Option-Based Evidence of the Nonstationarity of Expected S&P 500 Futures Price Distributions

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INTRODUCTION

Option pricing models and observed market prices are used often in efforts to recover information about an underlying futures contract's expected price distribution. The most prevalent examples of using options market prices to recover information about an underlying security's expected price distribution involve the Black–Scholes (1973) or Black (1976) option pricing models and recovery of an estimate of the ex ante price volatility (standard deviation) often termed the "implied volatility" (IV). Then, if an estimate of the mean of the ex ante distribution is available as well, an entire two-parameter expected price distribution may be constructed for the underlying asset using information from observed option prices. Using these two parameter distributions, many studies seek to either explain observed biases in pricing relative to implied measures, or more directly, to explain changes in the implied distributions through time series models of the implied volatilities.[1]

In deriving estimates of values of ex ante variables, ex post data are often used. More useful would be a set of ex ante distribution parameters, or an ex ante description of the stochastic process governing the realizations of the random variables. Unfortunately, direct elicitation of ex ante parameter expectations from market participants is often difficult, if not impossible.

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[1]For examples, see Latane and Rendleman (1976), Schmalensee and Trippi (1978), Chiras and Manaster (1978), Choi and Longstaff (1985), Park and Sears (1985), Anderson (1985), Milonas (1986), Shastri and Tandon (1987), Jordan et al. (1987), Beckers (1981), Ball and Torous (1986), and many others.

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Option's payoffs are contingent upon the entire range of possible outcomes for future price and, as such, contain an assessment of all relevant moments of a future price distribution. The option price (premium), therefore, implicitly contains the assessments and preferences of market participants over the expected distribution of the underlying security's price outcomes. This seemingly innocuous observation provides an avenue toward the recovery of the expected price distribution parameters without extensive surveys or direct elicitation.

In the case of S&P 500 futures markets, current prices may give a good deal of information about expected future prices. For example, the current futures price may correspond to the mean of the expected future price distribution. However, in the absence of relatively sophisticated descriptions of the price diffusion process and a risk premium, if any, the futures price may be silent about other moments of an expected price distribution. However, options may potentially be used to reveal information about future price distributions that may not be obtained from current futures prices.

A much less restrictive approach than those used in previous studies is available for recovering expected price distributions. This approach, based on the set of no-arbitrage restrictions first proposed by Ross (1978), can be used to recover estimates of the parameters of ex ante futures price distributions given a set of option premia.

The merits of allowing the price distribution to be parameterized in a more flexible manner are examined through the use of both lognormal and Burr distributions. The more common lognormal model provides a benchmark against which the Burr performance can be measured. This study uses transactions data for the S&P 500 futures and futures options and derives ex ante measures of expected price distributions independently of current futures prices.

An examination of the changes in the volatility of expected distributions over time then provides insights into the fundamental economic forces that may be reflected in these markets. If, for no other reason than the passage of time, expected distributions will collapse to the expiration date price. Therefore, the equilibrium implied expected distributions will also change as the agents' information sets change. The investigation of the equilibrium implied expected distributions would, therefore, be seriously suspect if, in addition to examining the static properties of the implied distributions, the issue of time-varying parameters is not also considered. An endogenous switchpoint regression model is employed to detect the number and timings of switchpoints in each contract's series of implied distributions allowing for the possibility of varying dependence on time-to-maturity and price level.

The purposes of this study are to offer evidence about the time-varying influences of price level and time-to-maturity on the volatility of expected price distributions. A less restrictive no-arbitrage option pricing model is used first to recover complete probabilistic descriptions of ex ante S&P 500 futures price distributions using a transactions data set to alleviate potential biases that have been noted with closing data. The expected distributions are parameterized as both the lognormal and a less-restrictive Burr distribution, and a flexible endogenous switchpoint regression is then used to characterize the evolution of the ex ante volatility over time with the possibility of time-varying influences of price level and time-to-maturity.

This article first discusses option pricing methods in light of the need to recover a complete probabilistic description of expected prices. Next, candidate parameterizations of expected distributions are given. The data are then described and the methods used and results are discussed. The article then turns to modeling changes of the implied distributions in a switching regression framework and documents the variable influences of time-to-maturity and price level on one parameter of the implied distributions.

OPTION PRICING: A NO-ARBITRAGE APPROACH

Ross (1978) and others [Breeden and Litzenberger (1978); Banz and Miller (1978); Cox and Ross (1976)] show that no-arbitrage implies the existence of a "supporting pricing function" which is denoted as $f(s)$. The pricing function may be interpreted as a set of prices of pure contingent claims that pay \$1 if and only if their particular state, s, occurs. A \$1 risk-free bond may be constructed by buying one pure contingent claim for each possible state. Hence, the price of a \$1 bond is equal to the sum of the state prices for all possible states. If the states are continuous, $f(s)$ corresponds to a continuous state price function rather than a discrete pricing function, but the arguments are otherwise analogous in that the current price of a bond that pays \$1 only at time T, regardless of what state occurs, has a current price equal to $b(T) = \int_s r_s f(s)ds$, where r_s is the return in state s (in this case equal to a constant of \$1) and $f(s)$ is the expected state price function at T. Given linearity of the pricing function across assets, any asset in the no-arbitrage economy with return r_s at time T may be valued like the bond by simply taking the expectation of its returns over the function $f(s)$. The bond is the simplest because its returns are \$1 in all states. Conversely, if the price of a risk-free bond were known, and all possible states were identified, a consistent $f(s)$ could be located. Note that a key distinction between this and the typical option pricing approach is that no assumptions need to be made about the underlying dynamics or changes in the economic environment prior to the options' expirations. Because the focus is on the use of the model, the discussion is somewhat limited. An excellent treatment of the complete set of no-arbitrage restrictions is given by Ingersoll (1987).

Given the system of (1) asset prices; (2) state-dependent payoffs of the assets; and (3) state pricing function, any one of the three may be determined if the other two are known.[2] The discussion above uses payoffs and a supporting price distribution to determine a no-arbitrage consistent bond price, but observed asset prices and payoff functions could be used to solve for a set of state probabilities that are proportional to the state prices. Or, if the state pricing function is parameterized as a continuous function, the parameters of that function can be estimated. Thus, a payoff function, r_s, and a distribution of time-dependent state prices yield a current value consistent with no arbitrage. Any asset, V_i, with return stream $r_{i,s}$ in the economy may, therefore, be priced if its return function is known via the relation:

$$V_i = \int r_{i,s} f(s)ds \tag{1}$$

Call and put options written on futures have clear return functions, $r_{i,s}$. For calls, the return at time T in the future is simply $\max\{Y_T - K_i, 0\}$ where K_i is the exercise price for call i and Y_T is the random futures price at time T. For puts, the function

[2]Strictly speaking, if the states are continuous but there are only a finite number of assets in the economy, there is no guarantee that $f(s)$ is unique, and a projection criterion is needed to recover or place bounds on the payoff space. The usual assumption is made here that no further adjustments are necessary to insure that the state prices are proportional to the state probabilities. Hence, factoring $f(s)$ as $b(T) * f(s)/b(T) = b(T) * f_T(s)$ allows the interpretation of $f_T(s)$ or $f(Y_T)$, for example, as the risk-neutralized probability function, because $b(T)$ is essentially the state price per unit probability and dividing $f(s)$ by $b(T)$ "inflates" the sum of the state prices to one.

is $\max\{K_j - Y_T, 0\}$. The outcome of Y at T completely determines the relevant state for an option's payoff. By noting that for $Y_T > K_j$, the value of the put option (P) is zero, and that for $Y_T < K_i$, the value of the call option (C) is zero, current values of puts and calls can be expressed as their discounted expected payoffs, with the expectation taken over the density $f(Y_T)$:

$$C_i = b(T)\int_{Ki}^{\infty} \{Y_T - K_i\} f(Y_T) dY_T \tag{2}$$

and

$$P_j = b(T)\int_{0}^{Kj} \{K_j - Y_T\} f(Y_T) dY_T \tag{3}$$

A key distinction between this and the typical option pricing approach is that no assumptions are made about the underlying price dynamics or changes in the economic environment prior to expiration. Only two known previous studies use the no-arbitrage option pricing method to derive complete probabilistic descriptions of uncertain price distributions. Fackler and King (1990) use closing-price option data to examine the calibration of no-arbitrage implied price distributions for four agricultural commodities and find evidence that the market implied distributions misstate the resulting variability and location of future prices. Sherrick et al. (1990) use the no-arbitrage option pricing model to investigate the expected soybean futures price distributions and find little evidence of miscalibration using transactions data. However, a slightly altered version of this approach employing an "empirically adjusted" $f(s)$ is suggested in Gastineau (1988). Garven (1985) and several other authors show the equivalence between this and the Black futures option pricing formula in the special case when $f(Y_T)$ is assumed to be lognormal. Again, the only assumption made is that there are no arbitrage opportunities, thus guaranteeing the existence of $f(Y_T)$.

ESTIMATION OF EXPECTED DISTRIBUTIONS

For obvious reasons, it is desirable to choose candidate distributions for $f(Y_T)$ that are as flexible as possible. To be practical, a distribution is needed that does not allow negative prices (simple arbitrage), has relatively few parameters, and allows a fairly wide range of shapes to emerge for the density function. Note that a subtle difference between the ex post price distribution and the expected price distribution exists that makes the question of the appropriate parameterizations largely one of convenience. Suppose that the "true" distribution includes several relevant higher moments that individuals ignore when making pricing decisions. Then, the choice of the parameterization of the implied distributions can be made on the basis of the fit to the option-pricing model, rather than through a verification with subsequent futures prices. But, to the extent that expected and empirical price distributions are similar (as one would expect), examination of prices can still reveal much information about the appropriate parameterization of the implied distributions. Many previous studies suggest that the lognormal distribution is not very descriptive of reality. Studies find empirical futures price distributions that are more leptokurtic and more or less skewed than that implied by a lognormal distribution of prices.[3]

This study uses two distributions as the primary candidates for $f(Y_T)$: (1) the two-parameter lognormal; and (2) the three-parameter Burr distribution [known

[3]See So (1987) or Hall et al. (1989) and the references therein.

also as the Singh–Maddala (1976) or Burr-XII distribution]. Both distributions allow for only positive values of Y_T, and the Burr may take on a wide range of skewness and kurtosis including all the region covered by the Pearson type IV, VI, and bell-shaped region of type I; the gamma; Weibull; normal; lognormal; logistic; and extreme value distributions [Tadikamalla (1980)]. Hence, the flexibility of the distribution for modeling data is extremely attractive. Additionally, the distribution function and its inverse exist in closed form, making it particularly attractive for modeling option-implied price distributions. Further, plots of the skewness and kurtosis of the ex post futures prices indicate that considerable flexibility in the ratio of the moments is necessary to capture the characteristics of the price data.[4] For example, the lognormal is always positively skewed and would not perform well if fit to a negatively skewed set of price expectations. The Burr does not suffer such restrictions.

The Burr cumulative distribution function (CDF) with parameters α, λ, and τ is:

$$F_{BR}(Y|\alpha, \lambda, \tau) = 1 - (\lambda/(Y^{\tau} + \lambda))^{\alpha} \quad \text{for} \quad \alpha, \lambda, \tau, Y > 0 \tag{4}$$

and thus, the density, or PDF is:

$$f_{BR}(Y|\alpha, \lambda, \tau) = \alpha\lambda^{\alpha}\tau Y^{\tau-1}(Y^{\tau} + \lambda)^{-(\alpha+1)} \tag{5}$$

The nth moment may be given as:

$$m'_n = \lambda^{n/\tau}\Gamma(\alpha - n/\tau)\Gamma(1 + n/\tau)/\Gamma(\alpha) \tag{6}$$

where Γ is the gamma function described by its properties as:

$$\Gamma(\alpha) = \int Y^{\alpha-1}e^{-Y}dY = (\alpha - 1)\Gamma(\alpha - 1) \tag{7}$$

The mode of the Burr is 0 for $\tau \leq 1$, and

$$[\lambda(\tau - 1)/(\alpha\tau + 1)]^{1/\tau} \quad \text{for} \quad \tau > 1 \tag{8}$$

The cumulative distribution function for the lognormal distribution with parameters μ and σ is:

$$F_{LN}(Y|\mu, \sigma) = N(\ln(Y - \mu)/\sigma) \tag{9}$$

where $N(\cdot)$ is the cumulative normal density function. The lognormal density is:

$$f_{LN}(Y|\mu, \sigma) = (2\pi)^{-1/2}(\sigma Y)^{-1}\exp[-(\ln Y - \mu)^2/(2\sigma^2)] \tag{10}$$

and the nth moment is given by

$$m'_n = \exp(n\mu + n^2\sigma^2/2) \tag{11}$$

When substituted into the option pricing formulas in eqs. (2) and (3), one obtains no-arbitrage option pricing formulas with which observed data are used to fit parameters of the two distributions. For example, usings the Burr distribution the value of a call is:

$$C_i = b(T)\int_{Ki}^{\infty}\{Y_T - K_i\}\alpha\lambda^{\alpha}\tau Y_T^{\tau-1}(Y_T^{\tau} + \lambda)^{-(\alpha+1)}dY_T \tag{12}$$

[4]Five of the contracts exhibit negative (centralized, unscaled) skewness, and seven of the contracts exhibit negative skewness in logarithms of price. These results are taken as evidence that flexibility in higher moments is desirable. More complete diagnostic tests of the distributions for the ex post futures data and plots of the skewness–kurtosis plane are available from the authors by request.

The parameters of the Burr may be easily estimated if observed prices are assumed to behave according to the model by simply applying a penalty function to the difference between observed prices and model prices conditional on the distribution parameters.

A comparison of the lognormal and Burr distributions is made to assess possible improvements in moving to a three-parameter distribution. There is no guarantee that the parameters of $f(Y_T)$ will conform to an ex post price distribution with either parameterization. Indeed, they are simply mathematical constructs with probability-like properties that are extracted from the option-pricing equations.

DATA

The data consist of a subset of all time-stamped transactions of S&P 500 futures and options from 1/11/84 to 9/30/88. The data are from tapes available from the Chicago Mercantile Exchange. The data from October 19–20, 1987 are not used due to some anomalous entries. In addition to all trades at which a price change occurs, the data set contains bids that exceed, and asks that fall below, the previous transactions. Volume data per se are not available, but an examination of the mean trading price-change time interval is examined as a flag for possible liquidity problems. Lack of volume does not appear to be a problem as evidenced by the number of usable option quotes. Some exclusion criteria are considered to alleviate induced biases. Deep in-the-money and out-of-the-money options are scrutinized carefully, although there is no theoretical reason for exclusion. Next, the time interval between trades of matched prices should be as short as possible. Two earlier studies [Whaley (1986); Ogden and Tucker (1987)] require that the futures price precede the option price. Jordan et al. (1987) simply require that the put, call, and futures prices each occur within a common 30-second interval. Since there is little reason (other than volume) to expect that one price necessarily leads or causes the other, no a priori imposition of order is made. Also, based on Bookstaber's (1981) arguments, synchronous futures and option prices are preferred, and hence, the matched futures and option price are required to be within 60 seconds of each other. And, to minimize the effects of changes in intraday volatility, the point in time during the day should be within a relatively stable trading interval and away from the opening and closing periods when price distortions due to heightened intraday volatility may be more prevalent [Edwards (1988); Cheung and Ng (1990)].

Synchronous option and futures prices are collected as follows: for each type of option (puts and calls), all strike prices that traded during the day are arranged according to their proximity to 11:00 (to avoid opening and closing distortions). Then, for each strike, the option that trades closest to 11:00 is matched to the nearest futures price. The result each day for each contract was a complete set of the strikes for both puts and calls that trade, with each observation as near as 11:00 as possible.

The resulting samples are further described in Table I. For example, the December 1987 contract distributions are computed on 77 different trading days. On average, 19.65 different options are used each day in the estimation for a total of 1513 total observed options with their matched futures prices. The average time difference between 11:00 AM and the option observation is 11.3 minutes, and the average length of time between the option observation and its matched futures price is 8.1 seconds. Approximately 53% of the options used for this contract are calls.

To solve for the parameters of the expected-price distributions, a risk-free rate is also needed that corresponds to $b(T)$. The daily rate represents a constant maturity nominally risk-free discount rate computed from the mean of the bid–ask spread of selected U.S. Treasury securities weighted to have an effective duration of 90 days.

Table I
DESCRIPTIVE STATISTICS OF OPTIONS SAMPLES

Contract	Days	Total Obs.	Dif11	DifFO	% Calls	Lognormal Index Pts.	Lognormal STD	Burr Index Pts.	Burr STD
Mar. 85	70	536	9.3	9.4	56.9	0.026	0.175	0.022	0.189
Jun. 85	79	539	9.9	11.7	59.2	0.021	0.129	0.030	0.161
Dec. 85	78	651	10.1	11.5	56.5	−0.005	0.113	−0.020	0.201
Mar. 86	87	836	8.7	9.7	52.3	0.040	0.156	0.004	0.306
Jun. 86	93	977	8.8	9.4	53.5	0.031	0.161	0.074	0.329
Sep. 86	93	1034	8.9	9.9	51.5	0.000	0.238	−0.005	0.261
Dec. 86	94	1075	7.8	9.6	56.2	0.012	0.253	0.067	0.591
Mar. 87	91	1176	8.2	8.4	48.2	0.011	0.467	0.068	0.266
Jun. 87	88	1365	8.8	8.0	51.2	0.082	0.205	0.047	0.187
Sep. 87	99	1352	9.2	9.0	50.4	0.033	0.279	0.004	1.480
Dec. 87	77	1513	11.3	8.1	53.3	0.495	1.206	−0.426	0.960
Mar. 88	71	888	12.2	8.4	47.7	0.220	1.449	−0.019	1.114
Jun. 88	72	804	12.1	8.2	39.1	0.625	2.163	−0.555	0.639

Days: total number of trade days for which option premia were sufficient to recover parameters of implied distributions; Total Obs.: total number of observations that remained in the contract after deletion/estimation criteria; Dif11: mean absolute difference in minutes of all strike prices used from 11:00; DifFO: mean absolute difference in seconds between option and futures prices; % Calls: percent of sample represented by calls; Index Pts.: mean of futures price index minus first moment of implied distribution; STD.: standard deviation of within-contract daily Index Pts.m

In the combined samples, 12,746 options are used to compute 1092 contract day's daily distributions for both the lognormal and Burr (or 2184 distributions) across 13 different S&P 500 futures options contracts.[5] On average, 11.7 options per day are used to derive an implied distribution for each contract, with 53% of the options being calls. The mean time difference between 11:00 and the option's trade is less than 9.5 minutes, and the mean difference between collected options prices and matched futures prices is 9.4 seconds.

METHODS AND RESULTS

If the no-arbitrage option pricing model is assumed to be correct, observed option prices may be used to recover estimates of the parameters of the candidate price distributions by searching for the values of the parameters that would most nearly result in the observed option prices. The procedure is similar to IV studies that search for a volatility measure that would equate observed prices to that predicted by the market, but the dimension of the choice variable vector is increased to the number of parameters of the candidate distribution.

To solve for the parameters of the candidate distributions that most nearly would result in the observed option prices, the expression

$$\min_{\beta}\left[\sum_{i=1}^{n}\left\{(c_i - b(T)\int_{K_i}^{\infty} f(Y_T|\beta)(Y_T - K_i)dY_T)^2\right\} + \sum_{j=1}^{m}\left\{(p_j - b(T)\int_{0}^{K_j} f(Y_T|\beta)(K_j - Y_T)dY_T)^2\right\}\right] \quad (14)$$

[5]Complete listings of parameters for these distributions are available from the authors upon request.

is solved. This procedure simply minimizes the sum of the squared differences between model and observed prices conditional on the choice vector, β. The procedure is applied twice daily to solve for the set of implied distribution parameters, β, that most nearly results in the observed option prices for both the Burr and lognormal distributions. Daily samples of "n" calls and "m" puts are used subject to the requirement that $(m + n)$ be greater than the dimension of β (greater than three for the Burr and greater than two for the lognormal).[6] While some studies weight the individual options differently based upon the elasticity of variance, or by the degree to which the option is in- or out-of-the-money, debate remains about the appropriateness of such weightings. Simple nonlinear least squares is used herein for simplicity.

An example of one of the daily estimated pairs of distributions is shown in Figure 1A which illustrates the implied ex ante price densities for the June 1988 S&P 500 futures contract that is implicit in the options prices on the day 60 prior to expiration of the option. Figure 1B shows the corresponding CDFs for both distributions.

Since the current futures prices do not enter directly into the estimation of the distributions [eq. (14)], comparisons of the first moment of the implied distributions with the current futures price gives an indication of the expected direction of futures price movements according to the options market. For example, if the futures price is lower (higher) than the mean of the implicit ex ante price distribution, it indicates that the futures price is expected to increase (decrease) according to the option-based estimates. This type of information is not available if the futures price is used in the estimation of the ex ante distribution. Table I also summarizes the mean difference between average futures price and $E(Y_T)$ as reflected in the implied distributions. For the positive (negative) differences in the variable "Index Pts.," the indication is that the options market expectation is for the price to fall (rise) as contract expiration approaches. The column marked STD gives the standard deviation of the daily Index Pts. differences. Strictly speaking, since the estimated distributions are not homoskedastic (in fact, they collapse at expiration to a point), it would be inappropriate to form a "t-statistic" to test for differences from zero. Nonethe-

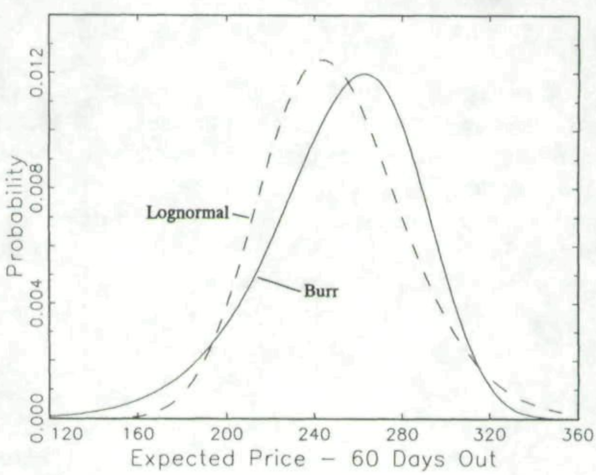

Figure 1a
Ex ante PDFs for June 1988.

[6]The GAUSS-386 programming language is used on an IBM-compatible computer.

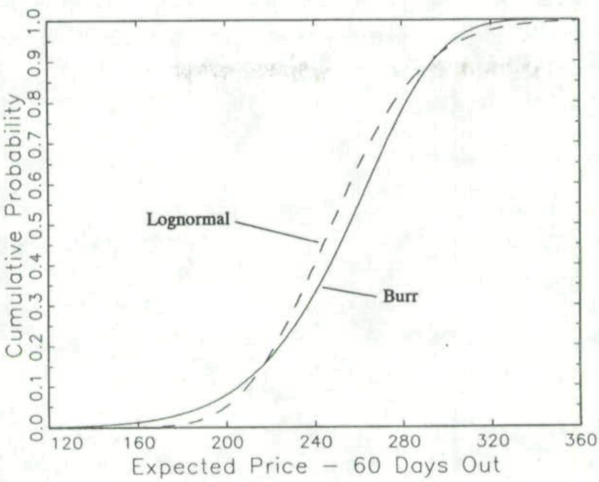

Figure 1b
Ex ante CDFs for June 1988.

less, the "pseudo-t" statistics formed by dividing the Index Pts. value by the standard deviation of its own series all fall in very insignificant ranges indicating close agreement in location between the mean of the implied distribution and the contemporaneous futures prices. The Burr distribution generally yields slightly smaller differences between futures prices and the first moments of the implied distributions than for the lognormal distribution. The true distribution of expected prices may be better reflected by the three-parameter Burr distribution than the more restrictive two-parameter lognormal distribution.

An alternative interpretation, given the apparent lack of significance in differences between the mean of the implied distributions and the contemporaneous futures prices is that the location predicted by both markets for a future price is similar. Or, if it is assumed that the futures price is an unbiased predictor of a future price, the options markets could be likewise judged to be accurate in assessing a distribution's location.

VARYING PARAMETER MODELS OF VOLATILITY

It is quite reasonable to expect the volatility of the implied distributions to be a function of time-to-maturity (TTM) by the very nature of the variable. Resolution of uncertainty, and the collapse of ex ante distributions to the realized price necessarily happen through time. Also, the price of the underlying variable may potentially influence volatility if there is a fairly constant coefficient of variation describing the uncertainty [consider the Cox (1975) Constant Elasticity of Variation model as a parallel]. Inferences about the speed and types of adjustments in expectations and expectation formation may be gleaned from an examination of the changes in market-implied expected distributions derived in an earlier section. Further, the "term structure" of uncertainty is revealed through a comparison of various maturity options at a point in time and by examining changes in a particular contract's implied distributions through time.[7] Therefore, a regression framework is

[7]As a simple analogy, suppose the variance of an expected price distribution for three months in the future is "k" and the variance of an expected price distribution for four months in the future is "$6 * k$," then the time during which the uncertainty is greatest is implicitly between the third and fourth months in the future.

imposed to describe the changes in volatility that takes direct recognition of the potential time dependence and inconsistent observation intervals. The posited model is quite simple in keeping some comparability to earlier approaches, while allowing very general parameter variation. It is assumed that the following model holds:

$$\theta_t = \beta_{1j} + \beta_{2j}(P_t) + \beta_{3j}(\mathrm{TTM}_t) + u_{jt} \tag{15}$$

where θ is the characteristic of the distribution being investigated (in this case, standard deviation from the Burr distribution), the j subscript is the regime index running from 1 to r implying $r - 1$ switchpoints, and P_t is the futures price on day t. If $r = 1$ (no switchpoints), the model reduces to ordinary least squares. If the index j depends upon some other possible stochastic variable z_i so that the regime of influence depends upon a threshold value of some $z_i \geq z^*$ for j to be in a new regime, then the model is of the class of switching regimes regressions. For this purpose, the most likely regime index is obviously time, but the cut off values, z^*, at which one regime supercedes the next must be estimated as well.

Brown, Durbin, and Evans (1975) (BDE) are among the first to investigate the detection of switchpoints in such a context. Because of potential non-normality problems, their CUSUM of squares test, while powerful at detecting the presence of a switchpoint, is not well suited at defining the precise location of that point. The BDE model makes use of recursive residuals and one-period-ahead forecast errors. Intuitively, each of the one-period-ahead forecast errors should contribute to the sum of squared residuals in about the same proportion.[8]

To test for the existence of multiple regime shifts, the BDE test is applied over sequentially longer intervals until a switchpoint is located. Then, the most likely point in the interval is found with the log-likelihood ratio [Johnston (1984)] by computing the test statistic for each possible switchpoint and picking the one that minimizes the ratio. Next, the interval up to and including the switchpoint is excluded from the sample and the process repeated. If no new switchpoints are located, then only two regimes are said to exist. If a new switchpoint is located, each possible subset of the original data, as partitioned by the switchpoints, is tested for the most likely occurrence. The process then searches for a third switch and so on until no new switchpoints are found. Since the BDE tests against an unspecified alternative (unknown location of change points), the LR tests are difficult to interpret. Furthermore, since the procedure outlined in Johnston searches across all possible LR statistics, it is potentially biased toward rejection.[9] The LR statistic searches are,

[8]BDE propose the following test statistic based on the series s_r:

$$s_r = \left(\sum_{j=k+1}^{r} w_j^2\right) \bigg/ \left(\sum_{j=k+1}^{T} w_j^2\right), \quad \text{for each} \quad r = k+1, \ldots, T \tag{16}$$

where w_j are the recursive residuals and, by definition, $s_k = 0$ and $s_T = 1$, and if all forecast errors are identically distributed, the s_r are shown to have a β distribution with mean $(r - k)/(T - k)$. Departures from the uniform line of the plot of s_r indicate a significant switch point is in the interval $(k + 1, T)$. A confidence interval can be set at $(r - k)/(T - k) \pm C_o$ with C_o a function of the level of significance (tabulated in BDE and elsewhere). Note that if the null of no switch holds, the w_j are independent and the use of such residuals avoids the serial correlation and non-normality problems associated with ordinary OLS residuals, and thus allows for much more powerful tests [Hays and Upton (1986)].

[9]The asymptotic properties cannot be relied upon because the likelihood function is defined only for integral values of "t." The authors are thankful to an anonymous reviewer for pointing this out to us and suggesting the fluctuations test as added evidence.

therefore, used not to detect the form of the parameter variation, but rather to help delineate subsets of data and provide good starting values for the endogenous switchpoint procedures. Next, the more powerful Ploberger et al. (1989) fluctuations test of the constancy of the regression coefficients is applied to each contract's data. Table II reports the location of the switchpoints and the results of the fluctuations test by contract. Note that the fluctuations test is highly significant in all contracts.

The most notable result is the prevalence of switches located within approximately two months prior to expiration. It is suspected that the expiration of the contract immediately preceding each contract has a spillover effect as activities are concentrated in what has become the new, nearby contract. For all contracts, there may also be a gradual change in volume over the life of the futures as different participants acquire new reasons to trade various maturity instruments. A changing volume may be associated with new information that manifests itself as a parameter change. The caution this finding raises is simply that there are frequent structural changes that require consideration before using long time series in other economic analyses.

At this point the switchpoints are still defined only in general terms. That is, if t^* is the switchpoint, the tests will detect either $\beta_{t<t^*} \neq \beta_{t>t^*}$ or nonconstant error variance across the regimes. Following Goldfeld and Quandt (1973, 1976), Quandt (1958, 1960), and Kane and Unal (1987, 1990), an endogenous switchpoint regression is used next to simultaneously estimate the $\hat{\beta}_{jt}$, equivalent t_i^*, and an additional parameter, σ_z^*, indicating the gradualness of change as one regime supercedes the previous one. The use of such a technique is also quite useful in event studies as the switch points are located <u>first</u> and then compared with plausible events. Running the test in this order conserves degrees of freedom and helps avoid the tendency to form illusive correlations.

Table II
LOCATION AND SIGNIFICANCE OF SWITCHPOINTS

Futures Contract	First Switch[a]	Second Switch[a]	BDE Statistic	Fluctuations Test
Mar. 1985	67	—	0.523[d]	10.11[c]
Jun. 1985	50	102	0.198[b]	13.10[c]
Dec. 1985	77	—	0.273[d]	16.64[c]
Mar. 1986	46	—	0.448[d]	9.94[c]
Jun. 1986	70	21	0.386[d]	13.32[c]
Sep. 1986	56	52	0.241[d]	9.97[c]
Dec. 1986	58	28	0.230[d]	12.53[c]
Mar. 1987	53	—	0.611[d]	7.42[c]
Jun. 1987	44	—	0.563[d]	10.24[c]
Sep. 1987	36	90	0.124	10.95[c]
Dec. 1987	62	23	0.462	10.72[c]
Mar. 1988	29	—	0.397[d]	8.87[c]
Jun. 1988	35	55	0.219[b]	10.05[c]

[a]Days from expiration.
[b]Significant at 0.05 level.
[c]Significant at 0.01 level.
[d]Significant at 0.005 level.
[e]October 19–20, 1987 omitted from December 1987, March 1988, and June 1988 contracts.

There are likely only a few relevant switchpoints if the model posited is, in fact, correct and relatively stable within given regimes. However, since the number and location of the switchpoints are unknown, a set of dummies to detect the switches cannot be used in any parsimonious fashion. That is, to detect the unknown location of one switchpoint with n observations and K regressors, there needs to be $(n - 2) * K$ sets of dummies, which obviously exceeds the total degrees of freedom. A feasible problem emerges, however, if a structure is imposed on the set of dummies such that a relatively small number of parameters describe an entire set of regime dummies. Then the parameters of the regression and the parameters of the dummy equations may be simultaneously estimated from the data. The model used is a variant of the "*D*-method" of Goldfeld and Quandt.

To allow the switchpoints to be endogenized, introduce transitional dummy variables, modeled as normal CDFs, D_{tj}:

$$D_{tj} = \int_{-\infty}^{Z_t} [(2\pi)(\sigma_j^*)^2]^{-1} \exp[-(1/2)\{(\Phi - z_j^*)/(\sigma_j^*)\}^2] d\Phi \tag{17}$$

where $j = 1, \ldots, k, \ldots r$ and, by definition, $D_{t0} = 1$ and $D_{tr} = 0$. Then the kth regime is multiplied by:

$$\tau_{tk} = \prod_{j=0}^{k-1} D_{tj} \prod_{j=k}^{r} (1 - D_{tj}) \quad \text{for} \quad t = 1, \ldots, T \tag{18}$$

approximating a step into and out of the regime. The "r" regime equations are then summed to get:

$$\sum_{k=1}^{r} \theta_t \tau_{tk} = \sum_{k=1}^{r} \{(\beta_{1k} + \beta_{2k} P_t + \beta_{3k} TTM_t + u_{tk})(\tau_{tk})\} \tag{19}$$

Assuming a normal distribution for θ with mean and variance of:

$$\mu_{t\theta} = \sum_{k=1}^{r} \{(\beta_{1k} + \beta_{2k} P_t + \beta_{3k} TTM_t)(\tau_{tk})\} \tag{20}$$

and

$$\sigma_{t\theta}^2 = \sum_{k=1}^{r} \sigma_{tk}^2 (\tau_{tk}^2) \tag{21}$$

where σ_{tk}^2 is the kth regime error variance and implies a log-likelihood function:

$$L = -(T/2)\log 2\pi - (1/2)\sum_{t=1}^{T} \log \sigma_{t\theta}^2 - (1/2)\sum_{t=1}^{T} \left[\left\{ \sum_{k=1}^{r} \theta_t \tau_{tk} - \mu_{t\theta} \right\}^2 \right] \Big/ \sum_{t=1}^{T} \sigma_{t\theta}^2 \tag{22}$$

Maximizing L w. r. t. its unknown parameters gives the switchpoints, z_j^*; the gradualness of change, σ_j^*; and the parameters of the original regression relation. The endogenous nature of the *D*-method lends itself to event study as the switchpoints are taken as events and estimated rather than specified a priori.

The above equation is estimated via maximum likelihood for the contracts which display regime dependence in the earlier tests of BDE and the likelihood ratio test. Starting values for β are taken as the OLS coefficients of the regression over the subperiods as delineated by the likelihood ratio test, and the error variance in separate regimes initially assumed identical. The value of z_i^*, or the most likely switchpoint, is started at the regime break located earlier, and is not permitted to vary beyond the ends of the sample. After some experimentation, the variance of the dummy equation is set to initial values near 0 and permitted to range to 20% of the maximum time index. Greater values tend to obliterate the discrimination between regimes. Note that a value of σ^* equal to zero corresponds to a perfect step function, and greater values indicate a slower speed of transition between regimes.

The switching regression results are given in Table III for the contracts that displayed regime dependence earlier. As expected, the switchpoints, z^*, correspond well to those located by the earlier methods. Further, all the estimates for σ^* are highly significant and indicate reasonably rapid transitions between regimes. Interpreting the σ^* as the gradualness parameter indicates that the two-SE confidence interval for the contract switchpoint varies from approximately 3.64 (June 1986) to 9.1 days (June 1988). Approximating the transition period as twice the gradualness parameter implies the switches complete themselves in the same amount of time [Kane and Unal (1987)].

The fact that the coefficients of the TTM and price level differ by regime indicates that there is a need to consider regime-dependent influences before using

Table III
SWITCHING REGRESSION RESULTS FOR MODEL OF EXPECTED S&P 500 FUTURES PRICE DISTRIBUTIONS

	Regime Nearest Expiration			Next Nearest Regime			Dummy Parameters	
Contract	β_{11}	β_{21}	β_{31}	β_{12}	β_{22}	β_{32}	σ^*	z^*
Mar. 86	−61.963	0.414	0.761	−52.843	0.475	0.117	2.425	28.331
	(0.227)	(0.058)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.683)
Jun. 86	−3.144	0.150	0.640	−84.390	0.628	0.152	1.824	65.316
	(0.820)	(0.010)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.002)
Sep. 86	6.605	0.131	0.521	8.217	0.203	0.190	2.058	59.286
	(0.500)	(0.001)	(0.000)	(0.525)	(0.000)	(0.000)	(0.000)	(0.027)
Dec. 86	−72.473	0.440	0.616	49.191	0.051	0.180	1.870	60.350
	(0.001)	(0.000)	(0.000)	(0.000)	(0.127)	(0.000)	(0.000)	(0.000)
Mar. 87	156.078	−0.410	1.164	−113.530	0.632	0.341	2.510	31.640
	(0.002)	(0.014)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
Jun. 87	112.914	−0.240	1.069	82.456	0.008	0.148	2.343	44.325
	(0.000)	(0.001)	(0.000)	(0.000)	(0.862)	(0.000)	(0.000)	(0.000)
Sep. 87	66.525	−0.077	1.569	201.843	0.780	0.712	3.010	30.072
	(0.136)	(0.586)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
Dec. 87	205.676	−0.682	1.757	6.268	0.222	0.355	4.319	62.334
	(0.000)	(0.000)	(0.000)	(0.854)	(0.054)	(0.000)	(0.000)	(0.000)
Mar. 88	−36.574	0.292	1.367	106.168	−0.115	0.368	3.199	38.783
	(0.501)	(0.148)	(0.000)	(0.000)	(0.006)	(0.000)	(0.000)	(0.000)
Jun. 88	58.236	−0.049	0.720	173.894	−0.405	0.424	4.561	67.914
	(0.033)	(0.634)	(0.000)	(0.003)	(0.077)	(0.000)	(0.000)	(0.000)

Entries are estimated coefficients from eq. (11), which correspond to the coefficients from eq. (4) and parameters from the dummy eq. (6). For example, β_{1j} is for the constant in regime j, β_{2j} is for price level in regime j and β_{3j} is for TTM in regime j.

P-values are given in parentheses.

these types of implied parameters in other economic contexts. Finally, the switching regression methods serve to confirm dates that may serve as events. For the present purposes, the most obvious and prevalent switches appear to be within bimonthly intervals prior to expiration. Further classifying the causes and modeling them directly present topics for further study.

Several other results are indicated. First, note that in each of the S&P 500 contracts that display multiple regimes, all the TTM coefficients are highly significant, and the coefficient on TTM_1 is always greater than on TTM_2, indicating that the rate of decline in volatility increases as the time-to-maturity decreases. The flexible model with endogenous switchpoints permits the regimes to be continuously mixed with the strength of each regime's variables depending on the dummy parameters, allowing a great deal of smoothing of the coefficients over time. Nonetheless, there appears to be high levels of discrimination among regimes, as evidenced by the low P-values for the coefficients on z^* and TTM_1 and TTM_2. One-half of the first regime price coefficients are positive and one-half are negative. In the second regime, four-fifths of the price coefficients are positive. The preponderance of evidence is weakly in support of a positive price level effect and rather strongly in support of a multiple regime influence of TTM on volatility.

The results can be considered also from an event-study standpoint. Each of the switches admits itself as a candidate event that corresponds to a temporal change in the coefficients of the model. The mere existence of significant switchpoints renders estimates unreliable if estimated from the entire sample. The lack of switches causes little concern, other than the loss of degrees of freedom in the current model, as similar and significant coefficients on TTM and price could exist across regimes and the discrimination by the dummy could be very poor. The interpretation of the transition parameter as σ^* allows confidence regions to be constructed to account for specified portions of the transition to be completed. To understand the changes in the market estimates of future volatility, it is important to allow this type of more general parameter change through time.

SUMMARY AND CONCLUSIONS

No-arbitrage pricing methods are used to derive estimates of expected future price distributions. The improved data and parameter estimation techniques of this study provide an important backdrop for empirical study. Risk-management techniques rely on accurate descriptions of uncertainty. This study investigates one such technique which works quite well at describing an uncertain price distribution. Thus, these techniques provide promising alternatives for deriving useful information about future price distribution.

A switching regression model is used then to describe the parameter variation in endogenously selected regimes. The notable findings include the fact that the switches tend to occur within approximately bimonthly intervals prior to expiration and that the volatility declines at a greater rate as the time to maturity decreases. No consistent price level effects are detected because they may, in fact, differ by time regime. A direct application is the generation of an ex ante benchmark for event studies. Further, the detection of numerous switchpoints in the implied volatility series suggests that inappropriate sample intervals may at times be candidate causes for anomalous results in previous studies that examine time-to-maturity effects without considering possibly varying parameters. Inappropriate conclusions may be drawn if the issues of parameter variability are not first considered.

A possible extension of this research involves pricing and trading strategies. If accurate descriptions of future price distributions are available, a simple option-pricing model emerges, as demonstrated for the Burr distribution. If low-cost estimates of future price distributions are available and reliable, they may be easily incorporated in forecasted outcomes of various trading strategies. Or, if "better" descriptions of future uncertainties are available, pricing models may be developed that accurately reflect the inner workings of the particular market.

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