{9}$$

Bibliography

Aitchison, J. (1981): "A New Approach to Null Correlations of Proportions," *Mathematical Geology,* 13:175–189.

Darroch, J. N. (1969): "Null Correlation for Proportions," *Mathematical Geology,* 1:221–227.

Darroch, J. N., and James, I. R. (1974): "F-Independence and Null Correlation of Continuous, Bounded-Sum Positive Variables," *Journal of the Royal Statistical Society,* B36:467–483.

Darroch, J. N., and Ratcliff, D. (1978): "No-Association of Proportions," *Mathematical Geology,* 10:361–368.

Ederington, L. (1979): "The Hedging Performance of the New Futures Markets," *Journal of Finance,* 34:157–170.

Holland, P., and Wang, Y. (1987): "Regional Dependence for Continuous Bivariate Densities," *Communication in Statistics: Theory and Methods,* 16:193–206.

Hull, J. (1989): *Options, Futures, and Other Derivative Securities.* Englewood Cliffs, NJ: Prentice-Hall.

Lien, D. (1985): "Moments of Truncated Bivariate Log-Normal Distributions," *Economics Letters,* 19:243–247.

Lien, D. (1986): "Moments of Ordered Bivariate Log-Normal Distributions," *Economics Letters,* 20:45–47.

Seaman, J. W., Young, D. W., and Marco, V. R. (1986): "The Variance of Bounded Continuous Unimodal Distributions," *Mathematical Scientist,* 11:103–106.

Snow, J. W. (1975): "Association of Proportions," *Mathematical Geology,* 7:63–73.

Appendix I

To derive eqs. (2a)–(2e), define $w_1 = \log F - \log S - (\mu_1 - \mu_2)$ and $w_2 = \log S - \mu_2$. Thus, (w_1, w_2) is a bivariate normal random vector with $E(w_1) = E(w_2) = 0$, $E(w_1^2) = \sigma_{11} + \sigma_{22} + 2\sigma_{12}$, $E(w_1 w_2) = \sigma_{12} - \sigma_{22}$, and $E(w_2^2) = \sigma_{22}$. The no-arbitrage condition, $F \leq \theta S$, may be rewritten as $w_1 \leq \log\theta - (\mu_1 - \mu_2)$. Consequently,

$$\begin{aligned} E_h(F^m S^n) &= E_g(F^m S^n | F \leq \theta S) \\ &= \exp(m(\mu_1 - \mu_2) + (m+n)\mu_2) \\ &\quad \cdot E(\exp(m w_1 + (m+n) w_2) | w_1 \leq \log\theta - \mu_1 + \mu_2) \end{aligned}$$

with the latter expectation being taken with respect to the joint density of w_1 and w_2. For nonnegative values of m and n, a general formula for the expectation is provided in Theorem 1 of Lien (1985). Upon successively substituting $(m, n) = (1, 0)$, $(0, 1)$, $(2, 0)$, $(0, 2)$, and $(1, 1)$ into the general formula, one obtains eqs. (2a)–(2e).

For the censoring case, first note that the marginal density function for S remains unchanged. Thus, $E_k(S) = E_g(S)$ and $E_k(S^2) = E_g(S^2)$. Moreover, Z is defined as $\min(F, \theta S)$ so that $E_g(Z) = E_k(F)$, $E_g(Z^2) = E_k(F^2)$, and $E_g(ZS) = E_k(FS)$. Note that $(F, \theta S)$ is a bivariate lognormal vector, the results of Lien (1986) are directly applicable to derive eqs. (3c)–(3e).

Appendix II

Here the no-arbitrage condition of the following form: $\theta_1 S \leq F \leq \theta_2 S$ with $0 \leq \theta_1 < \theta_2$ is considered. There is an upper as well as a lower bound on the futures price. For the truncation case, let $p(\cdot)$ denote the resulting density function. Then $p(F, S) = c' g(F, S)$ if $\theta_1 S \leq F \leq \theta_2 S$ and $p(F, S) = 0$, otherwise. The normalization constant, c', is established such that the integration of $p(F, S)$ over the area $\theta_1 S \leq F \leq \theta_2 S$ equals one. For any function of (F, S), say $f(F, S)$, one obtains:

$$\begin{aligned} E_p(f(F,S)) &= E_g(f(F,S) | \theta_1 S \leq F \leq \theta_2 S) \\ &= \{E_g(f(F,S) | F \leq \theta_2 S) \cdot \mathrm{Prob}(F \leq \theta_2 S)/\mathrm{Prob}(\theta_1 S \leq F \leq \theta_2 S)\} \\ &\quad - \{E_g(f(F,S) | F \leq \theta_1 S) \\ &\qquad \cdot \mathrm{Prob}(F \leq \theta_1 S)/\mathrm{Prob}(\theta_1 S \leq F \leq \theta_2 S)\} \end{aligned}$$

where $\mathrm{Prob}(A)$ denotes the probability that event A occurs. In this case, $f(F, S)$ takes the form of $F^m S^n$. While the expression of $E_g(F^m S^n | F \leq \theta S)$ for any $\theta > 0$ is presented in Appendix I, the probability terms may be calculated easily using the assumption that (F, S) is bivariately lognormally distributed. Consequently, $E_p(F^m S^n)$ can be easily evaluated.

Turning to the case of censoring, let $q(F, S)$ denote the resulting density function. Therefore: (i) $q(F, S) = g(F, S)$ if $\theta_1 S < F < \theta_2 S$; (ii) $q(\theta_1 S, S) = \int_{\theta_1 S}^{\infty} g(x, S)\,\mathrm{d}x$; (iii) $q(\theta_2 S, S) = \int_0^{\theta_2 S} g(x, S)\,dx$; and (iv) $q(F, S) = 0$, otherwise. Again, $E_q(S) = E_g(S)$ and $E_q(S^2) = E_g(S^2)$. Define $Q_1(\theta_i) = \Phi(-(\mu_1 - \mu_2 + \sigma_{11} - \sigma_{12} - \log\theta_i)/\sigma)$ and $Q_2(\theta_i) = \Phi((\mu_1 - \mu_2 - \sigma_{22} + \sigma_{12} - \log\theta_i)/\sigma)$,

$i = 1, 2$. Following the same procedure as described in Lien (1986), one derives:

$$\begin{aligned} E_q(F) = {} & \exp(\mu_1 + \sigma_{11}/2)[Q_1(\theta_2) - Q_1(\theta_1)] \\ & + \exp(\mu_2 + \sigma_{22}/2)[\theta_2 Q_2(\theta_2) + \theta_1(1 - Q_2(\theta_1))] \end{aligned}$$

Moreover, one obtains:

$$\begin{aligned} E_q(F^2) = {} & \exp(2\mu_1 + 2\sigma_{11})[R_1(\theta_2) - R_1(\theta_1)] \\ & + \exp(2\mu_2 + 2\sigma_{22})[\theta_2^2 R_2(\theta_2) + \theta_1^2(1 - R_2(\theta_1))] \\ E_q(SF) = {} & \exp(\mu_1 + \mu_2 + (\sigma_{11} + 2\sigma_{12} + \sigma_{22})/2)[S_1(\theta_2) - S_1(\theta_1)] \\ & + \exp(2\mu_2 + 2\sigma_{22})[\theta_2 R_2(\theta_2) + \theta_1(1 - R_2(\theta_1))] \end{aligned}$$

where $R_1(\theta_i) = \Phi(-(\mu_1 - \mu_2 + 2\sigma_{11} - 2\sigma_{12} - \log\theta_i)/\sigma)$, $R_2(\theta_i) = \Phi((\mu_1 - \mu_2 + 2\sigma_{12} - 2\sigma_{22} - \log\theta_i)/\sigma)$; and $S_1(\theta_i) = \Phi((\mu_2 - \mu_1 + \sigma_{22} - \sigma_{11} + \log\theta_i)/\sigma)$, $i = 1, 2$. Further details are available upon request from the author.