A Multiperiod Model for the Selection of a Futures Portfolio

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INTRODUCTION

Traditional portfolio theory selects an optimal portfolio from a universe of securities using a mean/variance approach to return–risk measurement and a well-behaved, usually quadratic, utility function. Unfortunately, the traditional model, developed by Markowitz (1952, 1959), is inadequate for portfolio optimization when applied in the context of a futures portfolio. There are at least five reasons for this inadequacy. First, futures positions are taken without "investment" in the usual sense. Second, futures trading involves considerable leverage, but the leverage multiplier, that is, the commodity value covered by a contract divided by the margin required to carry it, varies over time. Third, the symmetry of futures markets allows for long and short positions, but the margin requirement is positive in both cases. The optimization technique must explicitly recognize this peculiarity of futures trading by requiring all portfolio weights to be positive. Traditional optimization techniques built around the Markowitz model treat short positions as having negative weights. Fourth, the Markowitz model is single period while the futures portfolio problem is decidedly multiperiod. Finally, the traditional utility criterion is too abstract for most practical applications.[1]

As a side point, the capital market line (CML) extension of the Markowitz model does not hold in a futures portfolio framework, because the required separation theorem does not apply to a futures portfolio. The separation theorem, Tobin (1958), applicable in a securities portfolio context, allows for the investor to move wealth freely between a risky market portfolio and a risk-free asset. Thus, the size of the risk-free position is determined by the size of the risky market position. In a futures portfolio, trading capital is held in the form of risk-free securities, and these

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[1] Others who have looked at the problems with modeling a futures portfolio include Brorsen and Lukac (1990) and Lukac and Brorsen (1990). Related issues involving cost and performance of futures funds have been addressed by Elton, Gruber, and Rentzler (1990).

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same securities are used to meet margin requirements. The size of the risk-free position is, therefore, not determined by the size of the futures position.

This article offers a model for portfolio optimization in the context of a diversified futures portfolio and an illustration of its application. With the exception of the choice criteria, the model is a generalization of the Markowitz model. This article is targeted, first and foremost, to futures fund managers, but also to managers of other types of futures portfolios, and managers of portfolios that have a futures component.

This study is structured as follows: The next section lays out the model assumptions, introduces the bulk of the notation, and defines a number of underlying relationships. The underlying relationships are deliberately crafted to mimic, as closely as possible, actual trading practice. The third section formulates the optimization problem and solves for the minimum variance set in a single-period context. The fourth section adds a multiperiod dimension to the model and describes two equivalent multiperiod choice criteria to determine the optimal position sizes. The final section applies the model to data provided by a fund manager, obtains optimized values, and interprets these values.

ASSUMPTIONS, NOTATION, AND UNDERLYING RELATIONSHIPS

Since this article is directed to the futures fund manager, it is written in language as near to the industry as possible, while retaining its academic perspective. For simplicity, it is assumed that the fund manager employs "technical" methods of price analysis in the design of trading strategies.[2]

Define a "trading regime" to be a set of trading rules. Define a "strategy" to be a trading regime applied to a specific futures contract series. The same trading regime applied to two different futures series constitutes two different strategies. Similarly, two different regimes applied to the same futures series constitutes two different strategies.

Next, define a daily strategy return as the unlevered per-unit profit from that day to the next expressed as a percentage of that day's settlement price. That is, the return from strategy i on day t, denoted $R(i,t)$, is given by equation (1)

$$R(i,t) = 100 \times \frac{D(i,t) \cdot (Z(i,t+1) - P(i,t)) + D(i,t+1) \cdot (P(i,t+1) - Z(i,t+1))}{P(i,t)} \tag{1}$$

where $D(i,t)$ and $D(i,t+1)$ denote the side of the market strategy i is on on days t and $t+1$, respectively. These "dummy" variables take the values $+1$ when long, -1 when short, and 0 when neutral (out of the market). $Z(i,t+1)$ denotes the transaction price on day $t+1$ if a trade is made on day $t+1$. If no trade is made on day $t+1$ then $Z(i,t+1)$ is the settlement price on day $t+1$. $P(i,t)$ and $P(i,t+1)$ are the settlement prices on days t and $t+1$, respectively. This formulation allows for trading strategies that transact close-to-close, open-to-open, or any other time of the

[2] As shown by Brorsen and Irwin (1987), Irwin and Brorsen (1985); and Lukac, Brorsen, and Irwin (1988), most futures fund managers employ technical methods of price analysis in the design of trading rules and in the selection of their funds' positions.

day, provided that a strategy transacts no more than once per day.[3] This calculation is repeated daily to generate a sufficiently long daily strategy-return series. It is also repeated for each strategy the futures fund might employ.

The mean and variance of the return series i, denoted $\mu(i)$ and $\sigma^2(i)$, are assumed to be unbiased estimates of future strategy performance.[4] It is further assumed that each return series is mean/variance stationary, and that the successive returns in the return series are mutually independent—both within and across strategies. That is, $R(i,t)$ and $R(j,t+s)$ $(s \neq 0)$ are independent for all i and for all j. The covariance of daily returns on series i and series j is denoted $\sigma(i,j)$. It is important to appreciate that the means, variances, and covariances of the strategy returns will generally not be equal to the means, variances, and covariances of the various underlying price series.[5]

It is assumed that the size of a continuing position, that is, the number of contracts, can be adjusted without cost on any given day.[6] The portfolio manager possesses trading capital in the amount of $C(t)$ at time t, and the bulk of this trading capital is invested in interest-bearing securities. For purposes of this study, it is assumed that the interest-bearing securities are some form of overnight money—overnight repos, for example.[7] The fraction of $C(t)$ held in this form on day t is denoted by $a(t)$. The remaining portion of trading capital, $1 - a(t)$, is held as cash to provide for variation margin, that is, cash transfers to and from the margin ac-

[3] The notation in equation (1) can be modified to allow for multiple trades (intraday trading) between time t and time $t+1$. For example, partially suppress the day index (t) and the strategy index (i) and let k denote the kth trade made on day t, $k = 0, 1, 2, \ldots, T$. The return on the kth trade is given by:

$$R(k) = \frac{D(i,k) \cdot [Z(k+1) - Z(k)]}{P(i,t)} \qquad k = 0, 1, 2, 3, \ldots, T$$

where, if $k = 0$ then $Z(k) = P(i,t)$ and if $k = T$ then $Z(k+1) = P(i,t+1)$. The daily return for strategy i on day t is then given by:

$$R(i,t) = \sum_{k=0}^{T} R(k)$$

The remaining notation is the same as that employed in equation (1).

[4] This is not a trivial assumption. As noted elsewhere, many technical "systems" developers employ faulty development techniques that seriously bias the return–risk parameters of their strategies (Marshall, 1989). Others who have addressed these concerns, although in a different context, include Bodie and Rosansky (1980), Caudill and Holcombe (1987), Denton (1985), and Neftci (1983). In addition, if the individual return series are not normal, the unbiasedness property will not hold. Empirical evidence, discussed later, suggests that the individual return series are not normal. However, as also shown later, empirical evidence suggests that portfolio returns are approximately normal. In this study, it is the portfolio returns that are the key, and thus, the assumption of unbiasedness of the individual return parameters does no harm.

[5] The exception is a benchmark strategy that is always long.

[6] This is not to imply that there is no cost to trading. Transaction costs, including both commissions and market impact costs, are assumed to already be reflected in the return–risk parameters of each strategy. The proper handling of transaction costs is discussed later.

[7] While T-bills are a more common vehicle in which to hold trading capital, T-bill returns violate the serial independence assumption. As an operational matter, this violation is probably of little practical importance as the capital return component is a relatively small portion of total daily return. Furthermore, if the T-bill maturities precisely match the investment horizon, it would not be inappropriate to view the T-bill return as having zero variance and to be uncorrelated with the strategy return series.

count as a consequence of daily marking-to-market.[8] The daily return on invested trading capital held to meet margin requirements is denoted $R(c, t)$, and the mean and variance of this return are denoted $\mu(c)$ and $\sigma^2(c)$, respectively. The covariance of the daily return on invested trading capital and the return series for strategy i is denoted $\sigma(i, c)$.

The number of units covered by a futures contract, that is, contract size, is denoted $S(i)$. Contract value for strategy i on day t is then $P(i, t) \cdot S(i)$. The leverage multiplier for strategy i on day t, denoted $L(i, t)$, is given by equation (2)

$$L(i,t) = \frac{P(i,t) \cdot S(i)}{M(i,t)} \tag{2}$$

where $M(i, t)$ is the per-contract dollar margin required for a speculative position in the futures involved in strategy i on day t.[9] Finally, the portion of total trading capital committed to strategy i on day t, denoted $w(i, t)$, is given by equation (3a), which is equivalent to equation (3b)

$$w(i,t) = \frac{M(i,t) \cdot N(i,t)}{C(t)} \tag{3a}$$

$$N(i,t) = \frac{C(t) \cdot w(i,t)}{M(i,t)} \tag{3b}$$

where $N(i, t)$ denotes the number of strategy i futures held on day t. The notation used in this article is summarized in Appendix A.

THE OPTIMIZATION PROBLEM AND MINIMUM VARIANCE SET

Let there be $n - 1$ strategies available for inclusion in the futures portfolio and let there be one additional strategy defined to be the "null strategy." The null strategy is a strategy having a zero mean and a zero variance. The weight assigned to this strategy represents that portion of total trading capital not committed as margin on a futures position and may be regarded as trading capital held in reserve. The leverage multiplier for the null strategy is, by definition, 1.

The 1-day portfolio return for day t, denoted $R_p(1)$, is given by equation (4)

$$R_p(1) = \sum_{i=1}^{n} w(i,t) \cdot L(i,t) \cdot R(i,t) + a(t) \cdot R(c,t) \tag{4}$$

where $\Sigma\, w(i, t) = 1$ and $a(t) \leq 1$.

The mean 1-day portfolio return is given by equation (5).

$$\mu_p(1) = \sum_{i=1}^{n} w(i,t) \cdot L(i,t) \cdot \mu(i,t) + a(t) \cdot \mu(c,t) \tag{5}$$

The variance of the 1-day portfolio return is given by equation (6).

[8] As a practical matter, the value $a(t)$ will generally be close to 1.

[9] As reported by Brorsen and Irwin (1987), initial margins generally approximate 5% of contract value. This implies a leverage multiplier of about 20. Nevertheless, this varies from day to day because exchange margins are changed infrequently but contract value changes daily.

$$
\begin{aligned}
\sigma_p^2(1) = {} & \sum_{i=1}^{n} w^2(i,t) \cdot L^2(i,t) \cdot \sigma^2(i) \\
& + 2\sum_{i>j}^{n}\sum^{n} w(i,t) \cdot w(j,t) \cdot L(i,t) \cdot L(j,t) \cdot \sigma(i,j) \\
& + 2\sum_{i=1}^{n} w(i,t) \cdot a(t) \cdot L(i,t) \cdot \sigma(c,i) \\
& + a^2(t) \cdot \sigma^2(c)
\end{aligned} \tag{6}
$$

The requirement that $\Sigma\, w(i,t) = 1$ in equation (4) is overly restrictive. It only applies as an absolute condition if no two strategies involve the same commodity. When two strategies involve the same commodity *and* the two strategies happen to be on opposite sides of the market, then the margin requirements are partially offsetting.[10] Thus, in such situations, the sum of the weights can exceed 1. A more inclusive treatment would distinguish between regimes and commodities. For example, let $w(k,l)$ denote the weight on regime k applied to commodity l. Let there be $G(l)$ regimes applicable to commodity l and let there be H commodities. Then the constraint has the form given by equation (7) (in which the time subscript is suppressed but understood)

$$
\begin{aligned}
& \mathrm{ABS}\left[\sum_{k=1}^{G(1)} D(k,1) \cdot w(k,1)\right] + \mathrm{ABS}\left[\sum_{k=1}^{G(2)} D(k,2) \cdot w(k,2)\right] + \ldots \\
& \qquad \ldots + \mathrm{ABS}\left[\sum_{k=1}^{G(H)} D(k,H) \cdot w(k,H)\right] + w(\text{null}) = 1
\end{aligned} \tag{7}
$$

where ABS[—] denotes the absolute value of the term in brackets and w(null) denotes the weight on the null strategy. This less restrictive form of the constraint only needs to be considered in those optimization cases in which the weight on the null strategy is binding in the more restrictive form. In simple terms, if, when employing equation (4), the weight on the null strategy (reserve trading capital) is found to be zero, then the less restrictive form of the constraint, given by equation (7), should be employed instead. The more restrictive form given in equation (4) is used in the remainder of this study and in the computer program employed to solve the model.

The first term in the variance equation [equation (6)] is unsystematic risk. It is directly analogous to the unsystematic risk associated with holding a stock portfolio. It differs, however, from the latter in two ways: (1) excluding the weight assigned to the null strategy, the weights need not sum to 1.0 (and generally will not); and (2) the leverage multiplier term is included to account for the leverage provided by futures.

The second term in the variance equation is systematic risk. It is also analogous to the systematic risk of a stock portfolio. Again, it is modified to account for the presence of leverage.

[10] Suppose, for example, that strategy 1 requires a long position in three July wheat futures and strategy 2 requires a short position in two July wheat futures. Then, the net margin requirement is for one wheat futures. The lower margins on intracommodity spreads also present a problem. For example, if strategy 1 is long one July wheat and Strategy 2 is short one September wheat, then the margin requirements may again be regarded as partially offsetting.

The third and fourth terms are unique to a futures portfolio. The third term represents a special form of systematic risk. The presence of this risk is explained by the fact that a portion of trading capital is held in the form of an interest-bearing asset. A priori, one might expect the correlation between the returns on a trading strategy and the returns on invested trading capital to be near zero. However, this might not be the case for all futures strategies, particularly those involving short-term interest rate futures such as T-bill and Eurodollar futures. The fourth term is a special form of unsystematic risk. Unlike the unsystematic risk of a stock portfolio, this risk does not necessarily get smaller with increasing portfolio diversification since the weight, $a(t)$, on this return component is independent of the degree of diversification.

When the leverage multiplier, $L(i,t)$, equals 1 for all i, $a(t)$ equals 0, and $D(t,i) = +1$ for all i, then the model presented here reduces, in all of its dimensions, to the Markowitz model. It is for this reason, that the futures model may be viewed as a generalization of the Markowitz model.

While equations (5) and (6) provide the mean and variance of the one-day portfolio return, the distribution of this return is not clear due to the heavy weighting given to the $R(c,t)$ term. Fortunately, when extended to a multiperiod context, the distribution becomes clear. (This is explained in the next section.)

The single-period optimization problem can be defined now. The first step seeks to identify the weights $w(i)$, $i = 1, 2, 3, \ldots, n$, such that the portfolio variance is minimized for any required portfolio return. To simplify the notation, suppress the time subscript and make the following substitutions:

$$s(i,j) = L(i) \cdot L(j) \cdot \sigma(i,j)$$

$$x(i,c) = a \cdot L(i) \cdot \sigma(i,c)$$

$$v(i) = L(i) \cdot \mu(i)$$

Equation (8) is obtained by combining terms 1 and 2 of equation (6), and substituting the notation above.

$$\sigma_p^2(1) = \sum_i \sum_j w(i) \cdot w(j) \cdot s(i,j) + 2\sum_i w(i) \cdot x(i,c) + a^2 \cdot \sigma^2(c) \qquad (8)$$

To map the efficient set, portfolio variance must be minimized subject to the following two constraints:

$$\mathbf{\Sigma}\, w(i) = 1$$

$$\mathbf{\Sigma}\, w(i) \cdot v(i) + a \cdot \mu(c) = b$$

where b is the expected overall portfolio return (including any interest earned on trading capital). Formulate the appropriate Lagrangian with $h(1)$ and $h(2)$ denoting the Lagrangian multipliers:

$$\begin{aligned} \text{MIN } L(\mathbf{w}, \mathbf{h}) = {} & \mathbf{\Sigma\Sigma}\, w(i) \cdot w(j) \cdot s(i,j) + 2\mathbf{\Sigma}\, w(i) \cdot x(i,c) \\ & + a^2 \cdot \sigma^2(c) + h(1) \cdot (b - \mathbf{\Sigma}\, w(i) \cdot v(i) - a \cdot \mu(c)) \\ & + h(2) \cdot (1 - \mathbf{\Sigma}\, w(i)) \end{aligned}$$

The first order conditions with respect to the vectors **w** and **h** are as follows:

$$
\begin{array}{l}
1 \cdot s(1,1) \cdot w(1) + 2 \cdot s(1,2) \cdot w(2) + \cdots + 2 \cdot s(1,n) \cdot w(n) + 2 \cdot x(1,c) - v(1) \cdot h(1) - 1 \cdot h(2) = 0 \\
2 \cdot s(2,1) \cdot w(1) + 1 \cdot s(2,2) \cdot w(2) + \cdots + 2 \cdot s(2,n) \cdot w(n) + 2 \cdot x(2,c) - v(2) \cdot h(1) - 1 \cdot h(2) = 0 \\
\quad \cdot \qquad\qquad \cdot + \cdots + \qquad \cdot \quad + \quad \cdot \qquad \cdot \qquad \cdot = \cdot \\
\quad \cdot \qquad\qquad \cdot + \cdots + \qquad \cdot \quad + \quad \cdot \qquad \cdot \qquad \cdot = \cdot \\
\quad \cdot \qquad\qquad \cdot + \cdots + \qquad \cdot \quad + \quad \cdot \qquad \cdot \qquad \cdot = \cdot \\
2 \cdot s(n,1) \cdot w(1) + 2 \cdot s(n,2) \cdot w(2) + \cdots + 1 \cdot s(n,n) \cdot w(n) + 2 \cdot x(n,c) - v(n) \cdot h(1) - 1 \cdot h(2) = 0 \\
\quad -v(1) \cdot w(1) + \quad -v(2) \cdot w(2) + \cdots + \quad -v(n) \cdot w(n) + b - a \cdot \mu(c) = 0 \\
\quad -1 \cdot w(1) + \quad -1 \cdot w(2) + \cdots + \quad -1 \cdot w(n) + 1 = 0
\end{array}
$$

If a solution to this system of linear equations exists, that is, a feasible solution, it must be that which minimizes the original variance term. This is assured by the concavity of the original objective function. However, the solutions for the weights must be restricted to nonnegative values. This can be handled by employing a linear programming algorithm in which both the **w** and **h** values are restricted to be nonnegative. This approach requires formulating a new, yet null, objective function—with a dummy variable added for each strategy.[11] These dummy variables are denoted $d(1), d(2), \ldots d(n)$. The new problem to be solved is then maximize:

$$0 \cdot w(1) + 0 \cdot w(2) + \cdots + 0 \cdot w(n) + 0 \cdot h(1) + 0 \cdot h(2) + -d(1) + -d(2) + \cdots + -d(n)$$

subject to:

$$
\begin{array}{l}
1 \cdot s(11) \cdot w(1) + 2 \cdot s(12) \cdot w(2) + \cdots + 2 \cdot s(1n) \cdot w(n) - v(1) \cdot h(1) - 1 \cdot h(2) + 1 \cdot d(1) + 0 \cdot d(2) + \cdots + 0 \cdot d(n) = -2 \cdot x(1,c) \\
2 \cdot s(21) \cdot w(1) + 1 \cdot s(22) \cdot w(2) + \cdots + 2 \cdot s(2n) \cdot w(n) - v(2) \cdot h(1) - 1 \cdot h(2) + 0 \cdot d(1) + 1 \cdot d(2) + \cdots + 0 \cdot d(n) = -2 \cdot x(2,c) \\
\quad \cdot \quad + \quad \cdot \quad + \cdots + \quad \cdot \quad - \quad \cdot \qquad \cdot \qquad \cdot \quad + \quad \cdot \quad + \cdots + \quad \cdot \quad = \quad \cdot \\
\quad \cdot \quad + \quad \cdot \quad + \cdots + \quad \cdot \quad - \quad \cdot \qquad \cdot \qquad \cdot \quad + \quad \cdot \quad + \cdots + \quad \cdot \quad = \quad \cdot \\
\quad \cdot \quad + \quad \cdot \quad + \cdots + \quad \cdot \quad - \quad \cdot \qquad \cdot \qquad \cdot \quad + \quad \cdot \quad + \cdots + \quad \cdot \quad = \quad \cdot \\
2 \cdot s(n1) \cdot w(1) + 2 \cdot s(n2) \cdot w(2) + \cdots + 1 \cdot s(nn) \cdot w(n) - v(n) \cdot h(1) - 1 \cdot h(2) + 0 \cdot d(1) + 0 \cdot d(2) + \cdots + 1 \cdot d(n) = -2 \cdot x(n,c) \\
\quad v(1) \cdot w(1) + \quad v(2) \cdot w(2) + \cdots + \quad v(n) \cdot w(n) + \quad 0 \cdot h(1) + 0 \cdot h(2) + 0 \cdot d(1) + 0 \cdot d(2) + \cdots + 0 \cdot d(n) = b - a \cdot \mu(c) \\
\quad 1 \cdot w(1) + \quad 1 \cdot w(2) + \cdots + \quad 1 \cdot w(n) + \quad 0 \cdot h(1) + 0 \cdot h(2) + 0 \cdot d(1) + 0 \cdot d(2) + \cdots + 0 \cdot d(n) = 1 \\
\qquad w(i) \geq 0 \text{ and } d(i) \geq 0 \quad (i = 1, 2, 3, \ldots, n) \text{ and } h(i) \geq 0 \quad (i = 1, 2)
\end{array}
$$

For any selected value of b, the solution to the model will provide a point on the single-period (one-day) minimum variance set—provided of course that a solution exists. By varying b, the entire minimum variance set can be mapped (i.e., plotted in risk–return space).

THE MULTIPERIOD DIMENSION

The problem with the formulation thus far is that it is single-period and the length of a single period is but one day. If the portfolio manager's horizon is indeed one-day long, then the model, except for the final selection criteria, is complete as is. Alternatively, if the selection criteria are independent of the length of the horizon, then the length of the horizon is irrelevant to the selection decision, and again, the model is complete as is. Unfortunately, neither of these conditions generally hold. The

[11] This is essentially the quadratic programming procedure first described by Wolfe (1969), and applicable whenever the original objective function is concave.

portfolio manager's horizon is far more likely to be several months or even several years in length; and, the selection of an optimal portfolio by an individual with a well-behaved utility function is very much influenced by the length of the investment horizon.[12] This point is very often neglected in both theoretical discussions of portfolio theory and in applications by portfolio managers.

As shown by Tobin (1965), the single-period minimum variance set and the multiperiod minimum variance set are identical, although the single-period efficient set is a subset of the multiperiod efficient set. The portfolio return–risk measures for a length-T horizon are related to the portfolio return–risk measures for a length-1 horizon by equations (9) and (10). These relationships assume daily portfolio rebalancing.[13]

$$\mu_p(T) = (1 + \mu_p(1))^T - 1 \tag{9}$$

$$\sigma_p^2(T) = ((1 + \mu_p(1))^2 + \sigma_p^2(1))^T - (1 + \mu_p(1))^{2T} \tag{10}$$

The lognormal law assures that the length-T return, $R_p(T)$, will have a lognormal distribution (shifted left by 1). This will hold precisely if $R_p(1)$ has a lognormal distribution and approximately if $R_p(1)$ has any other type of distribution. The longer T, the better the approximation.[14] In any practical application (in which T is more than a few days in length), the approximation to the lognormal distribution should be quite good under the assumptions made.[15]

Knowledge of the distribution of $R_p(T)$ allows the construction of a confidence channel for any portfolio on the single-period minimum variance set. A confidence channel is defined by the upper and lower bounds of an appropriately specified confidence interval when the interval is computed for a continuum of investment horizons, that is, $T = 1, 2, \ldots, N$. The construction of a confidence channel is straightforward. In brief, the effective return for a one-day horizon is converted to its normally distributed return equivalent, that is, the continuous return. The mean and standard deviation of this distribution are easily inferred from the mean and standard deviation of the effective return. The confidence interval for the one-day horizon is then computed in the usual manner. Finally, the upper and lower bounds of the interval are converted back to their effective equivalents. This procedure is repeated for a two-day horizon, a three-day horizon, and so on, out to some preset

[12] The importance of the investment horizon in optimal portfolio selection has been considered by Gressis, Philippatos, and Hayya (1976); Lloyd and Haney (1980); Gilster (1983); and, more recently, Marshall and Wynne (1990). It is not unreasonable to expect that portfolio managers will have at least intermediate horizons that correspond to the frequency of their reports to their investors.

[13] For a discussion of the importance of the rebalancing assumption in multiperiod portfolio analysis see Marshall (1989, chap. 6).

[14] The lognormal law holds that any variable defined as the product of independently and identically distributed random variables will have a distribution which converges, as the number of variables included increases, to a lognormal distribution. For further discussion, see Aitchison and Brown (1957).

[15] Evidence from Fama (1965) and others have demonstrated that return series for assets traded in competitive markets often exhibit leptokurtic behavior. Studies by Hall, Brorsen, and Irwin (1989) and Lukac and Brorsen (1990) have shown that *individual* returns series for futures traders using technical trading models also exhibit leptokurtic behavior. However, Lukac and Brorsen have shown that *diversified* futures portfolios, even those employing technical trading systems, do not exhibit this perverse behavior when the portfolio returns are measured monthly. This finding justifies, to some degree, the assumptions of unbiasedness made earlier in this article.

limit.[16] For purposes of computing such a channel, and all other computations reported in this study, it is assumed that a month consists of 21 trading days and a year consists of 252 trading days. A typical confidence channel is depicted in Figure 1.

The confidence channel is important for two reasons. First, the mean of the channel is a direct measure of the "expected" performance of the portfolio over the continuum of possible horizons. Second, the lower bound of the channel represents the maximum loss of trading capital (called drawdown here) that might occur and the temporal frame within which this loss might occur for the chosen level of confidence.

The fact that the lower bound of a confidence channel represents the maximum drawdown at a prescribed level of confidence suggests a useful criterion for choosing an optimal portfolio. Call this criterion "drawdown criterion."[17] In using drawdown criterion, *the fund manager specifies the maximum acceptable drawdown of initial trading capital over some specified horizon.* For example, the maximum acceptable drawdown might be 25% and the horizon might be six months. Finally, the fund manager specifies some level of significance (say 5%). The level of significance, as employed here, is interpreted as the likelihood that actual drawdown of initial trading capital will exceed the maximum acceptable level of drawdown *at any point in time over the entire length of the trading horizon* as perceived at the start of the trading horizon. The goal for the fund manager is then to *find the efficient one-day port-*

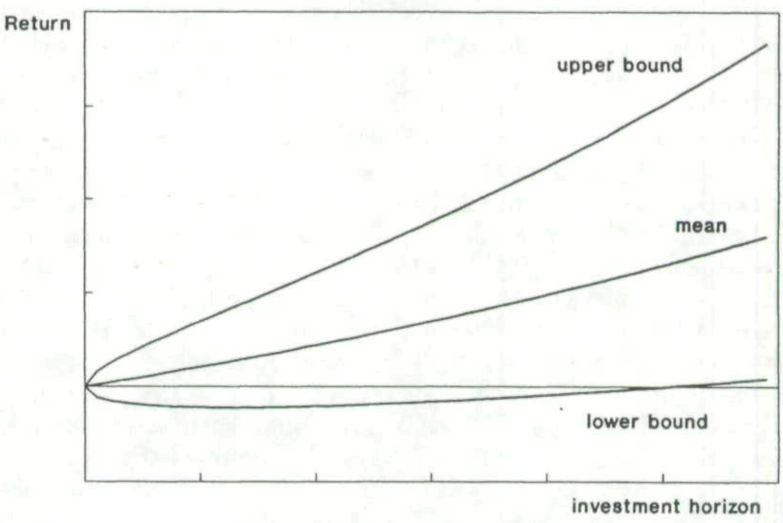

Figure 1
Confidence channel.

[16] The procedure for calculating a confidence channel when the return variable has a lognormal distribution is discussed more fully in Marshall (1989, chap. 16).

[17] Ad hoc measures and various rules-of-thumb with this same name have long been used by fund managers in designing portfolios; for example, see LaPorte Asset Allocation System, Burlington Hall Asset Management, or Managed Accounts Reports Quarterly Performance Reports. One such drawdown-based measure is the Sterling Ratio developed and used by Deane S. Jones, a futures portfolio manager, in Reno, Nevada. Many of these ad hoc measures, however, suffer from the defect that they are sample-size dependent. The drawdown criterion employed in this article is statistically stable in the sense that it is not dependent on the size of a sample.

folio that maximizes expected return over the trading horizon while not exceeding the drawdown limitation.[18]

An equivalent, but more intuitive, criterion for the selection of an optimal portfolio is "ruin criterion." Ruin criterion begins by selecting an *explicit percentage of trading capital that, if lost over some specified period of time, would constitute "ruin."* This might be 20% or 30%, or whatever percentage the portfolio manager deems appropriate. Then, using the multiperiod math described in the preceding section, and the properties of the lognormal distribution, the precise probability that a given single-period (one-day) portfolio will result in a loss of this magnitude or greater for each trading horizon from one day to the length of the actual horizon is determined. Call the graph of this probability of ruin (over the continuum from one day to the actual horizon) a "ruin curve." The goal for the manager using this criterion is to *maximize expected return while not exceeding some specified probability of ruin.*

While drawdown and ruin criteria are extremely attractive in terms of their appeal to a practitioner, they are also extremely tedious criteria to employ from a computational perspective. Fortunately, however, both readily lend themselves to a programming solution.

AN APPLICATION

To apply the model described in the preceding sections, a quadratic programming algorithm is modified to accept the input parameters (daily mean returns, standard deviations of returns, correlations coefficients between daily returns, leverage multipliers, and so on) in the form required. Further modification is made to allow for the specification of an initial required return and an incrementing factor. For example, an initial minimum required daily return of $z\%$ is specified with an incrementing factor of $x\%$. The program then solves for the optimal portfolio weights and the associated variance of return for the initial required return, increments the required return, resolves, increments again, and so on. By this process, the minimum variance set is mapped.

Additional modifications are made to the program to project the one-day portfolio parameters to their multiperiod equivalents up to one year. After completion of this latter step, a binomial search algorithm is superimposed on the quadratic programming algorithm to determine the efficient portfolio that maximizes return under the drawdown criterion. That is, once the drawdown parameters are specified, the program selects an initial portfolio on the efficient frontier and compares its return and drawdown parameters to the parameters of the desired portfolio. If the initial portfolio's drawdown is too high (low), relative to the specified drawdown, then the program increments downward (upward) on the efficient set until it finds a portfolio having too low (high) a drawdown, relative to the specified drawdown. Once two efficient portfolios are found, one having too high a drawdown and one having too low a drawdown, the returns on those two portfolios are averaged to generate a new required return. The efficient portfolio having this return is then solved for. The process is repeated until the optimal portfolio—that is, the one having the precise drawdown characteristics indicated—is found. This usually takes

[18] Zelney (1982) has shown that most institutional investors perceive risk as the probability of not achieving a minimum level of return. This is precisely what both the drawdown criterion and the ruin criterion, discussed later, measure.

from 12–15 iterations. The program is later altered to search, in the same manner, for the optimal portfolio using ruin criterion.

As a final step, two companion programs are developed. One generates confidence channels for any portfolio on the efficient set. The other generates a ruin curve for any portfolio on the efficient set.[19]

To test the efficacy of the portfolio model and the portfolio selection criteria, the assistance of a futures fund management firm was enlisted.[20] The fund managers provided return series data from 11 of their proprietary trading strategies.

The data cover the period January 23, 1973 through July 29, 1977 and represent in-sample return data adjusted for transaction costs.[21,22] There are a total of 1135 daily return observations on each series. These series are used to generate the required daily means, standard deviations, and correlation coefficients. For simplicity, margin requirements are assumed to be 5% of contract value for each contract traded under each strategy. Thus, the leverage multipliers are all 20.0. It is assumed that margin is held in the form of interest-earning securities that return 8.325% a year or 0.031746% per day compounded daily (252 trading days in a year), and that 90% of trading capital is held in this form, with the remainder held as cash. The return on trading capital is assumed to have zero variance and to be uncorrelated with each of the trading strategies. The commodities, together with the associated daily parameters, are reported in Table I.

The program noted in Table I then sequentially maps the portfolios on the efficient set. The same efficient set is mapped for investment horizons of one day, one month, three months, six months, and one year (these horizons were identified by fund managers to be of particular interest). A portion of the efficient set for a horizon of one year is depicted in Figure 2. Sample output from this program, for portfolio A on the efficient set, appears as Appendix B.

The drawdown criterion is tested next. The drawdown values are provided by the same futures fund managers that supplied the data. Specifically, the maximum accept-

[19] Interested parties may obtain copies of these programs from the authors.

[20] This was Brandywine Asset Management, Inc., which manages several futures funds.

[21] It is well understood that the use of in-sample return data—return data representing the same period on which the trading strategies were developed—will, generally, upwardly bias estimates of return performance. This problem has been extensively discussed elsewhere [see, e.g., Marshall (1989, chap. 16)]. The objective of this study, however, is not to judge system performance but rather to model portfolio behavior in a multiperiod context and to introduce practical selection criteria. No model can accurately predict portfolio performance if the input parameters of the individual trading strategies are biased. For these reasons, no claim is made that the portfolio performance presented here is at all realistic for this fund manager or for any other fund manager.

[22] For purposes of generating the return series, the fund managers assume a $50 per contract transaction cost. The correct adjustment for transaction costs is a two-step process. First, some assumption must be made as to per-contract transaction costs. After a trade is completed, the number of market days spanned by the trade is determined. The transaction cost is then restated on a per-unit per-day basis in the same monetary units as the futures price. In the case of soybean oil, for example, this is cents per pound and each contract covers 60,000 pounds. This value is computed as follows:

$$\mathrm{PTC}(t) = \frac{\text{Transaction cost (in dollars)} \times 100}{\text{Number of units} \times \text{number of days in trade}}$$

The value $\mathrm{PTC}(t)$ is then deducted from the numerator of equation (1) (for each day spanned by the trade) to arrive at the adjusted strategy return series as below (individual series subscript suppressed).

$$R(t) = 100 \times \frac{D(t) \cdot (Z(t+1) - P(t)) + D(t+1) \cdot (P(t+1) - Z(t+1)) - \mathrm{PTC}(t)}{P(t)}$$

Table I
STRATEGY RETURN PARAMETERS EMPLOYED AS INPUT VALUES TO THE SIMULATION

Commodity	Strategy	1	2	3	4	5	6	7	8	9	10	11
Deutsche mark	1	1.000	0.541	0.759	0.705	0.751	0.736	0.521	0.663	0.486	0.432	0.636
Swiss franc	2		1.000	0.550	0.509	0.536	0.653	0.589	0.575	0.277	0.266	0.747
British pound	3			1.000	0.906	0.820	0.750	0.554	0.661	0.422	0.400	0.722
Canadian dollar	4				1.000	0.818	0.745	0.525	0.653	0.450	0.413	0.701
Japanese yen	5					1.000	0.818	0.588	0.659	0.468	0.417	0.734
Silver	6						1.000	0.630	0.673	0.442	0.427	0.777
Live cattle	7							1.000	0.837	0.322	0.286	0.620
Live hogs	8								1.000	0.439	0.395	0.628
Sugar	9									1.000	0.703	0.340
Soybeans	10										1.000	0.281
Corn	11											1.000
Mean		0.062	0.077	0.097	0.093	0.083	0.079	0.072	0.074	0.061	0.070	0.079
St. dev.		0.632	0.681	0.670	0.661	0.643	0.631	0.507	0.516	0.630	0.664	0.682
Lev. multiplier		20.0	20.0	20.0	20.0	20.0	20.0	20.0	20.0	20.0	20.0	20.0
$a(t)$		0.90										
$R(c,t)$		0.031746%										

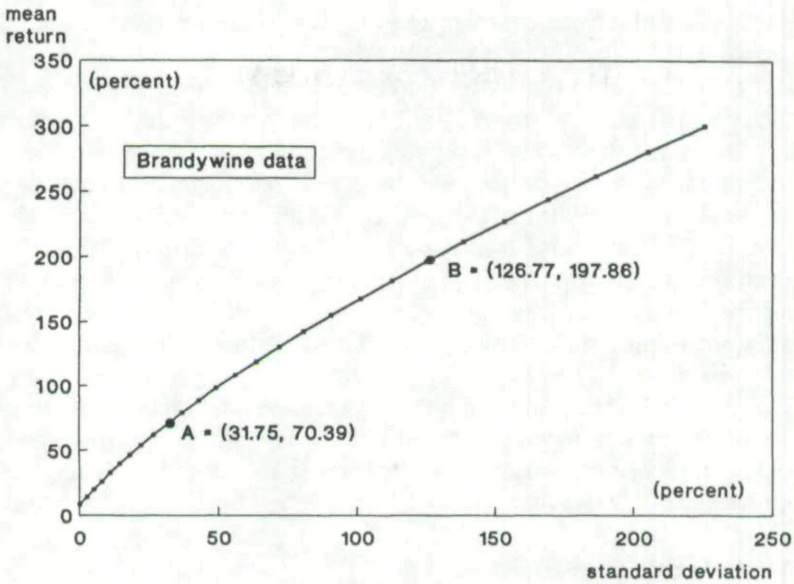

Figure 2
Efficient set of portfolios (one-year horizon).

able drawdown is set at 20% of trading capital over a one-year horizon with a level of significance of 1%. The algorithm then determines that the portfolio which maximizes return subject to drawdown not exceeding 20% at a 1% level of significance has the weights indicated in Table II. This particular portfolio has a mean annual return of 197.86% and a standard deviation of 126.77%. It is indicated as portfolio B on the mapping of the efficient set in Figure 2. The actual program output appears as Appendix C.

The percentage commitments indicated in Table II can be translated into the appropriate number of futures contracts using equation (3b). Suppose, for example, that the current per-contract margin requirements for silver, live cattle, and soybeans are $1000, $1500, and $1200, respectively. Suppose further that the fund has trading capital of $2 million. Then, the weights obtained suggest that the fund should hold 36.98 silver contracts, 209.82 live cattle contracts, and 178.22 soybean contracts.

A total of 28.3% of trading capital is committed as margin. The remainder of trading capital is uncommitted, as implied by the null strategy. As a practical mat-

Table II
OPTIMAL PORTFOLIO COMPOSITION USING DRAWDOWN CRITERION

Commodity	Strategy Number	Weight
Silver	6	1.849%
Live cattle	7	15.737%
Soybeans	10	10.693%
All others		0.000%
	Null	71.721%

ter, the indivisibility of futures contracts requires that the optimal size of the futures positions be rounded to the nearest integer value.[23]

As a next step, the ruin criterion is employed to select the optimal portfolio. In this case, ruin is defined as a loss of 20% of trading capital at any time over a one-year trading horizon. A maximum acceptable probability of ruin of 1% is indicated. As one would expect, the algorithm produces a portfolio identical to that produced under the drawdown criterion; thereby confirming the mathematical equivalence of the two criteria. The ruin curve, which is defined as the probability of ruin for any given trading horizon, is depicted in Figure 3. Notice that the probability of ruin rises for a time and then declines significantly. As defined, however, the probability of ruin is the maximum probability of experiencing ruin over the entire course of the trading horizon.

As a final step, the companion program is employed to generate a 90% confidence channel for the optimized portfolio. This is depicted in Figure 4. The lower bound of the confidence channel may be interpreted as a drawdown curve, at a 5% level of significance in this particular case.

SUMMARY AND CONCLUSION

This article discusses a portfolio model that allows for the determination of the efficient set of futures portfolios using the well-tested mean–variance approach. It extends the traditional Markowitz model by allowing for daily portfolio adjustments,

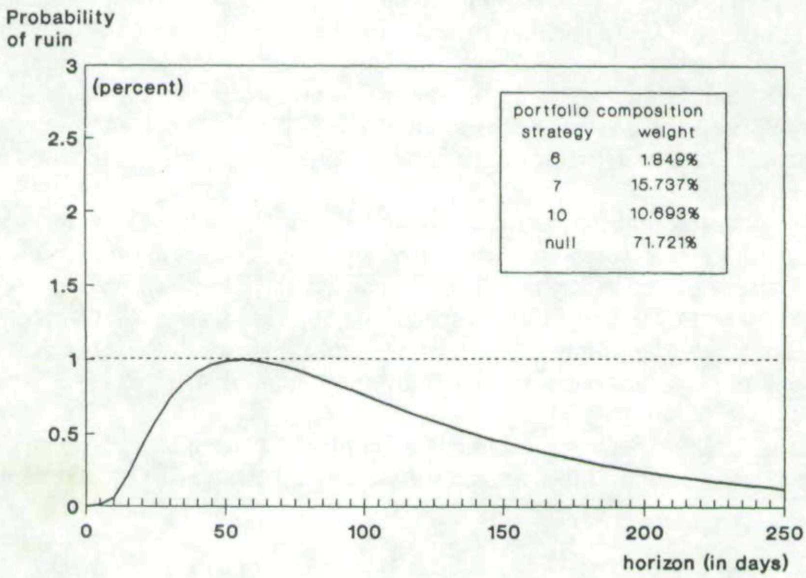

Figure 3
Ruin curve for portfolio B.

[23] The indivisibility of futures contracts requires that, after the weights are ascertained and translated into a given number of futures contracts, the number of futures should be rounded to the nearest integer value. It is well-known that an integer programming solution such as a Gomory cutting plane modification to a linear programming model can lead to a better solution than simple rounding. The use of such a procedure, however, requires considerable modification of the model and an extra layer of complexity. In repeated trials of the procedure on several data sets, no economically meaningful improvement is found by using integer programming over simple rounding. Nevertheless, some improvement might be obtained from such a modification for a different data set.

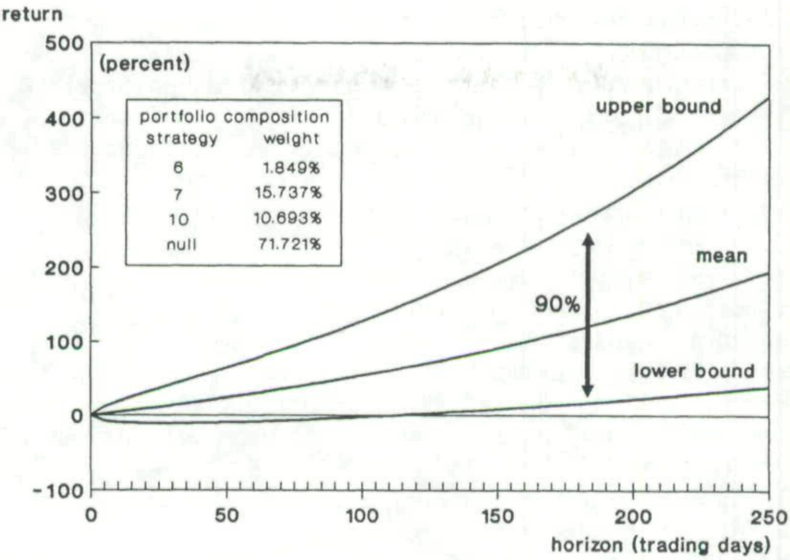

Figure 4
Confidence channel for portfolio B.

leverage, long and short positions, and multiperiod trading horizons. These enhancements make the model applicable in a futures portfolio context and, for this reason, should be of interest to managers of futures funds, managers of other futures portfolios, and managers of portfolios having a futures component. Under certain restrictive assumptions, the model reduces to the familiar Markowitz model.

This article also introduces two equivalent criteria for the selection of an optimal futures portfolio. These are called drawdown criterion and ruin criterion. These optimizing criteria are similar to, but more formally defined and have nicer statistical properties than, optimizing criteria currently employed by many futures fund managers.

In addition to its applicability in solving for the efficient set of futures portfolios, the model, when coupled with the selection criteria, lends itself to a rapid binomial search solution for determining the optimal portfolio weights. By specifying either a maximum acceptable drawdown or a maximum acceptable probability of ruin, an optimal portfolio can be found.

Appendix A

SUMMARY OF NOTATION

$P(i,t)$: the settlement price of the futures underlying strategy i on day t.

$Z(i,t+1)$: the transaction price for the futures underlying the ith strategy on day $t+1$ if a transaction has been made; otherwise $Z(i,t+1) = P(i,t+1)$.

$R(i,t)$: the return on strategy i for day t.

$\mu(i)$: the mean daily return for strategy i.

$\sigma(i)$: the standard deviation of daily return for strategy i.

$R(c,t)$: the daily return on overnight money on day t.

$\mu(c)$: the mean daily return on invested trading capital.

$\sigma(c)$: the standard deviation of daily return on trading capital.
$\sigma(i,j)$: the covariance of daily returns for strategies i and j.
$\sigma(i,c)$: covariance of daily return for strategy i and trading capital.
$C(t)$: the dollar amount of available trading capital on day t.
$M(i,t)$: the dollar amount of required margin on one futures contract for strategy i.
$S(i)$: the contract size for a specific futures contract (i.e., number of units covered by the contract).
$D(i,t)$: dummy variable representing the side of the market suggested by strategy i on day t. $D(i,t)$ is $+1$ for a long position, -1 for a short position, and 0 if flat.
$L(i,t)$: the leverage multiplier for strategy i on day t.
$N(i,t)$: the number of futures contracts constituting a position in strategy i on day t (for modeling purposes, contracts are assumed infinitely divisible).
$a(t)$: the fraction of trading capital held in interest-bearing form on day t. Remainder assumed to be in nonreturn-bearing cash.
$w(i,t)$: fraction of trading capital committed to strategy i on day t.
$R_p(T)$: the portfolio return for a T-day trading horizon.

Appendix B

SAMPLE OUTPUT FROM EFFICIENT SET MAPPING ALGORITHM (PORTFOLIO A)

Return and Volatility Analysis (all values are percentages)

	1-Day	1-Month	3-Months	6-Months	1-Year
Return	0.2117	4.541	14.252	30.53	70.39
Volatility	1.1661	5.580	10.577	17.13	31.75

Drawdown Analysis (all values are percentages)

	1-Day	1-Month	3-Months	6-Months	1-Year
LOS 10%	−1.28	−2.67	−2.67	−2.67	−2.67
LOS 5%	−1.69	−4.37	−4.37	−4.37	−4.37
LOS 1%	−2.47	−7.78	−8.55	−8.55	−8.55

Portfolio Composition (all weights are percentages)

Strategy Number	Strategy Name	Weight	Strategy Number	Strategy Name	Weight
1	ST01	0.000	7	ST07	7.108
2	ST02	0.000	8	ST08	0.000
3	ST03	0.000	9	ST09	0.000
4	ST04	0.000	10	ST10	4.830
5	ST05	0.000	11	ST11	0.000
6	ST06	0.835	12	Null	87.227

Appendix C

SAMPLE OUTPUT FROM DRAWDOWN CRITERION RETURN-MAXIMIZING ALGORITHM

Return and Volatility Analysis
(all values are percentages)

	1-Day	1-Month	3-Months	6-Months	1-Year
Return	0.4341	9.522	31.372	72.59	197.86
Volatility	2.5818	12.945	27.082	50.85	126.77

Drawdown Analysis
(all values are percentages)

LOS 1%: Maximum drawdown from 1-day to 1-year is 20.00%.

Portfolio Composition
(all weights are percentages)

Strategy Number	Strategy Name	Weight	Strategy Number	Strategy Name	Weight
1	ST01	0.000	7	ST07	15.737
2	ST02	0.000	8	ST08	0.000
3	ST03	0.000	9	ST09	0.000
4	ST04	0.000	10	ST10	10.693
5	ST05	0.000	11	ST11	0.000
6	ST06	1.849	12	Null	71.721

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