A Note on Constructing Spot Price Indices to Approximate Futures Prices

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INTRODUCTION

The broad-based futures contract, so called because it provides multiple delivery choices, is designed to mitigate the likelihood of various maturity problems [see Anderson (1984)] by ensuring the adequacy of a deliverable supply.[1] It is an unqualified success. This is indicated by its displacement of the narrow-based or traditional contract specifications—those that allow for the delivery of only a single grade at a particular location.[2] Nearly all contracts currently offered provide traders, usually sellers, with various delivery options. While the broad-based contract is less apt to suffer from delivery problems, either intentional or otherwise, this advantage is not without cost.[3]

The primary disadvantage with specifying broad-based delivery options is that all traders are exposed to varying amounts of delivery risk. [See Lien (1988, 1991) for a discussion on the trade-off between delivery options—and the resultant delivery risk—and the prospect of a long squeeze.] When traders contemplate initiating an open position in a broad-based futures contract, the inability to know exactly which grade will be delivered creates some additional uncertainty, i.e., delivery risk, in

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[1]More specifically, a broad-based contract is any contract that allows the delivery of multiple grades, or that allows delivery at various places, or both. Besides listing deliverable grades and/or locations, if nonpar delivery is permitted, the contract might also specify a system of premiums/discounts for settling when delivery is not at par.

[2]While the broad-based contract offers multiple grade and/or location of delivery, both the broad-based and narrow-based contracts may provide timing choices.

[3]Using broad-based delivery specifications is not a cure-all for contract delivery defects. Paul, Kahl, and Tomek (1981) recommend that, among other things, "the contract for white potatoes be broadened to permit delivery of all [grades of] white potatoes." Though the delivery specifications were expanded, the contract continued to suffer from delivery defects and eventually disappeared.

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terms of pricing the contract. (Delivery risk also subjects long hedgers to the prospect, and likely cost, of liquidating a delivered grade not matching their commercial needs.) As relatively attractive as the broad-based futures contract may be, the specification of multiple delivery options and the resultant delivery risk imply that its potential ability to perform as a hedging vehicle is reduced. In that futures markets are hedgers' markets, any impediment to hedging is a possibly serious drawback. It is clear that exchanges are faced with the challenge of designing and constructing contracts that support sufficient delivery flexibility and yet also provide an effective hedge.

In this article, a general formula for pricing broad-based futures contracts is derived. It is shown that the futures price may be decomposed into two elements: an index of deliverable grade spot prices, and a residual term or component which consists of certain covariance terms. In fact, all pricing formulas discussed in the literature can either be transformed so that they take the same appearance as this formulation, or are simply special cases of it.

To empirically evaluate any pricing formula it is necessary to impose some structure on spot prices. The standard approach is to assume the spot price series is generated by an underlying stochastic process, usually the Wiener or the lognormal.[4] Of course, one of the basic requirements of correctly applying a pricing formula is that it be consistent with the assumed stochastic structure. This requirement is not always satisfied. Some formulations are better than others. Although the parametric processes such as the Wiener or lognormal may fit the data well for certain price series, a different approach is used here which imposes restrictions on the parameters of the reduced form equations.[5] By doing so, the pricing formula is obtained with minimal assumptions on the underlying stochastic process, implying it is the most widely applicable. This approach also eases the empirical testing and, equivalently, the practical implementation of the formula.

Finally, there is the matter of how well the pricing formula approximates or tracks the futures price. The pricing method is subjected to an empirical test using Chicago Board of Trade (CBOT) wheat futures price data. It appears that it provides a basis for determining more effective hedge ratios, implying its use yields an improvement in the hedging performance of broad-based contracts, ceteris paribus.

THE BASIC ANALYSIS: A GENERAL PRICING FORMULA

Consider a futures contract that allows the delivery of multiple grades of the underlying asset. Let T denote the maturity date of the futures contract and assume there are K deliverable grades. Further, let $P_i(t)$ denote the spot price of the ith deliverable grade at time t, where $t \leq T$ and $i = 1, 2, \ldots, K$. Since every grade can be delivered against the contract, if delivery and transactions costs are zero, then by the usual arbitrage argument the price of the futures contract at maturity, $f(T)$, satisfies[6]:

$$f(T) = \min\{P_1(T), P_2(T), \ldots, P_K(T)\} \tag{1}$$

[4]The jump process is a third, and less frequently used, stochastic process. See Cox and Ross (1976) for details.

[5]This approach, in spirit, is motivated by the work of Bell and Krasker (1986).

[6]Provided the deliverable supply of each quality/grade is adequate, when transactions and delivery costs are nonzero it is possible that qualities other than the lest expensive, i.e., CTD, will be delivered. For details, see Kuhn (1988).

Using the relationships embodied in equation (1) one can construct K indicator variables, $I_j, j = 1, 2, \ldots, K$, such that $I_j = 1$ if the jth grade is cheapest to deliver (CTD) at maturity, otherwise $I_j = 0$. Mathematically:

$$I_j = \begin{cases} 1, \text{ if } P_j(T) \leq P_i(T) & \text{for every} \quad i = 1, 2, \ldots, K \\ 0, \quad \text{otherwise} \end{cases}$$

By definition, $\{I_j\}$ satisfies the following properties:

$$\text{(i)} \quad f(T) = \sum_{j=1}^{K} I_j P_j(T)$$

$$\text{(ii)} \quad \sum_{j=1}^{K} I_j = 1$$

In other words, $f(T)$ is a weighted linear combination of $P_1(T)$,

$$P_2(T), \ldots, \quad \text{and} \quad P_K(T)$$

For any date prior to the maturity date (say t), I_j is a random variable since the relative prices among deliverable grades, i.e., their rankings, at maturity, are unknown. The probability density function of I_j is appropriately modeled as a joint density function of $(P_1(T), P_2(T), \ldots, P_K(T))$ held at t.

Let $f(t)$ denote the futures price at time t. The futures price can be expressed as:

$$f(t) = E_t[f(T)] + b_t \tag{2}$$

where $E_t(\cdot)$ denotes the conditional expectation operator based on information available at t and b_t indicates the bias in the futures price. The futures market exhibits a martingale equilibrium whenever $b_t = 0$. A positive bias corresponds to a backwardation, while a negative bias is characteristic of a contango. A general case follows. However, it is worth noting that the pricing method of Cox and Ross (1976), though generally accepted in the literature [see, e.g., Boyle (1989)], is based on the assumption of a martingale equilibrium. Using equation (1), equation (2) may be rewritten as follows:

$$\begin{aligned} f(t) &= E_t\left[\sum_{j=1}^{K} I_j P_j(T)\right] + b_t \\ &= \sum_{j=1}^{K} E_t(I_j) E_t(P_j(T)) + \sum_{j=1}^{K} \mathrm{Cov}_t(I_j, P_j(T)) + b_t \end{aligned} \tag{3}$$

where $\mathrm{Cov}_t(\cdot, \cdot)$ is the covariance operator conditional on information available at t. The behavior of the deliverable-grade cash price series is characterized by assuming $E_t(P_j(T)) = \theta_j P_j(t) + c_j(t)$, for some positive $\theta_j, j = 1, 2, \ldots, K$. That is, the conditional expectation of the future cash price of grade j is a linear function of its current cash price. Many of the stochastic processes assumed to underlie cash prices infer $c_j(t) = 0$, implying the cash market is in a martingale equilibrium if $\theta_j = 1$. Otherwise, if $\theta_j > 1$ or $\theta_j < 1$, then, respectively, either a submartingale or supermartingale equilibrium are obtained. Generally, $c_j(t)$ may be positive or negative.

Given these assumptions, the futures price can be expressed as:

$$f(t) = \sum_{j=1}^{K} \theta_j E_t(I_j) P_j(t) + \left\{\sum_{j=1}^{K} E_t(I_j) c_j(t) + \sum_{j=1}^{K} \mathrm{Cov}_t(I_j, P_j(T)) + b_t\right\} \tag{4}$$

Note that, since I_j equals zero or one, $E_t(I_j)$ measures sthe probability at t that the jth deliverable grade will be CDT at the contract's maturity date. Accordingly, $E_t(I_j)$ indicates the index weight attached to the spot price of the jth deliverable grade (at t) with, of course, $0 \leq E_t(I_j) \leq 1$ and $\Sigma_{j=1}^{K} E_t(I_j) = 1$.

To be sure, equation (4) indicates that the futures contract is properly evaluated, prior to maturity, as the sum of a weighted linear combination of adjusted deliverable-grade spot prices (i.e., $\theta_j P_j(t)$) and a residual term. This term, shown in brackets, by incorporating b_t and $c_j(t)$, takes account of the influence created by bias in either the futures and cash markets, respectively. Theoretically, the bias terms may be zero, positive, or negative. However, it is usually assumed that both are zero for each deliverable grade. The residual term also shows how the futures price is influenced by expected changes in the ranking of deliverable cash prices and, subsequently, the probability that any one grade will CTD at time T. This is shown by the aggregation of various covariance terms which, unlike the market biases, cannot be assumed to be nonnegligible. In fact, by its definition, I_j is a monotonically decreasing function of $P_j(t)$. This implies $\mathrm{Cov}_t(I_j, P_j(t)) \leq 0$ for every $j = 1, 2, \ldots, K$. As a result, the residual term tends to depress the futures price. Hereafter, equation (4) is referred to as the general index pricing (GIP) formula.

SOME STOCHASTIC PROCESSES

To evaluate $\theta_j E_t(I_j)$, $c_j(t)$, and $\mathrm{Cov}_t(I_j, P_j(T))$ it is necessary to impose some structure on the underlying stochastic process generating $P_j(t)$. For example, Kamara and Siegel (1987) assume a Wiener process such that

$$P_j(t) = P_j(t-1) + u_j(t) \qquad j = 1, 2, \ldots, K \tag{5}$$

where $\{u_1(t), u_2(t), \ldots, u_K(t)\}$ is a K-variant normal vector with $E_t(u_j(t)) = 0$, $E(u_i(t)\, u_j(t)) = \sigma_{ij}$, and $E(u_i(t)\, u_j(s)) = 0$ whenever $t \neq s$. In this case

$$P_j(T) = P_j(t) + w_j(t,T) = P_j(t) + \sum_{m=t+1}^{T} u_j(m) \qquad j = 1, 2, \ldots, K, t < T \tag{6}$$

Hence, $P_j(T)$ is a normal random variable with mean $P_j(t)$ and variance $(T - t)\sigma_{jj}$ conditioned on information available at t. Thlus, $\theta_j = 1$ and $c_j(t) = 0$, assuming a Wiener process implies existence of a martingale equilibrium. For simplicity, Kamara and Siegel assume $K = 2$, in which case

$$\begin{aligned} E_t(I_1) &= Prob_t(P_1(T) \leq P_2(T)) = Prob_t(w_1(t,T) - w_2(t,T) \leq P_2(t) - P_1(t)) \\ &= \Phi((P_2(t) - P_1(t))(T-t)^{-1/2}\sigma^{-1}) \end{aligned} \tag{7}$$

where $\Phi(\cdot)$ denotes the cumulative density function of a standard normal variable and $\sigma^2 = \sigma_{11} - 2\sigma_{12} + \sigma_{22}$ is the variance of $(T-t)^{-1/2}(P_2(T) - P_1(T))$. Equation (7) clearly shows that one of the implications of assuming the two spot prices follow a Wiener process is that the index weights, $E_t(I_1)$ and $(1 - E_t(I_1))$, change over time in a nonlinear fashion. The residual term involves $\mathrm{Cov}_t(I_1, P_1(T))$ and $\mathrm{Cov}_t(I_2, P_2(T))$, both can be calculated by a formula provided in Kamara and Siegel (1987), and both are nonzero in general.

An alternative, and perhaps more frequently assumed, stochastic process is the lognormal. That is,

$$\log P_j(t) = \log P_j(t-1) + u_j(t) \qquad j = 1, 2, \ldots, K \tag{8}$$

Here, $\log P_j(T)$ is a normal random variable with mean $\log P_j(t)$ and variance $(T - t)\sigma_{jj}$, both conditional on information available at t. Using the moment generating function:

$$E_t[P_j(T)] = \exp\left(\log P_j(t) + \frac{(T - t)\sigma_{jj}}{2}\right)$$
$$= \exp\left(\frac{(T - t)\sigma_{jj}}{2}\right)P_j(t) \tag{9}$$

implying $\theta_j = \exp((T - t)\sigma_{jj}/2)$ and $c_j(t) = 0$. On the other hand, by supposing $K = 2$, the following is derived:

$$E_t(I_1) = \text{Prob}_t(P_1(T) \leq P_2(T)) = \text{Prob}_t(\log P_1(T) \leq \log P_2(T))$$
$$= \Phi\left(\log\left(\frac{P_2(t)}{P_1(t)}\right)(T - t)^{-1/2}\sigma^{-1}\right) \tag{10}$$

Again, index weights are variable over time. The residual term, which can be calculated using Lien (1986), is nonnegligible.

For the more general case where $K > 2$, some approximations to $E_t(I_j)$ and $\text{Cov}_t(I_j; P_j(T))$ are called for. Boyle and Tse (1990) apply the Clark approximation and conclude the results are reliable for certain parametric values.

ALTERNATIVE PRICING FORMULAS: NONINDEX METHODS

Kamara and Siegel (1987) and Stulz (1982) provide formulas, now well known, useful for pricing broad-based futures contracts with two deliverable grades. Each assumes a different stochastic process, Wiener and lognormal, respectively.[7] Neither formula makes light of the futures price components—an index composed of a weighted linear combination of deliverable-grade spot prices and a residual term—made explicit by the GIP formulation. Nevertheless, the appearance of both can be transformed so that each takes the form indicated by the GIP formulation. However, to apply the pricing formulas of Kamara and Siegel, and Stulz, it is necessary to verify the assumptions pertaining to the underlying stochastic processes, e.g., normality, constant variance, and independence over time. It appears verification is difficult to obtain in every data set, meaning the applicability of these models may be rather limited.

ALTERNATIVE PRICING FORMULAS: THE CTD AND CONSTANT INDEX METHODS

There are two other methods of pricing broad-based contracts: the CTD formula and what is referred to here as the constant index pricing (CIP) formula. The former is advanced by Figlewski (1985) and Toevs and Jacob (1987), the latter by Castelino and Chatterjee (1988). Both methods are essentially special cases of the GIP formulation. However, both employ the additional assumptions of $\theta_i = \theta_j = \theta$, for all i, j, and $c_j = 0$.

The CTD formulation is based on the argument that the broad-based contract is priced off the deliverable grade that is expected to be CTD at contract maturity time.

[7]In fact, the pricing formula Stulz provides is for options, but it is readily converted for use on futures.

More specifically, define $h(t)$ as follows:

$$h(t) = \theta \min\{P_1(t), P_2(t), \ldots, P_K(t)\} \tag{11}$$

The CTD method suggests using $h(t)$ to track the price of the broad-based contract. By assumption:

$$h(t) = \min\{E_t(P_1(T)), E_t(P_2(T)), \ldots, E_t(P_K(T))\} \tag{12}$$

Equation (12) is the CTD pricing formula. With the (implicit) assumption that the relative ranking of cash prices will not change, the minimum cash price at t receives a weight of one and all other prices receive a zero weight. Also with this assumption, the residual term is set essentially equal to zero. Hence, it is easy to interpret the CTD approach as a special case of the GIP formula.

To analyze the tracking ability of the CTD pricing method, return to equation (3) which states (ignoring the bias term, b_t):

$$f(t) = E_t[\min\{P_1(T), P_2(T), \ldots, P_K(T)\}] \tag{13}$$

Since $\min\{P_1(T), P_2(T), \ldots, P_K(T)\} \leq P_j(T)$ for every $j = 1, 2, \ldots, K$, one has $E_t[\min\{P_1(T), P_2(T), \ldots, P_K(T)\}] \leq E_t(P_j(T))$, which in turn implies $f(t) \leq h(t)$. Over the contract's life the futures price is less than the minimal spot price among deliverable grades,[8] meaning the CTD pricing formula theoretically overvalues the futures contract. The difference, $(h(t) - f(t))$, is generated by the value implicit in the contract's delivery options, reflecting the value shorts place on contract specifications that give them delivery choices.

For the CTD pricing method, like the pricing formulas discussed in the previous section, to be applicable, the assumptions embodied by the postulated stochastic process must be verified. To repeat, this is not always possible.

Despite its tendency to overestimate the value of the futures contract, the expected price of the CTD grade "might be adequate for tracking actual futures prices if the identity of this grade remains unchanged from the day a hedge is placed to the day the hedge is lifted" [Castelino and Chatterjee (1988)]. In this instance, the CTD grade at contract maturity is identical to that grade with the minimum price at time t. Letting c index the CTD grade, I_j becomes a degenerate random variable such that $I_j = 0$ with probability 1 if $j \neq c$, while $I_c = 1$ with probability 1. Equation (4) then reduces to $f(t) = \theta P_c(t)$ and the CTD pricing method proves to be well justified in terms of its tracking function.

However, in reality, the identity of the CTD grade changes over time with considerable frequency. This finding is discussed in the work of Barnhill and Seale (1988), Kamara and Siegel (1987), and Hegde (1989), where the use of the CTD pricing method is shown to result in an incorrect valuation of several financial and grain futures contracts.

Noting the tendency of the CTD method to produce an upward bias in the value of the futures contract, Castelino and Chatterjee (1988) propose an index pricing method that employs a linear combination of adjusted spot prices[9] of deliverable grades to approximate the futures price. Specifically, they consider the following index:

$$g(t) = \sum_{j=1}^{K} w_j \theta P_j(t) \quad \text{with} \quad \sum_{j=1}^{K} w_j = 1 \quad \text{where} \tag{14}$$

[8]This result is similar to the dominance relationship provided in Stulz (1982, p. 168).

[9]For their application, Castelino and Chatterjee employ implied forward prices.

w_j is time-invariant. Compared to equation (4), it is clear this pricing procedure does not utilize the information provided by the residual term[10] nor does it accomodate index weights that change with time. Consequently, this pricing method is referred to as the constant index pricing (CIP) formula. It is clear that the CIP formulation, like the CTD pricing formula, is a special case of the GIP approach. Again, the CTD and CIP approaches require the assumptions: $\theta_j = \theta_i = \theta$ for every $i, j = 1, 2, \ldots, K$ and $c_j(t) = 0$ for every j. Both of these assumptions (provided $\sigma_{ii} = \sigma_{jj}$ for every i, j) are supported by either the Wiener or lognormal processes. Of course, a general CTD or CIP method without the above assumptions may be analyzed analogously.

Since, as shown above, the residual term is generally negative, it holds that $f(t) \leq g(t)$. Thus, the index method of Castelino and Chatterjee, like the CTD pricing method, tends to overestimate the price of the contract prior to its maturity date.[11] Castelino and Chatterjee show empirically the CIP method outperforming the CTD in terms of tracking ability. Nonetheless, using the CIP method is not consistent with assuming either the Wiener or lognormal processes. This presents certain difficulties in terms of applying the CIP method.

However, it is possible that under certain conditions, use of the CIP method is satisfactory. For example, suppose, regardless of the price history of different deliverable grades, each has an equal chance of being the CTD grade at time T. As another example, because spot prices of deliverable grades tend to move together, it is reasonable to assume their relative ranking will not change in time; but, ex ante, any initial ranking of prices is equally likely. Either of these cases imply for every $j = 1, 2, \ldots K$, $I_j = 1$ with probability of $1/K$ and $I_j = 0$ with probability of $(K - 1)/K$; they also imply $\mathrm{Cov}_t(I_j, P_j(T)) = 0$. Consequently, ignoring the futures bias term, equation (4) can be written as:

$$f(t) = \sum_{j=1}^{K} \frac{\theta P_j(t)}{K} \tag{15}$$

which is equivalent to (14) upon choosing $w_j = 1/K$ for every j. Not only are the wieghts constant over time, they are also identical in size, making this a special case of the CIP formulation. In fact, equation (15) is the pricing formula suggested by Castelino and Chatterjee. To defend their proposed weighing scheme they invoke the "principle of insufficient information," citing the difficulty of exactly determining the density function of I_j. They proceed assuming that each deliverable grade is equally likely to be the CTD grade at the maturity date. Unfortunately, it is rather difficult to test the validity of this assumption. Perhaps a better approach is to test the time-invariant nature of the index weights directly.

In summary, both the CTD and CIP methods ignore the time-varying residual or intercept term and assign equal weights to spot prices of grades that are deliverable. Equivalently, each method assumes the probability that any one grade will be CTD remains unchanged over the life of the contract. This is a clear problem if reality is not to be ignored. The nonindex pricing formulas premitted under the two different stochastic processes—Wiener and lognormal—avoid this problem but create a separate problem. That is, these equations are highly nonlinear which makes their empirical implementation more difficult.

[10]Their approach sets the residual term equal to zero.

[11]Although both the CTD and CIP methods overestimate the futures price, Castelino and Chatterjee (1988) compare the two methods using a regression analysis on T-bond futures prices. They conclude the index method is superior based on the coefficient of determination, R^2.

For the GIP pricing formula to be an improvement over the other formulas it must be consistent with any assumptions made concerning the stochastic nature of prices or price relationships. For purposes of applicability these assumptions must be (preferably, widely) verifiable, they must be conformable to regression analysis (for the sake of practicability); and they must provide a good approximation of futures price movements. Basically, the task is to test the time-invariant nature of index weights directly, thereby avoiding some of the difficulties associated with using the other formulas.

SOME PRELIMINARIES: CHANGES IN FUTURES PRICES

In time series analysis, it is found that utilizing first differences rather than absolute levels of each variable provide better results. This finding is possibly due to the near random walk behavior of the times series for each variable. To test the ability of the GIP formulation to approximate the price of a broad-based futures contract, changes in the futures price are employed and restrictions on the stochastic nature of these changes are imposed.

The change in the futures price, denoted by $df(t)$, can be derived from equation (4) as follows[12]:

$$\begin{aligned} df(t) &= f(t) - f(t-1) \\ &= \sum_{j=1}^{K} \theta_j E_{t-1}(I_j) dP_j(t) + \sum_{j=1}^{K} [\theta_j E_t(I_j) - \theta_j E_{t-1}(I_j)] P_j(t) \\ &\quad + \sum_{j=1}^{K} [E_t(I_j) c_j(t) - E_{t-1}(I_j) c_j(t-1)] \\ &\quad + \sum_{j=1}^{K} [\mathrm{Cov}_t(I_j, P_j(T)) - \mathrm{Cov}_{t-1}(I_j, P_j(T))] + (b_t - b_{t-1}) \end{aligned} \tag{16}$$

where $dP_j(t) = P_j(t) - P_j(t-1)$ is the change in the spot prices of the jth deliverable grade occurring between time periods $t-1$ and t. Excluding the first term, let $v(t)$ denote the sum of all other terms in equation (16). Rewriting, one obtains:

$$df(t) = \sum_{j=1}^{K} \theta_j E_{t-1}(I_j) dP_j(t) + v(t) = dS(t) + v(t) \tag{17}$$

where $dS(t) = S(t) - S(t-1)$,

$$S(t) = \sum_{j=1}^{K} \theta_j E_{t-1}(I_j) P_j(t), \quad \text{and} \quad S(t-1) = \sum_{j=1}^{K} \theta_j E_{t-1}(I_j) P_j(t-1)$$

In words, $S(t)$ is an index composed of a linear combination of adjusted spot prices of deliverable grades. Furthermore, at time t, the weights used in the computation of the index are constant, being based on information available at time $t-1$.

To facilitate their empirical research, Castelino and Chatterjee assume $E_{t-1}(I_j) = 1/K$ and ignore the $v(t)$ term. Their approach can be justified if it is supposed that the joint density function of $P_1(T), P_2(T), \ldots,$ and $P_K(T)$ at time $t-1$, for any $t < T$, is independent of new information available subsequently. This implies the joint density function remains unchanged from $t-1$ to t. Also, in this case, $E_t(I_j) = E_{t-1}(I_j)$ and $\mathrm{Cov}_t(I_j, P_j(T)) = \mathrm{Cov}_{t-1}(I_j, P_j(T))$ for every $j = 1, 2, \ldots, K$.

[12]Additional terms concerning θ_j should be included in equation (16), for example, $\Sigma_{j=1}^{K} E_{t-1}(I_j) P_j(t-1) d\theta_j$. However, since $d\theta_j$ is generally rather small, these terms are ignored here.

Hence, ignoring the biases in both the futures and cash markets, equation (17) reduces to: $df(t) = dS(t)$. The sufficiency of their pricing formula is obtained if it is further assumed that $E_t(I_j) \doteq 1/K$. Of these two assumptions, they make only the latter, and it is not sufficient to eliminate the varying coefficient problem—meaning the residual term cannot be neglected.

To illustrate this a stronger assumption is adopted. Consider the possibility that the conditional density function of $P_1(T), P_2(T), \ldots, P_K(T)$ changes over time, but always remains symmetric in all of its arguments. Then,

$$E_t(I_j) = \frac{1}{K} \quad \text{and} \quad \mathrm{Cov}_t(I_j, P_j(T)) = \mathrm{Cov}_t(I_K, P_K(T))$$

for every $j = 1, 2, \ldots, K$, and equation (16) becomes:

$$\begin{aligned} df(t) &= \sum_{j=1}^{K} \theta_j dP_j(t)/K \\ &\quad + K[\mathrm{Cov}_t(I_K, P_K(T)) - \mathrm{Cov}_{t-1}(I_K, P_K(T))] \end{aligned} \tag{18}$$

The latter term varies over time and hence enables, in the context of regression analysis, modeling either a varying intercept term [which is easily facilitated by adding a disturbance term to equation (18)] or a heteroscedastic error term (if it is treated as the error term). While the CIP formulation may not be exact, it provides a good approximation in some cases.

Determining the quality of the approximation of a pricing formula is, of course, an empirical matter; but its ease of application is also important. In any case, selecting a pricing formula for practical purposes requires choosing between a convenient specification, such as the CTD and CIP formulations, and the more complicated nonindex formulations. However, using the GIP formulation provides a useful compromise to these alternatives.

AN EMPIRICAL STUDY

This empirical analysis considers a special case where $K = 2$. Also, it is assumed that $\theta_1 = \theta_2 = \theta$. This assumption is easily justified in an equilibrium framework. Equation (17) then becomes

$$\begin{aligned} df(t) &= \theta E_{t-1}(I_1)dP_1(t) + \theta E_{t-1}(I_2)dP_2(t) + v(t) \\ &= \theta E_{t-1}(I_1)dP_1(t) + (\theta - \theta E_{t-1}(I_1))dP_2(t) + v(t) \end{aligned}$$

which implies

$$df(t) - dP_2(t) = \theta E_{t-1}(I_1)(dP_1(t) - dP_2(t)) + (\theta - 1)dP_2(t) + v(t)$$

Note that $E_{t-1}(I_1)$ depends on information available at $t - 1$ and $v(t)$ on information available at t. In principle both are determined by the assumed underlying stochastic process. For empirical purposes, it is possible to construct relatively simple functions to approximate equation (19). For instance, it is possible to let $E_{t-1}(I_1) = \alpha_0 + \alpha_1 t + \alpha_2(P_1(t-1) - P_2(t-1))$ and $v(t) = \beta_0 + \beta_1 t + \beta_2 (P_1(t-1) - P_2(t-1))$. However, a quadratic function of t and $dP_1(t-1) - dP_2(t-1)$ is specified, which covers the index method as a special case.

To test the pricing formula a portion of the data base provided by the CBOT wheat futures contract is used. Despite the fact that this contract allows the delivery of 11 grades, only the two most popular, No. 2 red hard and No. 2 red soft, are used.

All data are taken from the *CBOT* 1981 *Statistical Annual*. Let $P_1(t)$ and $P_2(t)$ denote the cash prices for hard and soft wheat respectively. The time period considered is from January to March 1981.[13]

For the first modeling attempt, $P_1(t)$ and $P_2(t)$ are assumed to follow a Wiener process. The results obtained are unsatisfactory since this specification fails the normality test for all attempts. This implies that the pricing method of Kamara and Siegel (1987) is not applicable.

For the second modeling attempt, the spot price differences are assumed to follow the lognormal process. Using this specification, a satisfactory model is derived. Let $y_i(t) = \log P_i(t) - \log P_i(t-1)$ and let $z_i = y_i(t) - y_i(t-1)$, $i = 1, 2$. The results obtained for this model are:

$$z_1(t) = \underset{(0.5920)}{0.001778} - \underset{(5.35)}{0.6345}\, z_1(t-7)$$

$$R^2 = 0.389, \quad \text{D.W.} = 2.257$$

$$z_2(t) = \underset{(0.5004)}{0.001595} - \underset{(2.59)}{0.2974}\, z_2(t-1) - \underset{(4.51)}{0.3914}\, z_2(t-7)$$

$$R^2 = 0.437, \quad \text{D.W.} = 1.904$$

The numbers in parentheses indicate t-statistics. Both equations pass all the tests for serial correlation, functional form, normality, heteroscedsticity, and ARCH (2). A joint estimation via the Seeming Unrelated Regression method improves efficiency a little.[14] At this point, it is possible to derive $E_t(I_1)$ and $\text{Cov}_t(I_1, P_1(T))$. But it is rather burdensome because that process involves nine lags.

As an alternative, it is possible, by using equation (19), to specify a quadratic function of t and $dP_1(t-1) - dP_2(t-1)$ for $E_t(I_1)$ and $v(t)$. Using the identical data and, after certain model selection procedures, the following results are obtained[15]:

$$\begin{aligned} df(t) - dP_2(t) = {} & \underset{(3.89)}{0.0088}\,(dP_1(t-1) - dP_2(t-1))^2 \\ & + (\underset{(20.35)}{1.0623} + \underset{(4.88)}{0.0439}\,(dP_1(t-1) - dP_2(t-1)) \\ & - \underset{(3.58)}{0.0023}\, t(dP_1(t-1) - dP_2(t-1)))(dP_1(t) - dP_2(t)) \\ & R^2 = 0.9222, \quad D.W. = 2.245 \end{aligned}$$

Note that the coefficient associated with the term $dP_1(t) - dP_2(t)$ estimates $E_{t-1}(I_1)$. The CIP method is justified if $E_{t-1}(I_1)$ is a constant, which is clearly not the case here. In fact, the time-varying nature of $E_{t-1}(I_1)$ plays an important role in tracking futures manifest by the significance of the intercept term, meaning the residual term of equation (4) is, likewise, significant.

[13] Kamara and Siegel (1987) note several mistakes in reported cash prices after March 1981.

[14] The estimation results using the Seeming Unrelated Regression method are as follows:

$$z_1(t) = \underset{(0.58)}{0.0017} - \underset{(5.92)}{0.5636}\, z_1(t-7), \qquad R^2 = 0.3840$$

$$z_2(t) = \underset{(0.47)}{0.0015} - \underset{(2.40)}{0.1943}\, z_2(t-1) - \underset{(6.03)}{0.4142}\, z_2(t-7), \qquad R^2 = 0.4261$$

[15] Equation (19) contains the term $(\theta - 1)dP_2(t)$. For this term, the regression results show an estimate of -0.0528 with a t-statistic of 0.8928. Consequently, it is withdrawn from the final form.

CONCLUSION

This study's method of evaluating the value of the futures contract by utilizing a weighted linear combination of deliverable spot grade prices with time varying weights is easier to implement. It precludes the problem of modeling and validating any of the stochastic process assumptions. It also maintains enough flexibility in terms of possible specifications, which eliminates the problem associated with the CIP method. The usefulness of the proposed pricing formula is illustrated through an ex post analysis using CBOT wheat data. The approach also applies to ex ante evaluations of futures contracts.

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