

The Theoretical Source of Autocorrelation in Forward and Futures Price Relationships

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INTRODUCTION

When investigated, evidence of autocorrelation consistently is revealed by empirical analyses of comparable forward and futures contracts.¹ For example, Vignola and Dale (1979) and Rendleman and Carabini (1979) find that time series of Treasury-bill forward and futures price differences are autocorrelated. In this regard, they note the systematic patterns of sign reversals that occur when sequences of delivery-matched Treasury-bill forward and futures rates are subtracted for the purpose of examining a mean rate differential. French (1983) confirms that such patterns arise as well for copper and silver forward-futures time series. In documenting that resettlement causes systemic pricing differences between currency forward and futures contracts, Polakoff and Grier (1991) note the same finding. Rubinstein (1987) concludes that the autocorrelation phenomenon is shown to be prevalent in studies contrasting forward values of the S&P 500 index against corresponding index futures prices. In his view, no satisfactory explanation yet has been offered that accounts for this relationship. The absence of insight into the autocorrelation property is not surprising, inasmuch as theory posits that forward cash flows can be replicated with futures contracts in a manner that will result in irrelevant and nonsystemic differences between the instruments' prices (Levy,

¹To be considered strictly comparable, forward and futures contracts must not only be written for identical underlying commodity units and quality, but they must also share an identical term to delivery [Chang and Chang (1990)]. The term-to-delivery constraint tends to limit the use of reported price data for commodities where forward contracts exist on a nonsynthetic basis. For this reason, (nonsynthetic) currency forward and futures prices cannot be contrasted on a daily basis [Chang and Chang (1990)]. Thus, while a number of studies examine the time series properties of daily currency forward and futures rates, [e.g., Hall, Brorsen, and Irwin (1989); So (1987); Hodrick and Srivastava (1987, 1984); Westerfield (1977)], such analyses cannot be used to make direct inferences about forward-futures price relationships.

In markets where forward contracts are structured on an implied basis (such as Treasury-bills or the S&P 500 index), delivery matching poses far fewer difficulties.

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1989). Under such circumstances, there is no theoretical basis for autocorrelation to be displayed by time series of such differences.

This article demonstrates, however, that autocorrelation is to be expected when futures (and forward) contracts are modeled as indivisible. Employing the assumption that capital markets are perfect, Cox, Ingersoll, and Ross (CIR) (1981) and Levy (1989) treat futures contracts as resettled daily, but infinitely divisible. Arbitrage pricing models presented by CIR remain the most widely cited premises underlying the conduct of empirical inquiries into the resettlement issue, yet the forward and futures data employed in these studies are reported as single contract units. From a theoretical perspective, there is no rationale for the existence of futures contracts when markets are held to be perfect [Polakoff (1991); Silber (1985); Telser and Higinbotham (1977)]; therefore, models that omit the indivisibility feature are misspecified economically. When futures contracts are treated both as resettled daily and indivisible, theoretical divergences in forward and futures prices are rendered by distinct and less restrictive conditions than previously used in the literature. Incorporating futures resettlement and indivisibility into a forward-futures pricing framework, Polakoff and Diz (1991) demonstrate that the contracts' prices should differ even when interest rates are constant. As a corollary to this condition, it also can be shown that autocorrelation in time series of these differences is a logical consequence. This finding not only provides a theoretical explanation for the autocorrelation phenomenon, but has implications for empirical work inasmuch as measurements of forward-futures price differences often employ tests unadjusted for the presence of autocorrelation.

LITERATURE REVIEW AND DISCUSSION

Overview of the Resettlement Mechanism

Resettlement represents the basic contractual distinction between forward and futures contracts (Black, 1976) and provides the impetus for numerous studies addressing forward-futures relationships. It is appropriate, therefore, to begin by considering the pricing influence presented by this institutional mechanism.

Unlike the forward, a futures contract is resettled, or marked-to-market, at the end of each business day. That is, the settlement prices of open futures contracts from successive days are contrasted. With open futures positions artificially offset by the clearinghouse exchange, the difference between settlement prices gives rise to losses or gains. These losses or gains are reconciled before the next trading session to limit default risk to a single day. As a result, actual cash profits and losses occur on a daily basis. Futures trades cannot be initiated without the commitment of original margin, but these funds can be made available in the form of a Treasury-bill. As a result, no opportunity cost need be incurred in the process of opening new positions. Thus, original margin usually is ignored in theoretical treatment of resettlement. However, the daily variation margin payments associated with resettlement must be made available on a cash-equivalent basis. It is this condition that is relevant to the pricing process and accounts for the discrete-time perspective employed throughout this study.

Factors leading to differences between comparable forward and futures prices are readily identified through a brief comparison of the instruments' payout schedules. Suppose a pair of forward and futures contracts are purchased simultaneously at the

same price, are held until their respective delivery dates, and are assumed to trade in perfect, frictionless capital markets. If the instruments' delivery dates coincide, the commodities underlying the maturing contracts must converge (through arbitrage) to a single future cash price. Disregarding the resettlement friction for the moment, these forward and futures contracts could be arbitrated so as to generate the same net cash flows since their opening and liquidation prices are the same [see Levy (1989); French (1983); Jarrow and Oldfield (1981); and CIR (1981)]. By the law of One Price, therefore, they must be priced indentially.

Under stochastic conditions, resettlement of a given futures position generally gives rise to daily cash flows. These cash flows can be invested or financed at prevailing rates of interest, depending on whether a gain or loss occurs. Unless the summed products of all resettlements and corresponding interest rates (henceforth, this summed product is referred to as the net interest payout) are zero, the futures contract cannot have the same total payout as its forward counterpart. According to CIR, a nonzero net interest payout will occur provided uncertain futures prices and interest rates display nonzero local covariances. This condition is to be expected in the general case (as demonstrated later, however, divergent forward and futures prices still will result, despite the presence of zero local covariances, when the contracts are modeled as indivisible). Consistent with the Law of One Price, any nonzero net interest payout should generate price differences between a pair of comparable forward and futures contracts. There is, however, an additional complication to be considered.

All net interest payouts are determined after corresponding futures positions have been initiated. Hence, there is current uncertainty associated with these expected payouts, which, in turn, arise from the daily resettlement process. For this reason, a futures contract is not characterized "as the corresponding forward contract rewritten at the end of each day..." [Jarrow and Oldfield (1981)]. Instead, a futures contract is viewed as representing the current value of "the total return which will be earned from a policy of continual investment in one-period bonds" [CIR (1981)]. Since expected net interest payouts are pivotal in determining resettlement influences on forward-futures price relationships, the assumptions surrounding treatment of this payout component have far-reaching methodological implications.

Indivisibility: The Foundation Underlying Divergent Resettlement Inferences and Biases toward Autocorrelation

CIR assume "perfect frictionless capital markets." Prices and interest rates are treated as stochastic. Employing arbitrage arguments, they demonstrate that the expected net interest payout engendered by resettlement precludes riskless arbitrage between comparable forward and futures contracts. When interest rates are stochastic, it is impossible to determine (in the current time period) the precise amount of the net interest payout that will accrue over the life of the futures contract, even when futures positions are treated as perfectly divisible and thus can be adjusted on a fractional basis in each time period. Consequently, CIR view current forward-futures price differentials as a function of expected movements in stochastic interest rates. In a continuous time framework, CIR demonstrate directly that the sign of forward-futures price differences depends on the signs of local covariances between stochastic futures prices and interest rates. Thus, the relationship between current forward and futures prices is time dependent in the CIR framework.

With a single amendment, Levy (1989) uses the same set of assumptions employed by CIR. Levy points out that, if only the next day's interest rate were known with certainty, forward and futures prices could be arbitrated risklessly. Foreknowledge of tomorrow's prevailing rate of interest would be used to structure a perfect hedge ratio using fractional futures positions. This hedge ratio would be rebalanced each day. Thus, in contrast to Black's (1976) approach,² the entire time path of interest rates need not be deterministic for forward and futures contracts to be arbitrated without risk. Acknowledging the impossibility of possessing the foreknowledge necessary to implement a riskless hedge ratio, Levy (1989) nevertheless argues that the uncertainty associated with one-day interest forecasts should be sufficiently small to relegate forward-futures pricing differentials to the realm of empirical inconsequence. Thus, Levy shifts the focus on the relation between forward and futures prices from effects rendered by an entire time path of stochastic interest rates to consideration of the next day's rate influence alone. Consequently, and in contrast to CIR, Levy maintains that forward and futures prices should be indistinguishable. It follows that, through an elimination of the interest rate time path as a pricing factor, Levy provides a theoretical basis for the time independency of forward-futures price differentials. In Levy's view, empirical findings support the contention that actual price differentials that do arise will be statistically and nonsystematic.

Levy's theoretical insight has significant implications. For example, Chang and Chang (1990), Park and Chen (1985), and Cornell and Reinganum (1981) test for resettlement influences in the FX markets. They report insignificant pricing effects. The event methodologies employed in these studies are based on an assumption that forward-futures price differences are independent with respect to time. The results obtained are claimed to be consistent with inferences attributable to CIR. CIR contend, however, that "with a constant interest rate, the two are essentially equivalent and forward prices are equal to futures prices, but in general this is not true." The apparent conflict between theory and application is reconciled most reasonably by concluding that the preceding empirical investigations appear to document the practical irrelevance of resettlement. They do not necessarily affirm the CIR results. On the other hand, if these tests of resettlement are interpreted from the perspective provided by Levy's framework, it appears that a far stronger, demonstrated linkage between theory and application could be claimed.

Also from an arbitrage perspective, Morgan (1981) models forward-futures price relationships under the specification that capital markets merely are efficient. He states that forward and futures prices differences "must" occur; and, the current futures price is in fact dependent on the entire future time paths of stochastic interest and futures rates. Consequently, autocorrelation in time series of the contracts' price differences is to be expected. Unfortunately, the causes underlying this condition are not given explicit consideration. Either uncertain interest or futures movements are sufficient to generate spreads between forward and futures prices. The current price of a futures contract, however, will be influenced by both conditions. Morgan's pricing inferences are attainable only when futures contracts are modeled as indivisible [Polakoff (1991)]. In Polakoff's view, the CIR, Morgan, and Levy models can be rendered equivalent only through a reconciliation of the divisibility assumption, with Morgan's framework emerging as the most relevant to the conduct of tests for reset-

²Flesaker (1991) demonstrates that Levy's treatment is tantamount to the assumption of a deterministic term structure.

tlement effects. As demonstrated later, the theoretical implications arising from indivisibility extend beyond treatment accorded by Morgan.

Kane (1980) discusses the role of asymmetric taxes and contract performance guarantees as potential contributors to forward-futures price divergence. In limiting the focus of this study to the implications implied by resettlement and indivisibility, such additional imperfections are not dealt with here. Instead, the analysis is confined to states documented in markets such as FX. In the FX markets, there is disagreement as to whether resettlement constitutes an empirically significant pricing influence. However, there is a consensus that the reported prices of FX forward and futures contracts are not influenced statistically by the presence of asymmetries attributable to differential cross-market states of efficiency, taxation, segmentation, default risk, transactions costs, or liquidity [see Polakoff and Grier (1991); Cornell and Reinganum (1981); and Denis (1976)]. On the other hand, the preceding imperfections are documented extensively by empirical studies of Treasury-bill forward-futures price relationships [Allen and Thurston (1988); Kamara (1988); Capozza and Cornell (1979); Branch (1978); Lang and Rasche (1978)].

Using documented states of the FX markets as the basis, forward-futures states of informational efficiency, taxation, segmentation, default risk, transactions costs, and liquidity are treated as symmetric. Resettlement and indivisibility remain as pertinent pricing frictions.

THE PURE RESETTLEMENT MODEL

CIR's Proposition 6 is used to illustrate the theoretical factors influencing the relation between forward and futures prices, to provide a linkage among disparate theoretical treatments, and serve as the underlying foundation for the discussion of the autocorrelation property. While the final treatment accorded is consistent with the absence of imperfections documented in markets for FX,³ the generality of inferences obtained is not compromised by the incorporation of such additional frictions.

Arbitrage pricing constitutes the underpinnings for the cash-flow relationship depicted by CIR's Proposition 6. Accordingly, suppose a short position in a single forward contract is taken at time t . Against this short position, a long position is taken in $A(j)$ futures contracts in each period $j, j = t, \dots, s - 1$. CIR use $P(j)$ to represent simultaneously each successive futures position and the price of a corresponding default-free discount bond that matures at date s . Thus, it is implicit that futures divisibility is assumed, since the maturity value of all discount bonds in the model is \$1. That is, at times $j, j = t, \dots, s - 1$, the current value of the discount bonds is less than \$1. To avoid confusion and underscore the role of indivisibility, $A(j)$ is substituted for the corresponding term in the CIR treatment. In expression (1) below, $A(j)$ represents the rebalanced futures position at time $j, j = t, \dots, s - 1$, and $A(j)$ precisely corresponds to the fractional equivalent of the current price of a single, default-free discount bond that matures at time s . Each fractional futures position is liquidated after one period. The (possibly negative) proceeds are placed in bonds with

³ Interest rates for borrowing and lending diverge when capital markets are not perfect; however, this condition serves to strengthen the final inferences obtained in this section and does not compromise the generality of our analysis. In assuming that markets are perfect, CIR use Treasury-bills to represent the financing (reinvestment) costs (returns) arising from futures resettlement. In actual practice, Capozza and Cornell (1979) note that government securities dealers charge 50 basis points over the Treasury-bill rate for financing costs, although Allen and Thurston (1988) maintain that the term repo rate is a more accurate reflection of financing costs in the government bill market.

maturity date s . This process replicates the daily resettlement mechanism associated with futures contracts. At time s , the payoff to the preceding arbitrage strategy is

$$\begin{aligned}
 & G(t) - G(s) + \sum_{j=t}^{s-1} A(j) [H(j+1) - H(j)] [1/P(j+1)] \\
 &= G(t) - G(s) + \sum_{j=t}^{s-1} [H(j+1) - H(j)] \\
 &\quad + \sum_{j=t}^{s-1} [H(j+1) - H(j)] [[A(j)/P(j+1)] - 1] \\
 &= H(s) - G(s) + G(t) - H(t) \\
 &\quad + \sum_{j=t}^{s-1} [H(j+1) - H(j)] [[A(j)/P(j+1)] - 1]
 \end{aligned} \tag{1}$$

where:

- $A(j)$ = futures position at time j ,
- $P(j)$ = price at time j of a default-free discount bond with \$1 maturity value,
- $G(t)$ = forward price at period t ,
- $H(t)$ = futures price at period t ,
- s = maturity date of the forward and futures contracts, and
- t = current time period.

Following the CIR presentation, the above strategy requires no investment, hence, the current value of its corresponding payoff must be zero. Through arbitrage, $G(s)$ and $H(s)$ must converge to the identical cash-market price, and the current value of a certain payment of $G(t) - H(t)$ at time s is $P(t) [G(t) - H(t)]$. Consequently, $G(t) - H(t)$ is the current value at time s of

$$- \sum_{j=t}^{s-1} [H(j+1) - H(j)] [[A(j)/P(j+1)] - 1] / P(t) \tag{2}$$

Expression (2) corresponds to CIR's Proposition 6. In the CIR treatment, expression (2) is to be considered stochastic, and it is CIR's position that forward and futures prices generally must differ when interest rates are nondeterministic. Although expression (2) is treated as stochastic by CIR, inferences from various segments in the literature can be replicated by altering the assumptions underlying this cash-flow relationship.

If futures prices are stochastic and interest rates are assumed to be constant, successive bonds will have one less period to maturity than is the case for their immediate predecessors. Hence, the prices of two succeeding bonds will differ slightly, despite the fact that interest rates are constant on an annualized basis. Nevertheless, as long as interest rates are known to be constant, at each time, j , a perfect futures hedge ratio can be taken in $P(j+1)$ contracts. Thus, each of the summed $[A(j)/P(j+1)] - 1$ terms from expression (2) can be made equal to zero. Therefore, the entire relationship can be made equal to zero, and $G(t) = H(t)$. By the Law of One Price, forward and futures prices must be the same because their payouts could be arbitrated so as to be identical. This result forms the basis of Black's conclusions, which are predicated on the twin assumptions of futures divisibility and a constant interest rate.

Let futures contracts remain divisible, futures prices be stochastic, and interest rates be deterministic but nonconstant. This set of assumptions does not alter Black's results. The individual $[A(j)/P(j+1)] - 1$ terms from expression (2) again can be made equal to zero, because $A(j)$ is adjustable each day across the entire time path of interest rates so as to render $A(j) = P(j+1)$. Hence, current forward and futures prices must remain the same. From Levy's perspective, if both futures prices and interest rates are stochastic and futures contracts are divisible, only the next day's interest rate need be known for $A(j)$ to be set equal to $P(j+1)$. Knowledge of the next day's stochastic interest rate is unattainable according to Levy. However, the variance associated with forecasting this rate will be sufficiently small to render inconsequential and nonsystemic differences between forward and futures prices.⁴ As long as futures contracts are divisible, therefore, Levy maintains that the entire time path of stochastic interest rates becomes irrelevant with respect to processes that determine forward-futures price differentials.

Because the existence of futures contracts fundamentally depends on the instrument's standardization features, a theoretical model must incorporate indivisibility to provide consonance with economic theory. When futures contracts are treated as indivisible, the $A(j)$ term in expression (2) equals 1,⁵ and $G(t) - H(t)$ is the current value at time s of

$$- \sum_{j=t}^{s-1} [H(j+1) - H(j)] [[1/P(j+1)] - 1]/P(t) \quad (3)$$

Assume, for the moment, that futures prices and interest rates are stochastic, but futures contracts are indivisible. Given the relationship depicted in expression (3), neither the individual $[H(j+1) - H(j)]$ terms nor their sum can be taken as zero

⁴Fleasaker (1991) questions whether Levy's pricing inferences (with respect to the variance of one-day forecasts) can be considered correct; however, he does not present an alternative explanation in the context of a discrete-time model.

⁵We represent $A(j)$ on a one-contract basis; however, consideration of several institutional characteristics will demonstrate why scale effects do not alter the generality of the analysis presented.

Discount rates on Treasury-bills are quoted to four decimal places in the market. If expression (3) is multiplied by 10,000, then, in a deterministic setting, $A(j)$ can be positioned repeatedly on a nonfractional basis so as to be set equal to $P(j+1)$. Under these circumstances, the $[A(j)/P(j+1) - 1]$ term will be zero, and forward and futures prices must be the same. However, multiplying expression (3) by 10,000 results in open futures positions that typically exceed arbitrageurs' regulatory limits. For example, class B members of the International Monetary Market are the institutional counterpart of the theoretical relationship represented by expression (3). They engage exclusively in FX forward-futures arbitrage. However, the maximum number of open FX futures contracts permitted by the International Monetary Market is 6000 for all maturity dates. If discount rates were 10% at the initiation of a six-month forward-futures arbitrage, an original position in 9524 futures contracts would be required to circumvent the effects of futures indivisibility. All such contracts would share a single maturity date. Such transactions do not occur in practice.

Thus, the positions required to negate the pricing effects of indivisibility bear no relation to existing market microstructures. We note that, at no time prior to 1982, did FX futures daily volume totals cumulatively exceed 10,000 contracts for an underlying currency. Moreover, as of 1989, cumulative daily volume has never reached 1000 contracts for any FX futures 180-day maturities. FX forward-futures price differentials do not display significantly different characteristics subsequent to 1982, as further underscored by the similarity of results obtained in the Cornell and Reinganum (1981), Park and Chen (1985), and Chang and Chang (1990) studies in this area; therefore, it is reasonable to infer that market volume has not played a statistically measurable role in determining the size of these particular spreads. Thus, only on a noneconomic basis can indivisibility be demonstrated to be irrelevant with respect to fundamental theoretical influences on forward-futures price relationships.

in the general case. The individual $[1/P(j + 1) - 1]$ terms and their sum cannot equal zero. Thus, the individual crossproducts of the futures and interest rate terms and the crossproduct sum of these terms usually are not equal to zero.⁶ As a result, the time paths of stochastic interest rates and futures prices are shown to exert an influence on the net resettlement payout, hence, the current forward-futures rate differential. This is the conclusion reached by Morgan, whose findings rely on the implicit assumption that futures contracts are indivisible. The preceding conditions also underscore a significant point of departure from the CIR inferences. Stochastic futures prices and interest rates can display zero local covariances, yet forward and futures prices still must differ in the general case as a direct consequence of indivisibility.

It is unnecessary, however, for futures prices and interest rates to be stochastic to conclude that forward and futures prices should diverge. To illustrate the conditions sufficient to generate price divergence in a cost-of-carry model of forward and futures prices,⁷ futures contracts continue to be considered as indivisible. Markets are assumed to correspond to conditions documented in FX. For the time being, and for the sake of illustration, both futures prices and interest rates are assumed to be constant. When this condition applies, each $[H(j + 1) - H(j)]$ term in expression (3) must be zero.⁸ Hence, the crossproduct of this term and the interest rate term must be zero, and $G(t)$ must equal $H(t)$. Forward prices must equal futures prices.

If futures prices are held constant and interest rates are deterministic but nonconstant, then $G(t)$ must still equal $H(t)$, because each $[H(j + 1) - H(j)]$ term will be zero. The crossproduct of these futures and interest rate terms must equal zero.

Now let futures prices be deterministic but nonconstant. If interest rates are held constant, forward prices will not equal futures prices. The individual $[H(j + 1) - H(j)]$ terms will no longer be zero in the general case. Therefore, the crossproduct of these and the constant interest rate terms, or $[1/P(j + 1) - 1]$, generally will be nonzero as well. Thus, all that is required to render theoretical divergences in price between forward and futures contracts is for futures prices to be nonconstant and for the underlying contracts to be indivisible. This result will remain unaffected whether interest rates are constant, deterministic but nonconstant, or stochastic.

THE THEORETICAL BASIS UNDERLYING AUTOCORRELATION IN TIME SERIES OF FORWARD-FUTURES PRICE DIFFERENCES

It is shown above that, when futures prices are deterministic and interest rates are constant, the net resettlement payout is a function of the summed products of the

⁶In the purely deterministic case, the crossproduct sum will be zero only if linearly independent futures prices and the sequence of coefficients, $[1/P(j + 1) - 1]$, form the unique combination that results in a zero sum. Under stochastic states, a zero sum similarly will occur only if an analogous, unique combination is formed. This condition is not to be confused with CIRs' inferences, which posit that zero local covariances between interest rates and futures prices will result in zero-valued price differentials. When indivisibility is introduced into the modeling process, forward prices generally will not equal futures prices, regardless of local covariance relationships.

⁷It is implicit in cost-of-carry representations that underlying spot commodities are storable. When commodities are not storable costlessly, the difference between their storage rates and interest carrying charges determines forward and futures price relationships.

⁸This state implies a restrictive relationship governing the behavior of spot prices, namely, that they increase at the risk-free rate (in a zero storage model where rental rates equal the risk-free rate).

(indivisible) resettled futures and interest rate time series over a given interval. Under these circumstances, current forward and futures prices will differ precisely by the value of the net resettlement payout. Put differently, the characteristics surrounding the net interest payout correspond exactly to those displayed by the current forward-futures price differential. Generally, the net interest payout will be nonzero, but this result will be attained only when futures contracts are indivisible. Since the nonzero resettlement value is determined by processes that occur over a future time interval, individual, deterministic forward-futures price differences informally are shown to be time dependent. This result is consistent with Morgan's position.

To provide insight into properties documented by several empirical studies of forward-futures price differentials, the fundamental bases underlying the manner in which autocorrelation tends to be induced across a time series of such differences are explored now. To this end, the analysis again begins under an assumption of deterministic conditions.

Digressing for a moment, if time series of individual forward and futures prices are to be considered strictly comparable, the underlying contracts must not only be written for identical underlying commodity units, but their term-to-delivery must coincide as well in each distinct time period. Over these separate periods, the futures contract will be resettled daily. The resettlement process should affect the current forward-futures price differential in the manner described in the preceding sections. Subtracting delivery-matched futures from forward prices yields a single time series that corresponds to treatment accorded by empirical analyses in this area.⁹

Accordingly, consider a single series of deterministic futures prices that is decomposed into two sequences. Both sequences have identical terms to delivery and occur over successive time periods. The initial futures prices in each time series are $H(t)$ and $H(y)$. Prices $H(t + 1), H(t + 2), \dots, H(s)$ and $H(y + 1), H(y + 2), \dots, H(z)$ represent the daily resettlement paths of original positions $H(t)$ and $H(y)$, respectively. For the sake of simplicity, a flat interest rate term structure is assumed for the time being. That is, all interest rates are constant on an annualized basis. Equation (1) represents an extension (over two distinct, equal time periods) of expression (3) and demonstrates that the joint product of two resettled futures price series and constant interest rates will be linearly independent only when futures contracts are treated as divisible.

$$\sum_{j=t}^{s-1} [[H(j) - H(j + 1)][A(j)/P(j + 1) - 1]] \times \sum_{k=y}^{z-1} [[H(k) - H(k + 1)][A(k)/P(k + 1) - 1]] = 0 \quad (1)$$

⁹Forward and futures prices on the same day are strictly comparable only in instances where the contracts' deliveries coincide [Chang and Chang (1990)]. For an entire time series of prices corresponding to the life of a particular futures contract, at most, a single datum matches the term to delivery of similar sequence of forward prices. In the empirical portion of this study, forward and futures contracts are maturity matched in accordance with the procedure implemented by Park and Chen and Cornell and Reinganum. While this process does not represent contract matching in the manner advocated by Chang and Chang, we believe our results are not compromised since the same conclusions are reached in the Chang and Chang, Park and Chen, and Cornell and Reinganum studies. Matching on a delivery basis provides roughly half the sample points that can be obtained when FX forward and futures contracts are maturity matched. In turn, for the sample period covered, maturity matching yields only 32 to 39 sample points for separate maturity categories in each currency.

where:

$$\begin{aligned}
 H(t) &= \text{futures price today, with maturity at time } s, \\
 H(z) &= \text{futures price at time } y, \text{ with maturity at time } z, \\
 P(j+1) &= P(k+1), \\
 A(j) &= A(k), \\
 t &= 0, 1, 2, \dots, s, \\
 y &= 0, 1, 2, \dots, z, \\
 y &> s, \text{ and} \\
 s - t &= y - z.
 \end{aligned}$$

As demonstrated before, $A(j)$ can be set equal to $P(j+1)$ when interest rates are known to be constant. Similarly, $A(k)$ can be set equal to $P(k+1)$. As a result, the product of each $H(j)$ resettlement and its corresponding interest rate expression is zero. The same condition applies to each $H(k)$ resettlement and its corresponding interest rate expression. Therefore, the summed crossproducts of all terms in equation (1) must be zero. The individual, underlying net interest payouts are zero; equivalently, a time series of forward-futures price differentials must also be zero. When futures contracts are modeled as indivisible, however, $A(j) = A(k) = 1$, and equation (1) must be expressed as an inequality.

$$\begin{aligned}
 &\sum_{t=j}^{s-1} [[H(j) - H(j+1)][1/P(j+1) - 1]] \\
 &\quad \times \sum_{y=k}^{z-1} [[H(k) - H(k+1)][1/P(k+1) - 1]] \neq 0 \quad (2)
 \end{aligned}$$

Each interest rate expression in the above inequality no longer can equal zero. Provided futures prices are deterministic but nonconstant, each futures resettlement cannot equal zero. Therefore, the product of each interest rate and resettlement expression does not equal zero. Thus, the summed crossproducts of all terms in the inequality generally will be nonzero as well. Consequently, the joint product of resettled futures prices and constant interest rates will be linearly dependent in the general case when futures contracts are indivisible, and a nonzero time series of forward-futures price differentials generally must result.

Suppose futures contracts remain indivisible, but futures prices and interest rates are taken to be stochastic. For the (now stochastic) series considered previously, let

$$\begin{aligned}
 &\text{cov}[j_t^*, k_y^*] = \quad (3) \\
 &E \left[\sum_{t=j}^{s-1} [[H(j^*) - H(j+1^*)][1/P(j+1^*) - 1] - \bar{j}] \right. \\
 &\quad \left. \times \sum_{y=k}^{z-1} [[H(k^*) - H(k+1^*)][1/P(k+1^*) - 1] - \bar{k}] \right]
 \end{aligned}$$

where:

$$\begin{aligned}
 \bar{j} &= \text{the mean of } \sum_{t=j}^{s-1} [H(j^*) - H(j+1^*)][1/P(j+1^*) - 1] \\
 \bar{k} &= \text{the mean of } \sum_{y=k}^{z-1} [H(k^*) - H(k+1^*)][1/P(k+1^*) - 1]
 \end{aligned}$$

$$j_t^* = [H(j^*) - H(j + 1^*)][1/P(j + 1^*) - 1], \text{ over } t, t + 1, \dots, s - 1$$

$$k_y^* = [H(k^*) - H(k + 1^*)][1/P(k + 1^*) - 1], \text{ over } y, y + 1, \dots, z - 1$$

E = expectations operator

$*$ = stochastic

The autocorrelation function, $\rho(q)$, provides a measure of interdependency among neighboring data points in a time series. For the stochastic series set forth above, $\rho(q)$ is represented by the following inequality.

$$\rho(q) = \text{cov}[j_t^*, k_y^*] / \sigma_{j_t} \sigma_{k_y} \neq 0 \quad (4)$$

where: σ_{j_t} = one standard deviation about $\bar{j}$, and σ_{k_y} = one standard deviation about $\bar{k}$.

Properties underlying linear dependency (between the joint futures and interest rate series) in the deterministic case tend to induce autocorrelation when conditions are stochastic. Consistent with deterministic states, stochastic movements within $H(j^*)$ and $H(k^*)$ generally will yield nonzero values for these separate terms when futures contracts are indivisible. As before, each futures resettlement generally will be nonzero. Each interest rate term, $[1/P(j + 1^*) - 1]$ and $[1/P(k + 1^*) - 1]$, must be nonzero. Therefore, $\text{cov}[j_t^*, k_y^*]$, is nonzero as well in the general case. Both σ_{j_t} and σ_{k_y} are nonzero because futures prices are stochastic; hence, there is dispersion about the means $\bar{j}$ and $\bar{k}$, respectively. As a consequence, the autocorrelation function generally cannot be equal to zero. The futures resettlement sequences underlie a time series of nonzero, expected net interest payouts. In turn, these payouts form the theoretical basis for an individual time series of nonzero forward-futures price differentials. As demonstrated, this time series is internally biased toward an autocorrelated state, a condition fundamentally attributable to the indivisibility of futures contracts. Again, in contrast to the CIR model, the preceding condition applies even when futures prices and interest rates display zero local covariances. That is, the preceding condition does not serve to induce a zero-valued autocorrelation function. This finding is used in the following section as a motivation for testing whether time series of FX forward-futures price differences display empirical properties consistent with the theoretical bias described.

EVIDENCE FROM THE FX MARKETS

Examined separately, daily FX forward price distributions are identified as nonstationary by Westerfield (1977). Hall, Brorsen, and Irwin (1989) find that time series of daily FX futures price distributions correspond to the mixture of normals hypothesis. As noted earlier, however, the forward-futures price-differential component addressed in this study does not represent a daily difference. Unfortunately, direct comparisons of FX forward and futures prices cannot be properly conducted on such a basis. Because the maturities for forward and futures contracts are reported on different bases, there is, at best, one day during the life of the futures when its maturity can be matched to a sequence of otherwise comparable forwards. To illustrate, suppose that two sequences of daily three-month forward and futures prices are compared. Each successive forward contract matures in three months. Each successive futures contract declines in maturity until its settlement date is reached. When three-month contracts or those of any term are used, four compa-

table forward-futures maturities at most can be retrieved from a multiyear list of rates. This restriction on available data points is a constraint to which all direct comparisons of FX forward and futures prices are subject.

To date, the Chang and Chang, Park and Chen, and Cornell and Reinganum studies are the most noteworthy investigations into whether forward-futures price differences are caused by resettlement. Their studies employ FX data, since the relative absence of imperfections displayed in these markets is considered ideal for the purpose of isolating resettlement pricing influences [Cornell and Reinganum (1981)]. In each of these analyses, an ordinary *t*-statistic is employed to determine if mean price differentials of FX forward and futures rates differ significantly from zero. Mean differences are obtained by summing subtracted values of term-matched forward and futures rates across a time series sample. While this procedure may generate distributional stationarity for the differenced series, there is reason to suspect that autocorrelation remains.

In their investigations of Treasury-bill forward and futures prices, Vignola and Dale (1979), Rendleman and Carabini (1979), and Lang and Rasche (1978) note the inappropriateness of using the ordinary *t*-statistic as a measure when classical distributional assumptions are violated. They cite the frequent sign reversals that occur when bill forward and futures rates are subtracted from each other for the purpose of obtaining a mean differential. Polakoff and Grier (1991) note that these reversals are also common when term-matched FX forward and futures rates are subtracted. This condition is suggestive of autocorrelation, a property not accounted for by the ordinary *t*-statistic.

In Tables I through IV below, time series are calculated in the usual manner for 30-, 60-, 90-, and 180-day British pound, German mark, Japanese yen, and Swiss franc forward-futures price differentials. As of 1980, single futures contracts for British pounds, German marks, Japanese yen, and Swiss francs represent 25,000, 125,000, 12,500,000, and 125,000 underlying foreign currency units, respectively. On a dollar basis (using prevailing spot rates on January 2, 1980), these foreign currency units correspond to \$56,075, \$72,950, \$52,459, and \$79,213. Forward and futures prices are obtained from the International Monetary Market for the period May 1976 through December 1984. Consistent with Cornell and Reinganum, one-year contracts are not included due to the potential for tax-induced pricing distortions. A total of 16 forward-futures time series separately are subject to the Lyung-Box Q-test. This statistic is designed to assess whether all underlying autocorrelation coefficients within a given series can be rejected for white-noise properties. At the 10% cutoff level traditionally employed, only two series (180-day Japanese yen and 180-day Swiss franc) cannot be rejected as being white noise.

In view of the properties characterizing FX forward and futures prices (i.e., no documented cross-market imperfections other than resettlement and indivisibility),

Table I
BRITISH POUND FORWARD-FUTURES PRICE-DIFFERENTIAL TIME SERIES

	30 Days	60 Days	90 Days	180 Days
Observations	39	39	39	38
Q-probability	0.400	0.320	0.660	0.138

Q-probability indicates the significance level for the Lyung-Box Q-statistic.

Table II
GERMAN MARK FORWARD-FUTURES PRICE-DIFFERENTIAL TIME SERIES

	30 Days	60 Days	90 Days	180 Days
Observations	37	37	37	37
Q-probability	0.014	0.854	0.003	0.808

Q-probability indicates the significance level for the Lyung-Box Q-statistic.

Table III
JAPANESE YEN FORWARD-FUTURES PRICE-DIFFERENTIAL TIME SERIES

	30 Days	60 Days	90 Days	180 Days
Observations	36	35	37	32
Q-probability	0.012	0.385	0.784	0.909*

Q-probability indicates the significance level for the Lyung-Box Q-statistic.

*Nonrejection of null hypothesis that series is white noise (at 10% cutoff level).

Table IV
SWISS FRANC FORWARD-FUTURES PRICE-DIFFERENTIAL TIME SERIES

	30 Days	60 Days	90 Days	180 Days
Observations	38	38	39	38
Q-probability	0.083	0.757	0.754	0.999*

Q-probability indicates the significance level for the Lyung-Box Q-statistic.

*Nonrejection of null hypothesis that series is white noise (at 10% cutoff level).

the above finding is most reasonably attributed to factors specified in this article's theoretical treatment of the autocorrelation bias.

SUMMARY AND CONCLUSION

Autocorrelated time series of forward-futures price differentials remain inadequately explained at the theoretical level. This study attempts to demonstrate that futures indivisibility constitutes the fundamental condition underlying the autocorrelation property.

Futures contracts should be treated as indivisible on theoretical grounds and are actually traded in this manner. Major participants usually conduct forward trades in a similar fashion. In fact, both forward and futures contracts are tested empirically using data reported on a single-contract basis. Absent the indivisibility constraint, the time paths of interest and futures rates should not exert systemic influences on forward-futures price differentials modeled in discrete time. As a logical consequence, no theoretical explanation exists for the documented presence of autocorrelation within actual time series of such differences.

In theory, net interest payouts arising from futures resettlement represent the foundation underlying price differences between (indivisible) forward and futures contracts. Under deterministic states, net interest payouts will be nonzero only when indivisibility is incorporated. Given such circumstances, the joint time paths of resettled futures prices and interest rates precisely determine forward-futures price

differentials. When conditions are stochastic and futures contracts are divisible, arguments have been advanced that net interest payouts will be insignificant and nonsystemic. Consistent with these arguments, time series of forward-futures price differences probably should not manifest significant time-dependency traits. This theoretical inference, however, is contradicted by applied time-series analyses. As a careful theoretical examination of the autocorrelation function reveals, however, a bias toward time dependency in time series of forward-futures price differentials does appear when futures contracts are treated as indivisible. This bias is to be expected, even where futures prices and interest rates display zero local covariances. Once the preceding perspective is established, it is not surprising that actual time series of forward-futures price differences usually are found to be autocorrelated. Empirical and theoretical harmony in this area is achieved only when futures indivisibility is taken explicitly into consideration.

Bibliography

- Allen, L., and Thurston, T. (1988): "Cash-Futures Arbitrage and Forward-Futures Spreads in the Treasury Bill Market," *The Journal of Futures Markets*, 8:563-573.
- Black, F. (1976): "The Pricing of Commodity Contracts," *Journal of Financial Economics*, 3:167-179.
- Branch, B. (1978): "Testing the Unbiased Expectations Theory of Interest," *The Financial Review*, 13: 51-66.
- Cappozza, D., and Cornell, B. (1979): "Treasury Bill Pricing in the Spot and Futures Markets," *The Review of Economics and Statistics*, 61:513-520.
- Chang, C., and Chang, J. (1990): "Forward and Futures Prices: Evidence from the Foreign Exchange Market," *The Journal of Finance*, 45:1333-1336.
- Cornell, B., and Reinganum, M. (1981): "Forward Prices and Futures Prices: Evidence from the Foreign Exchange Market," *The Journal of Finance*, 36:1035-1045.
- Cox, J., Ingersoll, J., and Ross, S. (1981): "The Relation between Forward Prices and Futures Prices," *Journal of Financial Economics*, 9:321-346.
- Denis, J. (1976): "How Well Does the IMM Track the Interbank Forward Market?" *Financial Analysis Journal*, 50-54.
- Diz, F., and Polakoff, M. (1991): "A Comparison of Foreign Exchange Forward and Futures Prices: New Evidence." *Working Paper*, Syracuse University, Syracuse, NY.
- Fama, E. (1976): "Forward Rates as Predictors of Futures Spot Rates," *Journal of Financial Economics*, 361-377.
- Flesaker, B. (1991): "The Relationship between Forward and Futures Contracts: A Comment," *The Journal of Futures Markets*, 11:113-115.
- French, K. (1983): "A Comparison of Futures and Forward Prices," *Journal of Financial Economics*, 12:311-342.
- Hall, J., Brorsen, B., and Irwin, S. (1989): "The Distribution of Futures Prices: A Test of the Stable Paretian and Mixture of Normals Hypothesis," *Journal of Financial and Quantitative Analysis*, 24:105-116.
- Hodrick, R., and Srivastava, S. (1987): "Foreign Currency Futures," *Journal of International Economics*, 22:1-24.
- Hodrick, R. (1984): "An Investigation of Risk and Return in Forward Foreign Exchange," *Journal of International Money and Finance*, 3:5-29.
- Hsieh, D. (1990): "Implications of Observed Properties of Daily Exchange Rate Changes," *Journal of International Financial Markets, Institutions, & Money*, 1:61-71.

- Hsieh, D. (1984): "An Investigation of Risk and Return in Forward Foreign Exchange," *Journal of International Money and Finance*, 3:5-29.
- Jarrow, R., and Oldfield, G. (1981): "Forward Contracts and Futures Contracts," *Journal of Financial Economics*, 9:373-382.
- Kamara, A. (1988): "Market Trading Structures and Asset Pricing: Evidence from the Treasury-Bill Markets," *The Review of Financial Studies*, 1:357-375.
- Kane, E. (1980): "Market Incompleteness and Divergences Between Forward and Futures Interest Rates," *The Journal of Finance*, 35:221-234.
- Lang, R., and Rasche, R. "A Comparison of Yields on Futures Contracts and Implied Forward Rates," *The Federal Reserve Bank of St. Louis Review*, 60:21-30.
- Levy, A. (1989): "A Note on the Relationship between Forward and Futures Contracts," *The Journal of Futures Markets*, 9:171-173.
- Margrabe, W. (1976): "A Theory of Forward and Futures Prices." *Working Paper*, The Wharton School, University of Pennsylvania, Philadelphia, PA.
- Morgan, G. (1981): "Forward and Futures Pricing of Treasury Bills," *Journal of Banking and Finance*, 5:483-496.
- Park, H., and Chen, A. (1985): "Differences Between Forward and Futures Prices: A Further Investigation of the Marking-to-Market Effects," *The Journal of Futures Markets*, 5:77-83.
- Polakoff, M. (1991): "A Note on the Role of Futures Indivisibility: Reconciling the Theoretical Literature," *The Journal of Futures Markets*, 11:117-120.
- Polakoff, M., and Diz, F. (1991): "The Effect of Futures Indivisibility on Forward-Futures Price Relationships." *Working Paper*, Syracuse University, Syracuse, NY.
- Polakoff, M., and Grier, P. (to appear) "A Comparison of Foreign Exchange Forward and Futures Prices," *Journal of Banking and Finance*.
- Rendleman, Jr., R., and Carabini, C. (1979): "The Efficiency of the Treasury Bill Futures Market," *The Journal of Finance*, 34:895-914.
- Richard, S. F., and Sundaresan, M. (1981): "A Continuous Time Equilibrium Model of Forward Prices and Futures Prices in a Multigood Economy," *Journal of Financial Economics*, 9:373-382.
- Rubinstein, M. (1987): "Derivative Assets Analysis," *Economic Perspectives*, 1:73-93.
- Sevenson, R., and Bear, R. (1970): "Commodity Futures—Trends or Random Walks," *The Journal of Finance*, 65-81.
- Silber, W. (1985): "The Economic Role of Financial Futures," in Peck, A. E. (ed.), *Futures Markets: Their Economic Role*, American Enterprise Institute for Public Policy Research, pp. 83-114.
- So, J. (1987): "The Sub-Gaussian Distribution of Currency Futures: Stable Paretian or Non-Stationary?" *The Review of Economics and Statistics*, 69:100-107.
- Telser, L., and Higinbotham, H. (1977): "Organized Futures Markets: Costs and Benefits," *Journal of Political Economy*, 85:969-1000.
- Vignola, A., and Dale, C. (1979): "Is the Futures Market for Treasury Bills Efficient?" *The Journal of Portfolio Management*, 5:78-81.
- Westerfield, J. (1977): "An Examination of Foreign Exchange Risk Under Fixed and Floating Regimes," *Journal of International Economics*, 7:181-200.

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