A New Look at Interest Rate Futures Contracts

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INTRODUCTION

The difference between the forward price and the futures price has been discussed in a number of papers [e.g., Cox, Ingersoll, and Ross (1981); Cornell and Reinganum (1981); and French (1983)]. Although this difference results from a number of reasons, most of them are institutional. Among the few economic factors that contribute to the difference, marking to market of the futures contracts is the most important and significant one.[1] This study reinvestigates this marking-to-market activity and derives a closed-form solution for the futures price under discrete marking market with a powerful methodology developed recently by Jamshidian (1988, 1989). Although the solution is derived under the assumption that the instantaneous risk-free rate follows the Ornstein–Uhlenbeck (O-U) process, it can be seen that the methodology can be applied to any stochastic process for the interest rate.

To derive the discrete marking-to-market futures price, it is necessary to derive a forward price on a pure discount bond. The derivation may be less interesting since it is well known that a forward price can be expressed as the ratio of two pure discount bond prices. However, it is useful in deriving the futures price under discrete marking to market.

The following symbols and notations are used throughout the article:

r = instantaneous risk-free rate,
α = mean reverting parameter in the O-U process,
μ = long-term mean to which r converges in the O-U process,
σ = instantaneous interest rate volatility in the O-U process,
$P(r, t; T)$ = pure discount bond price at time t maturing at time T,
$\Phi(r, t; T_F, T)$ = bond futures price at time t maturing at time T_F with the underlying bond maturing at time T, and
$\Psi(r, t; T_F, T)$ = bond forward price at time t maturing at time T_F with the underlying bond maturing at time T.

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[1] If interest rates are not stochastic, then this marking to market makes no difference to the two prices.

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PRICING FORWARD AND FUTURES CONTRACTS

In their seminal article, Cox and Ross (1976) show that any European-style contingent claim price can be found by taking the conditional expectation of its maturity-day payoff if a properly transformed process can be identified. Furthermore, Cox, Ingersoll, and Ross (1981) show that the futures price is the expectation of the end payoff under the risk-neutralized process, which deviates from the true process by a risk premium in the drift term. Recently, Jamshidian (1989) shows that the forward price can be found in the similar manner, but the expectation should be taken under a "forward rate process," which deviates from the risk-neutralized process by another term in the drift term. In this section, with the Cox–Ross–Jamshidian methodology, forward and futures prices are derived under the assumption that the only state variable, the instantaneous riskless rate, follows an O-U process.

Let r, the instantaneous riskless rate, follow the following O-U process:

$$dr = \alpha(\mu\text{-}r)dt + \sigma dz \tag{1}$$

where α, μ, and σ are positive constants representing mean-reverting speed, long-term mean, and volatility, respectively, and $z(t)$ is a standard Wiener process. It is clear that this O-U process is mean-reverting. If $r(t) > \mu$, then the drift term $\mu - r$, is less than zero. Thus, in the next instant, $t + dt$, the interest rate, will be pulled down. On the other hand, if $r(t) < \mu$, then $\mu - r > 0$, and the interest rate will be pulled up in the next instant. There is a major problem associated with this particular process: there exists a nontrivial probability of having negative interest rates. However, it can be shown that this probability can be controlled to be arbitrarily small with plausible parameters. Moreover, it can be seen that the methodology presented in this article is so general that it can be applied to any other stochastic process. The O-U process is chosen simply for its mathematical tractability.

To find the futures price under continuous marking to market, a risk-neutralized process needs to be identified:

$$dr = \alpha(\bar{\mu} - r)dt + \sigma dz \tag{2}$$

where $\bar{\mu} = \mu - \rho/\alpha$ and ρ represents the risk premium associated with trading futures contracts.

Vasicek (1977) shows that the conditional density for $r(s)$ given $r(t) = r$ for all $s > t$ under the risk-neutralized process is Normal (μ', σ') = Normal $(\bar{\mu} + (r - \bar{\mu})e^{-\alpha(T_F - t)}, \sigma^2/2\alpha(1 - e^{-2\alpha(T_F - t)}))$ and the bond price is:

$$P(r, t; T) = e^{-rF(t,T) - G(t,T)} \tag{3}$$

where:

$$F(t,T) = \frac{1 - e^{-\alpha(T-t)}}{\alpha}$$

$$G(t,T) = [T - t - F(t,T)]\left(\mu - \frac{\rho}{\alpha} - \frac{\sigma^2}{2\alpha^2}\right) + \frac{\sigma^2 F(t,T)^2}{4\alpha}$$

Since the futures price is the price paid at the maturity date of the futures contract for the underlying asset, the pure discount bond, by the Cox–Ross methodology, it must be the conditional expectation taken under the risk-neutralized process of the bond price at the maturity date, T_F.

$$\begin{aligned}\Phi(r,t;T_F,T) &= \bar{E}[P(r,T_F;T)|r(t) = r]\\ &= \bar{E}[e^{-r(T_F)F(T_F,T)-G(T_F,T)}|r(t) = r]\\ &= \bar{E}[e^{-r(T_F)F(T_F,T)}|r(t) = r]e^{-G(T_F,T)}\end{aligned}$$

By the moment generating function, one may explicitly express the futures price as follows.

$$\begin{aligned}\Phi(r,t;\,T_F,T) &= \exp\left(-F(T_F,T)\mu' + \frac{F(T_F,T)^2(\sigma')^2}{2} - G(T_F,T)\right)\\ &= e^{-rX(t)-Y(t)}\end{aligned} \tag{4}$$

where:

$$X(t) = F(t,T) - F(t,T_F)$$

and

$$Y(t) = [T - T_F - X(t)]\left(\mu - \frac{\rho}{\alpha} - \frac{\sigma^2}{2\alpha^2}\right) - \frac{\sigma^2}{2\alpha^2}\left(X(t) - \frac{\alpha}{2}X(t)^2 - F(T_F,T)\right)$$

This solution is shown by Chen (1992).

A similar calculation is done for the forward price, except that this time the conditional expectation is taken under the forward rate process:

$$dr = \alpha(\tilde{\mu} - r)dt + \sigma dz \tag{5}$$

where:

$$\tilde{\mu} = \bar{\mu} - \frac{q}{1 - e^{-\alpha(T_F-t)}} \quad \text{and} \quad q = \frac{1}{2}\sigma^2F(t,T_F)^2$$

Notice that the conditional distribution for the interest under the forward rate process is also normal, but with a different mean: Normal$(\tilde{\mu} + (r - \tilde{\mu})e^{-\alpha(T_F-t)}$, $\sigma^2/2\alpha(1 - e^{-2\alpha(T_F-t)}))$ = Normal$(\bar{\mu} + (r - \bar{\mu})e^{-\alpha(T_F-t)} - q, \sigma^2/2\alpha(1 - e^{-2\alpha(T_F-t)})$.[2] Jamshidian shows that the forward price is the conditional expectation taken under this transformed process. Again, using the moment-generating function of the normal density, one obtains:

$$\begin{aligned}\Psi(r,t;T_F,T) &= \tilde{E}[P(r,T_F;T)|r(t) = r]\\ &= \tilde{E}[e^{-r(T_F)F(T_F,T)-G(T_F,T)}|r(t) = r]\\ &= \exp\left(-F(T_F,T)(\mu' - q) + \frac{F^2(\sigma')^2}{2} - G(T_F,T)\right)\\ &= \exp(-r(F(t,T) - F(t,T_F)) - (G(t,T) - G(t,T_F)))\\ &= \frac{P(r,t;T)}{P(r,t;T_F)}\end{aligned} \tag{6}$$

Now, it is proved that the forward price is the ratio of two pure discount bonds. The main purpose of this proof is not to derive the forward price, but to show that any

[2] See Jamshidian (1988, 1989).

forward price can be derived in this way. In the next section, it will be shown that a discrete marking-to-market futures price can be broken down into a series of forward prices. Hence, to solve for such a futures price is to solve for the series of forward prices. Since those forward prices are not written on any bond, the solutions are not ratios of pure discount bonds. As a consequence, one can only obtain the discrete marking-to-market futures price with the help of the method presented in equations (5) and (6).

Finally, a useful relationship will be shown between two pricing formulae. The futures price can be written in terms of the forward price in a simple way:

$$\Phi(r, t; T_F, T) = e^J \Psi(r, t; T_F, T) \tag{7}$$

where $J = -\frac{1}{2}\sigma^2 F(t, T_F)^2 F(T_F, T)$.

THE FUTURES PRICE UNDER DISCRETE MARKING TO MARKET[3]

This section derives a closed-form expression for the futures price subject to discrete marking to market. This generalization is useful for several reasons. First, the solution is still closed form, which provides simple calculation and great insight. Second, the discrete marking-to-market feature fits the real world much better than the continuous marking to market. Third, the model offers the flexibility of changing the number of times a futures contract marks to market. Changing the value of n enables one to model various kinds of discrete marking to market. Fourth, the model converges in both directions: it converges to continuous time futures price as n approaches infinity and to forward price as n approaches unity.

The previous section shows that the futures price and the forward price are both expected values of the bond price at T_F; but the expectations are taken under different diffusion processes. In this section, the same technique is used to derive futures prices under discrete marking to market.

As Figure 1 demonstrates, if a futures contract marks to market n times in its lifetime, then one must cut the time segment from t to T_F into n pieces with each piece equal to $(T_F - t)/n \equiv k$. Then, one starts with the maturity date of the futures contract and works backward, just as in dynamic programming. First, the futures price at time $T_F - k$ is found, using the *bond* price available at T_F. Then an attempt is made to find the price at $T_F - 2k$, using the *forward* price available at $T_F - k$. The procedure is repeated until $T_F - nk$ is reached, which is the current time, t.

It should be noticed that at $T_F - k$, the futures price and the forward price should be the same, i.e., $\Phi(r, T_F - k; T_F, T) = \Psi(r, T_F - k; T_F, T)$. Since the forward price on a discount bond is a ratio of two discount bonds, both of them are equal to $\exp\{-r(F(T_F - k, T) - F(T_F - k, T_F)) - (G(T_F - k, T) - G(T_F - k, T_F))\}$. In this

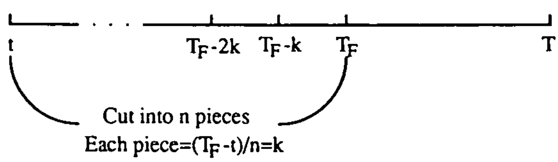

Figure 1
Discrete marking to market time line.

[3] Flesaker (1989) was the first to show solutions for futures prices under discrete marking to market.

period, the futures contract only marks to market once at the end. The only cause for the two prices to differ is the daily marking to market in futures contracts. If a futures contract just marks to market once, then it should be priced exactly the same as the forward contract.

At $T_F - 2k$, Φ starts to differ from Ψ, since the futures contract now marks to market twice, one at $T_F - k$ and another at T_F. To find the futures price at $T_F - 2k$, it is necessary to recognize that $\Phi(r, T_F - 2k; T_F, T)$ is not only the futures price maturing at T_F, but also a forward price maturing at $T_F - k$, with the maturity payment equal to $\Phi(r, T_F - k; T_F, T)$. Recall that the forward price is the expected value taken under the forward rate process of the maturity value at the maturity date. Therefore, to find the futures price at $T_F - 2k$ is equivalent to pricing the forward contract maturing one period later (at $T_F - k$) with $\Phi(r, T_F - k; T_F, T)$ the cash price. The same argument can be applied to all the times between t and T_F. The following theorem results:

A pure discount bond futures price, $\Phi^{(n)}(r, t; T_F, T)$ that marks to market n times during its lifetime (from t to T_F) can be expressed as follows:

$$\begin{aligned}
&\Phi^{(n)}(r, t; T_F, T) \\
&= \exp\Bigg\{-rF^{\delta}_{nk} + F^{\delta}_{nk}(\mu - \rho/\alpha) - (\mu - \rho/\alpha - \sigma^2/2\alpha^2)(T - T_F) \\
&\quad + \frac{\sigma^2(F^{\delta}_{(n-1)k} + \ldots + F^{\delta}_{k})F(T_F - k, T_F)^2}{2} \\
&\quad + \frac{\sigma^2}{4\alpha}(F^{\delta 2}_{k} - F^{\delta 2}_{nk}) - \frac{\sigma^2}{2\alpha^2}F^{\delta}_{k} - \frac{\sigma^2}{4\alpha}(F(T_F - k, T)^2 - F(T_F - k, T_F)^2)\Bigg\}
\end{aligned}$$

where $F^{\delta}_{ik} = F(T_F - ik, T) - F(T_F - ik, T_F)$ and $F(\cdot,\cdot)$ is defined in equation (3). See Appendix for Proof.

Although the solution is lengthy, it is closed form, and therefore the computation is easy and fast. The solution involves no infinite sum nor integral sign. Simulations based upon the above theorem are presented in the next section.

Two special cases to be noted are $n \to \infty$ and $n \to 1$. Continuous marking to market is implied by $n \to \infty$, while $n \to 1$ implies the equality between forward and futures. Thus, the forward price and the continuous marking-to-market futures price are the two extremes. One can show that the above theorem will be broken down to equation (4) as $n \to \infty$ and equation (6) as $n \to 1$. An easier way to verify this result is through an alternative expression for $\Phi^{(n)}$: $\Phi^{(n)}(r, t; T_F, T) = e^{J(n)}\Psi(r, t; T_F, T)$, where:

$$\begin{aligned}
J(n) = \exp\Bigg\{&\frac{\sigma^2(F^{\delta}_{(n-1)k} + \ldots + F^{\delta}_{k})F(T_F - k, T_F)^2}{2} + \frac{\sigma^2}{4\alpha}(F^{\delta 2}_{k} - F^{\delta 2}_{nk}) \\
&- \frac{\sigma^2}{2\alpha^2}(F^{\delta}_{k} - F^{\delta}_{nk}) - \frac{\sigma^2}{4\alpha}(F(T_F - k, T)^2 - F(T_F - k, T_F)^2 \\
&- F(T_F - nk, T)^2 + F(T_F - nk, T_F)^2)\Bigg\}
\end{aligned} \tag{8}$$

a parallel form with the one in continuous marking to market. One can see that when $J \to 0$ as $n \to 1$ and $J \to -\frac{1}{2}\sigma^2 F(t, T_F)^2 F(T_F, T)$ as $n \to \infty$ (i.e., $k \to 0$ and nk remains constant). Then, a continuous marking-to-market futures price is reached.

THE IMPACT OF THE DISCRETE MARKING TO MARKET

From equation (7), it is known that the difference between the forward price and the futures price is characterized by the J function, which is a function of: two parameters (α and σ), the contract's time to maturity ($T_F - t$), and the underlying bond's time to maturity ($T - T_F$). Note that $F(t, s)$ is a monotonically decreasing function of α with two limits: $s - t$ (as $\alpha \to 0$) and 0 (as $\alpha \to \infty$). Thus, the absolute value of J is a decreasing function of α and increasinsg function of σ. As a consequence, the degree of mean reversion of the instantaneous rate has negative impact on the difference between the forward price and futures price and, on the other hand, the instantaneous volatility has positive impact on the difference. In one extreme case, where $\alpha \to \infty$ or $\sigma \to 0$, the futures price is the same as the forward price. This is because both conditions imply deterministic interest rates. α approaching ∞ means the current interest rate can never deviate from the long-term mean (if it ever deviates, it comes back to μ instantaneously). This implies constant interest rate: the interest rate r remains at the level of μ always. σ approaching 0 means the instantaneous rate does not vary and, therefore, should be deterministic. In the other extreme case where $\alpha \to 0$, the biggest difference between the two prices is found: $-\frac{1}{2}\sigma^2(T_F - t)^2(T - T_F)$ which is determined by the level of the volatility and two times to maturity.

In his empirical study, Tse (1989) shows the parameter estimates for α, μ, and σ to be 0.634 ($t = 1.75$), 0.08 ($t = 4.40$), and 0.034 ($t = 30.55$), respectively, using weekly three-month T-bill data from January 1980 to June 1988. Chen and Scott (1991) find the risk premium in T-bill markets to be -0.09 but insignificant.[4] Thus, the simulation here is created using parameter values based on Tse's and Chen and Scott's estimates. Table I shows discrete and continuous futures prices of different α, σ, and time to maturity ($T_F - t$) values. The underlying interest rate instrument is a three-month pure discount bond.[5] The long-term mean and current instantaneous rate are chosen to be 8% and 6% to represent the common case of upward sloping yield curve. The upper panel assumes risk neutrality, i.e., $\rho = 0$; while the lower panel assumes some degree of risk aversion, i.e., $\rho = -0.1$. For the daily marking-to-market futures prices in the table, n is assumed to be 65 for three-month futures contracts, since there are approximately 65 trading days. $n = 130$ and $n = 260$ are assumed according to the similar rule.

The numerical results in both panels confirm what has been found previously: the difference between forward (or discrete futures) and futures prices increases as α decreases, σ increases, and $T_F - t$ increases. In the risk-neutral case (upper panel, $\rho = 0$), differences between discrete and continuous futures prices are small. In most cases, differences cannot be detected until the fifth decimal place, suggesting less than 0.01 penny for \$100 face value bond. However, as the degree of risk aversion of investors increases, differences increase. In the risk-averse case (lower panel, $\rho = -0.1$), even for low volatilities, differences can be detected in the fourth decimal place. For high volatilities, differences can be observed in some contracts in the third decimal place.

[4] Note that the real risk premium should be the absolute value of ρ. The negative sign comes from the Ito transformation of the interest rate process. See Vasicek (1977) for an excellent discussion of this issue.

[5] Contracts of this kind include T-bill futures and Eurodollar futures.

Table I
FUTURES PRICES UNDER DISCRETE MARKING TO MARKET

	$\sigma = 0.03$		$\sigma = 0.05$		$\sigma = 0.07$		$\sigma = 0.09$	
$T_F - t =$ **3 months**[a]								
	$\Phi^{(65)}$	Φ	$\Phi^{(65)}$	Φ	$\Phi^{(65)}$	Φ	$\Phi^{(65)}$	Φ
$\alpha = 1.7$	98.2817	98.2817	98.2826	98.2826	98.2839	98.2839	98.2856	98.2855
0.9	98.3716	98.3716	98.3728	98.3727	98.3745	98.3744	98.3768	98.3767
0.1	98.4940	98.4940	98.4956	98.4955	98.4980	98.4979	98.5012	98.5011
$T_F - t =$ **6 months**[a]								
	$\Phi^{(130)}$	Φ	$\Phi^{(130)}$	Φ	$\Phi^{(130)}$	Φ	$\Phi^{(130)}$	Φ
$\alpha = 1.7$	98.1913	98.1913	98.1924	98.1924	98.1941	98.1940	98.1963	98.1962
0.9	98.3011	98.3010	98.3027	98.3027	98.3052	98.3051	98.3086	98.3084
0.1	98.4829	98.4829	98.4856	98.4856	98.4897	98.4896	98.4952	98.4950
$T_F - t =$ **12 months**[a]								
	$\Phi^{(260)}$	Φ	$\Phi^{(260)}$	Φ	$\Phi^{(260)}$	Φ	$\Phi^{(260)}$	Φ
$\alpha = 1.7$	98.0935	98.0935	98.0948	98.0948	98.0967	98.0966	98.0992	98.0990
0.9	98.1997	98.1997	98.2019	98.2019	98.2053	98.2051	98.2097	98.2095
0.1	98.4615	98.4615	98.4664	98.4662	98.4736	98.4734	98.4833	98.4829
$T_F - t =$ **3 months**[b]								
	$\Phi^{(65)}$	Φ	$\Phi^{(65)}$	Φ	$\Phi^{(65)}$	Φ	$\Phi^{(65)}$	Φ
$\alpha = 1.7$	96.5049	96.5048	96.5081	96.5081	96.5131	96.5130	96.5196	96.5195
0.9	96.7015	96.7015	96.7064	96.7063	96.7137	96.7136	96.7235	96.7233
0.1	96.0016	96.0016	96.0093	96.0092	96.0208	96.0207	96.0361	96.0359
$T_F - t =$ **6 months**[b]								
	$\Phi^{(130)}$	Φ	$\Phi^{(130)}$	Φ	$\Phi^{(130)}$	Φ	$\Phi^{(130)}$	Φ
$\alpha = 1.7$	96.3582	96.3582	96.3621	96.3620	96.3680	96.3679	96.3758	96.3756
0.9	96.5772	96.5772	96.5837	96.5836	96.5935	96.5933	96.6065	96.6062
0.1	96.9813	96.9812	96.9932	96.9931	96.0112	96.0110	96.0352	96.0348
$T_F - t =$ **12 months**[b]								
	$\Phi^{(260)}$	Φ	$\Phi^{(260)}$	Φ	$\Phi^{(260)}$	Φ	$\Phi^{(260)}$	Φ
$\alpha = 1.7$	96.1997	96.1996	96.2040	96.2039	96.2104	96.2103	96.2191	96.2188
0.9	96.3986	96.3986	96.4068	96.4067	96.4191	96.4189	96.4355	96.4351
0.1	96.9419	96.9418	96.9619	96.9617	96.9919	96.9914	96.0318	96.0311

[a] The underlying security is 3-month pure discount bond; $r = 6\%$ and $\mu = 8\%$ represent a typical upward sloping yield curve; $\rho = 0$, the risk-neutral case.
[b] The underlying security is 6-month pure discount bond; $r = 6\%$ and $\mu = 8\%$ represent a typical upward sloping yield curve; $\rho = -0.1$, the risk-averse case.

SUMMARY

Daily marking to market is observed in the real marketplace. The model under this realistic assumption is certainly useful in empirical testing since it better resembles the real world. The model derived in this study is also attractive in terms of its functional form. The closed-form expression is easy to understand and less expensive to work with. Results generated by simulations suggest that unless investors are highly risk averse, discrete futures prices are little different from the continuous ones, implying that either the Chen (1992) or the Cox et al. (1981) futures model should work well for real contracts.

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Appendix

PROOF OF THE THEOREM

At $T_F - k$:

$$\begin{aligned}\Phi^{(1)}(r, T_F - k; T_F, T) &= \Psi(r, T_F - k; T_F, T) \\ &= \exp\{-r(F(T_F - k, T) - F(T_F - k, T_F)) \\ &\qquad - (G(T_F - k, T) - G(T_F - k, T_F))\}\end{aligned}$$

At $T_F - 2k$, the futures price is equal to the forward price maturing at $T_F - k$ with the maturity payment $\Phi^{(1)}(r, T_F - k; T_F, T)$:

$$\begin{aligned}
\Phi^{(2)}(r, T_F - 2k; T_F, T) &= \tilde{E}_{T_F-2k}[\Phi^{(1)}(r, T_F - k; T_F, T)] \\
&= \tilde{E}_{T_F-2k}[\exp\{-r(F(T_F - k, T) - F(T_F - k, T_F)) \\
&\qquad - (G(T_F - k, T) - G(T_F - k, T_F))\}] \\
&= \exp\{(F(T_F - k, T) - F(T_F - k, T_F))\,(\mu^r - q) \\
&\qquad + \frac{(F(T_F - k, T) - F(T_F - k, T_F))^2(\sigma^r)^2}{2} \\
&\qquad - (G(T_F - k, T) - G(T_F - k, T_F))\}
\end{aligned}$$

For notational convenience, let us define $F^{\delta}_{ik} = F(T_F - ik, T) - F(T_F - ik, T_F)$, for all $i = 1, \ldots, n$. Then,

$$\begin{aligned}
\Phi^{(2)}(r, T_F - 2k; T_F, T) &= \exp\Bigg\{-rF^{\delta}_{2k} - (F^{\delta}_{k} - F^{\delta}_{2k})\,(\mu - \rho/\alpha) \\
&\qquad + \frac{\sigma^2 F^{\delta}_{k} F(T_F - k, T_F)^2}{2} + \frac{\sigma^2}{4\alpha}(F^{\delta 2}_{k} - F^{\delta 2}_{2k}) \\
&\qquad - (G(T_F - k, T) - G(T_F - k, T_F))\Bigg\}
\end{aligned}$$

At $T_F - 3k$, the futures price is again equal to the one period (from $T_F - 3k$ to $T_F - 2k$) forward price with the maturity payoff $\Phi^{(2)}(r, T_F - 2k;\ T_F, T)$:

$$\begin{aligned}
\Phi^{(3)}(r, T_F - 3k; T_F, T) &= \tilde{E}_{T_F-3k}[\Phi^{(2)}(r, T_F - 2k; T_F, T)] \\
&= \tilde{E}_{T_F-3k}\Bigg[\exp\Bigg\{-rF^{\delta}_{2k} - (F^{\delta}_{k} - F^{\delta}_{2k})\,(\mu - \rho/\alpha) \\
&\qquad + \frac{\sigma^2 F^{\delta}_{k} F(T_F - k, T_F)^2}{2} + \frac{\sigma^2}{4\alpha}(F^{\delta 2}_{k} - F^{\delta 2}_{2k}) \\
&\qquad - (G(T_F - k, T) - G(T_F - k, T_F))\Bigg\}\Bigg] \\
&= \exp\Bigg\{-rF^{\delta}_{3k} - (F^{\delta}_{k} - F^{\delta}_{3k})\,(\mu - \rho/\alpha) \\
&\qquad + \frac{\sigma^2(F^{\delta}_{2k} + F^{\delta}_{k})F(T_F - k, T_F)^2}{2} \\
&\qquad + \frac{\sigma^2}{4\alpha}(F^{\delta 2}_{k} - F^{\delta 2}_{3k}) - (G(T_F - k, T) \\
&\qquad - G(T_F - k, T_F))\Bigg\}
\end{aligned}$$

If this process is allowed to go on, at t, which is $T_F - nk$, one obtains:

$$\begin{aligned}
\Phi^{(n)}(r, t; T_F, T) = \exp\Bigg\{ & -rF^{\delta}_{nk} - (F^{\delta}_{k} - F^{\delta}_{nk})(\mu - \rho/\alpha) \\
& + \frac{\sigma^2(F^{\delta}_{(n-1)k} + \ldots + F^{\delta}_{k})F(T_F - k, T_F)^2}{2} + \frac{\sigma^2}{4\alpha}(F^{\delta 2}_{k} - F^{\delta 2}_{nk}) \\
& - (G(T_F - k, T) - G(T_F - k, T_F))\Bigg\} \\
= \exp\Bigg\{ & -rF^{\delta}_{nk} - (F^{\delta}_{k} - F^{\delta}_{nk})(\mu - \rho/\alpha) \\
& + \frac{\sigma^2(F^{\delta}_{(n-1)k} + \ldots + F^{\delta}_{k})F(T_F - k, T_F)^2}{2} + \frac{\sigma^2}{4\alpha}(F^{\delta 2}_{k} - F^{\delta 2}_{nk}) \\
& - (\mu - \rho/\alpha - \sigma^2/2\alpha^2)(T - T_F - F^{\delta}_{k}) \\
& - \frac{\sigma^2}{4\alpha}(F(T_F - k, T)^2 - F(T_F - k, T_F)^2)\Bigg\} \\
= \exp\Bigg\{ & -rF^{\delta}_{nk} + F^{\delta}_{nk}(\mu - \rho/\alpha) - (\mu - \rho/\alpha - \sigma^2/2\alpha^2)(T - T_F) \\
& + \frac{\sigma^2(F^{\delta}_{(n-1)k} + \ldots + F^{\delta}_{k})F(T_F - k, T_F)^2}{2} + \frac{\sigma^2}{4\alpha}(F^{\delta 2}_{k} - F^{\delta 2}_{nk}) \\
& - \frac{\sigma^2}{2\alpha^2}F^{\delta}_{k} - \frac{\sigma^2}{4\alpha}(F(T_F - k, T)^2 - F(T_F - k, T_F)^2)\Bigg\}
\end{aligned}$$