Hedge Effectiveness: Basis Risk and Minimum-Variance Hedging

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INTRODUCTION

Hedgers historically view hedging in terms of the basis.[1] This is because hedging is seen much more as an act of arbitrage between cash and futures prices rather than as an action undertaken specifically to reduce risk. In a sharply critical appraisal of the view that hedging is undertaken to reduce risk, Holbrook Working states:

> Perhaps the main reason that hedging, as commonly practiced on futures markets, has been so widely misunderstood and misrepresented is that economists have tried to deal with it in terms of a concept that seemed to cover all sorts of hedging. This would be desirable if it were feasible, but the general concept of hedging as taking offsetting risks wholly, or even primarily, for the sake of reducing net risks, serves so badly as applied to most hedging in futures markets that we need another concept for that most common sort of hedging. To put it briefly, hedging in commodity futures involves the *purchase or sale of futures in conjunction with another commitment, usually in the expectation of a favorable change in the relation between cash and futures prices.*[2]

The essence of Working's view of hedging is that hedging is a form of arbitrage between cash and futures prices. It is undertaken to profit from predictable changes in the relationship between cash and futures prices and not specifically to reduce risk. This is not to say that hedgers are not concerned with risk, but rather that risk reduction is *incidental* to the hedging decision and *not* the motivation for it. A hedger is not a risk averter, as tradition would have it; but rather a risk selector. A hedger prefers to profit from a skillful prediction of changes in the basis rather than from predictions of price levels. In so doing, the hedger becomes exposed to "basis risk." Hence, the view of hedging as "speculating on the basis."

The view of hedging that appears to prevail today is the one that draws from portfolio theory. It originates from Johnson (1960) and Stein (1961). Extensions or applications of it are seen in Heifner (1972); Ederington (1979); Anderson and Danthine (1980); Beninga, Eldor, and Zilcha (1984); Brown (1985); Adler and

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[1]For extensive coverage of this view, see Peck (1978).

[2]In Working (1953). Italics in the original.

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Detemple (1988); Briys, Crouhy, and Schlesinger (1990); and others. The rationale underlying the theory is that hedgers are risk-averse utility of wealth maximizers. Given the hedger's degree of risk aversion, the hedger chooses to hedge partially or fully in an attempt to trade-off risk against return.

An important result developed from the portfolio theory approach is the concept of the minimum-variance hedge ratio, i.e., the hedge ratio that minimizes risk of the hedged position. A hedger initiates a minimum-variance hedge if he is extremely risk averse. Estimation of the minimum-variance hedge ratio is the subject of much research. [See, e.g., Ederington (1979); Hill and Schneeweiss (1981); Kamara and Siegel (1987); Witt, Schroeder, and Hayenga (1987); Myers and Thompson (1989); Castelino (1990a, 1990b); Myers (1991).] Although there's no consensus regarding an appropriate method to estimate the minimum-variance hedge ratio, a distinct bias in favor of regression analysis exists.

The view of hedging as expressed by Working appears to diverge significantly from the portfolio view in a very important aspect—motivation. According to Working, the primary motive for undertaking a hedge is profit. Risk reduction is, at best, a secondary consideration. The portfolio view elevates risk reduction to a level of primary importance or to a level of importance equal to that of profit. If futures prices are unbiased, then the portfolio view of hedging reduces to minimum-variance hedging, thereby giving risk reduction prime consideration in the hedging decision.[3] This is in sharp contrast to Working's view.

The underlying theme of this study is to relate minimum-variance hedging to Working's view of hedging as "speculating on the basis." Given the diametrically opposite motives of the arbitrage hedger (hedging for profit) and the minimum-variance hedger (hedging to reduce risk), this study attempts to relate the hedging strategies of the two hedgers. It is shown that, in unbiased futures markets, hedgers should be driven to hedge by expected profit. If the decision to hedge is made based on the expected profit, the hedger may consider whether it is worthwhile placing a minimum-variance hedge. The reason for this is that a minimum-variance hedge has *no effect* on expected profit, it only serves to reduce risk.

The risk reduction afforded by minimum-variance hedging is put in perspective by comparing it to the risk of a full-hedge (basis risk). If minimum-variance hedging can reduce risk substantially below basis risk, then it ought to be considered; otherwise, it should not. In principle, a minimum-variance hedge ratio possesses a "time dimension." The minimum-variance hedge ratio is low for hedges lifted far from contract expiration and monotonically increases as the hedge-lifting date approaches contract expiration. [See Castelino (1989, 1990a) for an explanation of the relationship of "time dimension" to basis risk and for an explanation of the need to adjust the hedge ratio in response to it.]

This study reformulates the hedging decision from the perspective of the hedger, i.e., in terms of the basis. The role played by basis risk in influencing the minimum-variance hedge ratio is next addressed, followed by a theoretical development of the time dimension to the minimum-variance hedge ratio. Tests for the existence of a time dimension to the minimum-variance hedge ratio in the wheat, corn, Treasury-bill, and Eurodollar futures markets are conducted. A significant time dimension is revealed for T-bills and Eurodollar futures. An insignificant one is revealed for wheat and corn futures. The significantly different behavior of the minimum-

[3]See Beninga et al. (1984).

variance hedge ratios for interest rate futures compared to the grain futures is explained by Working's theory of the price of storage [Working (1948, 1949)].

THEORETICAL DISCUSSION

Hedging Model

Consider a hedger taking a position in the cash market at time "0" when the spot price is $P_c(0)$ and the futures price for delivery at time "T" is $P_f(0,T)$. If a hedge is placed on the cash position with the intent of lifting it at time "t," $(0 < t \leq T)$, the profit or loss (Hedge P/L) from the hedge is given by:

$$\begin{aligned} \text{Hedge } \tilde{P}/L &= \text{Cash } \tilde{P}/L - \text{Hedge Ratio} \times [\text{Futures } \tilde{P}/L] \\ &= [\tilde{P}_c(t) - P_c(0)] - h(t)[\tilde{P}_f(t,T) - P_f(0,T)] \end{aligned} \tag{1}$$

where $h(t)$ is the fraction of the spot position hedged.

Since hedgers often view hedging in terms of the basis, the Hedge P/L is reformulated to reflect the view of the hedger. This is shown in eq. (2).

$$\text{Hedge } \tilde{P}/L = [\tilde{B}(t,T) - B(0,T)] + [1 - h(t)][\tilde{P}_f(t,T) - P_f(0,T)] \tag{2}$$

where $\tilde{B}(t,T) = \tilde{P}_c(t) - \tilde{P}_f(t,T)$ is the basis at time "t"

The expected Hedge P/L and variance of Hedge P/L are given in eqs. (3) and (4), respectively.

$$E[\text{Hedge } \tilde{P}/L] = \{E[\tilde{B}(t,T)] - B(0,T)\} + [1 - h(t)]\{E[\tilde{P}_f(t,T)] - P_f(0,T)\} \tag{3}$$

$$\begin{aligned} \text{Var}[\text{Hedge } \tilde{P}/L] &= \text{Var}[\tilde{B}(t,T)] + [1 - h(t)]^2\text{Var}[\tilde{P}_f(t,T)] \\ &\quad + 2[1 - h(t)]\text{Cov}[\tilde{B}(t,T), \tilde{P}_f(t,T)] \end{aligned} \tag{4}$$

Eq. (3) reveals that expected Hedge P/L consists of two components:

1. A basis component.
2. A price biasedness, or speculative component.

The basis component of Hedge P/L is not affected by the hedge ratio. The price biasedness component, however, is affected by it.

There are two distinct theories regarding what motivates the hedger to hedge. For the sake of brevity the theories are referred to as Working's Arbitrage Theory, and the Utility Maximization or the Portfolio Theory of hedging. The implications for hedging drawn from eqs. (3) and (4) will be different, depending upon which theory the hedger subscribes to.

Working's Arbitrage Theory. In Working's arbitrage theory, hedging is viewed as an act of arbitrage between cash and futures prices. Hedges are placed if the hedger believes that futures prices reflect an attractive profit opportunity when compared to the spot price. What represents an "attractive profit opportunity" is something that only individual hedgers can decide. The reasons for hedging are so varied that it would be difficult, if not impossible, to come up with a precise form for it that would apply to all hedgers. The important point, though, is that most hedgers are motivated by profit and not by the desire to reduce risk. This does not mean that hedgers are unconcerned with risk, but rather that risk reduction defers to profit as the primary motive in undertaking a hedge.

Additionally, Working argues that expectations regarding future events are embedded in spot prices just as much as they are in futures prices. So much so that, at least for storable commodities, the difference between spot and futures prices (the basis) does not depend on events forecasted to occur in the future. To quote him once again [Working (1948)]:

> The business of futures markets, so far as it may differ from any other, is to anticipate future developments as best it may and to give them due expression in present prices, spot and near futures as well as distant futures.

The implication drawn from his analysis is that the current futures price is as good an estimate as any of a future futures price. Hence, substituting $P_f(0,T)$ for $E[\tilde{P}_f(t,T)]$ in eq. (2), the expected Hedge P/L is given in eq. (5). If the hedge is placed (i.e., $h(t) = 1$), then the risk of the hedge is given in eq. (6).

$$E[\text{Hedge } \tilde{P}/L] = E[\tilde{B}(t,T) - B(0,T)] \tag{5}$$

$$\text{Var}[\text{Hedge } \tilde{P}/L] = \text{Var}[\tilde{B}(t,T)] \tag{6}$$

Eqs. (5) and (6) represent Working's view of hedging as "speculating on the basis." Since a hedger expects the basis, $B(t,T)$, to narrow as the hedge lifting date, (t), approaches the contract expiration date, (T); the profit from the hedge can be anticipated. For example, if a hedge is initiated on a commodity *underlying* the futures contract with the intent of lifting the hedge at contract expiration, the profit on the hedge is highly predictable. It will equal the opening basis, $-B(0,T)$; since, theoretically, the closing basis, $B(T,T)$, should be zero. If, in the hedger's judgment, the opening basis represents an attractive profit opportunity, a hedge should be placed (i.e., $h = 1$). This hedge is riskless. If the opening basis is unattractive, a hedge should not be placed ($h = 0$). The size of the basis at the time of hedge initiation, therefore, crucially influences the decision to hedge or not. The issue of a hedge ratio other than zero or unity does not arise.

If the hedge-lifting date is prior to contract expiration, the closing basis will not be zero which guarantees that the hedge will have some residual risk, "basis risk." However, the decision to hedge or not depends on the hedger's forecast of the closing basis and whether it represents an attractive profit opportunity when compared to the opening basis. The important point is that the hedger is not concerned with forecasting the futures price at the hedge-lifting date, but only its relationship to the then spot price.

Portfolio Theory—Utility Maximization. The portfolio theory or utility maximization approach views hedging as a trade-off between risk and return. Such a trade-off can exist only if futures prices are believed to be biased, and the hedger is risk averse. For example, if a (short) hedger believes futures prices are downward biased, the hedger may choose to under-hedge ($h(t) < 1$ in eq. (3)] to profit from the upward drift anticipated for futures prices. The reverse is true if futures prices are upward biased. Hence, by varying the hedging fraction in line with a hedger's degree of risk aversion, an efficient frontier representing a risk-return trade-off can be drawn with eqs. (3) and (4).

This line of reasoning fosters a shift in the focus of the hedger away from the basis component of return in eq. (3) to the price biasedness component. The hedging fraction is critically influenced by the price biasedness component. This is valid and consistent with the theory, but is it useful in practice?

The empirical evidence to date suggests that prices in active futures markets display no significant bias one way or the other.[4] If such is the case, one could argue that the portfolio approach to hedging tends to divert the attention of the hedger from what should be focused on, the basis, to something that should be avoided, price forecasting, in an attempt to extract a share of the perceived risk premium.[5]

The concept of a minimum-variance hedge does not depend on any notion of biasedness in futures prices. The role it plays in the hedging decision is best seen if futures prices are assumed to be unbiased.

Minimum-Variance Hedging. If futures prices are unbiased [$E(\tilde{P}_f(t,T)) = P_f(0,T)$ in eq. (3)] then the expected Hedge *P/L* is unaffected by the hedge ratio. It depends only on the current basis and the predicted basis on the hedge-lifting date. In such a case it may be worthwhile for the hedger to consider a hedge that minimizes risk since profit is unaffected. However, prior to considering a minimum risk hedge, the hedger ought to consider whether it is worthwhile placing a hedge at all. The basis should be considered here. The hedger should hedge if he feels the basis reflects an attractive profit opportunity. Once the decision to hedge is made, the hedger may then consider placing a minimum-variance hedge. The minimum-variance hedge ratio is obtained by differentiating eq. (4) with respect to $h(t)$ and equating to zero. It is displayed in eq. (7):

$$\begin{aligned}\text{MVHR}(t) &= h(t)^* = 1 + \text{Cov}[\tilde{B}(t,T), \tilde{P}_f(t,T)]/\text{Var}[\tilde{P}_f(t,T)]\\ &= 1 + \rho_{Bf}\,\sigma_B(t,T)/\sigma_f(t,T) \qquad (7)\end{aligned}$$

where ρ_{Bf} is the correlation between the basis and the futures price. $\sigma_B(t,T), \sigma_f(t,T)$ are, respectively, the standard deviations of the basis and the futures price at the hedge-lifting date (t). The residual risk (RESID) of the minimum-variance hedge is the variance of Hedge *P/L*, after substituting MVHR(t) for $h(t)$ in eq. (4). It is shown in eq. (8).

$$\text{RESID}(t)^* = \sigma_B^2(t,T)[1 - (\rho_{Bf})^2] \qquad (8)$$

The attractiveness of eqs. (7) and (8) lies in the fact that the hedging fraction and the risk of the hedge are expressed in terms of *basis risk,* the quantity the hedger is intimately concerned and familiar with.

Basis Risk and Minimum-Variance Hedge Ratio

Eqs. (7) and (8) reveal that basis risk [$\sigma_B(t,T)$] has a profound effect on the minimum-variance hedge ratio and its residual risk. Theoretically, if basis risk is zero, then the minimum-variance hedge ratio is unity and residual risk is zero. If the

[4]The term "significant" is used in a material sense rather than a statistical one. Some studies do lay claim to the existence of "statistically significant" bias in some futures markets [see, e.g., Carter, Rausser, and Schmitz (1983); Raynaud and Tessier (1984); Beck (1987)]. Of importance to the hedger, however, is the amount of (hedging) cost being paid because of biasedness [see Gray (1960a, 1960b)]. If the cost is trivially small compared to the basis, as it appears to be in most active futures markets, then the hedger is well advised to ignore the issue of bias in deciding whether to hedge or not.

[5]The term "risk premium" is used interchangeably with "price biasedness." This is not strictly correct, because biasedness in prices does not necessarily imply the existence of a risk premium. For an excellent discussion of this issue, see Gray (1960b, 1961).

basic risk exists, then the minimum-variance hedge ratio may depart from unity. The direction of departure from unity depends critically on the correlation between the basis and the futures price. If $\rho_{Bf} < 0$, then the MVHR(t) < 1. If $\rho_{Bf} > 0$ then MVHR(t) > 1. Additionally, if basis risk is a function of time (to expiration) then so must be the minimum-variance hedge ratio.

Even though the hedging model suggests that the minimum-variance hedge ratio can exceed unity [if $\rho_{Bf} > 0$], theoretically, as well as in practice, it should not exceed 1.00 because ρ_{Bf} should be negative or, at most, equal to zero. Consider the following arguments supporting a negative correlation between the basis and futures prices for interest rate futures and commodity futures.

In the case of interest rate futures, an unanticipated rise in interest rates causes a fall in both cash and futures prices. However, the rise in interest rates implies an increase in the cost of carry which is manifested through a widening of the basis. Thus, opposite movements in the basis and the futures price implies negative correlation between the two. The reverse argument can be made if interest rates fall.

For commodities, the case for negative correlation can be deduced from Working's theory of the price of storage. Working (1948, 1949, 1958) argues that the difference between spot and futures prices depends only on the levels of existing supplies and not on events forecasted to occur in the future. A negative basis [$P_f(0,T) > P_c(0)$] reflects high current supplies and a positive return to storage. A positive basis [$P_f(0,T) < P_c(0)$] reflects low current supplies, thereby discouraging storage. The degree to which the basis is negative or positive depends on the excess or shortage of current supplies. Now consider an event which reveals that current supplies are severely underestimated. This information immediately causes both spot and futures prices to drop sharply. However, spot prices should drop further than futures prices (i.e., the basis should widen) to reflect the need for an increased return to storage in the face of the increased supply. The reverse is true if the new information reveals an overestimate of current supply. In either case, however, opposite movements in the basis and the futures price imply negative correlation between the two.

Hence, negative correlation between the basis and the futures price implies that the minimum-variance hedge ratio should, at most, equal unity. Its departure from unity is guaranteed if the time pattern of basis volatility differs from that of futures volatility. The following section reveals that such is the case under some well specified assumptions of the behavior of cash and futures prices.

The Time Dimension to the Minimum-Variance Hedge Ratio

This section is devoted to showing that minimum-variance hedge ratios possess a time dimension in response to the time dimension of basis risk. It should be understood that basis risk possesses several dimensions, the familiar ones being those arising from quality and location differentials between the commodity hedged and the one underlying the futures contract. Without a doubt, these dimensions of basis risk have their own effects on the minimum-variance hedge ratio. The concern here is specifically with the impact of hedge timing (the time dimension of basis risk) on the minimum-variance hedge ratio. Therefore, to abstract from the other dimensions of basis risk, it is assumed that the commodity hedged is the one underlying the futures contract. This permits the assumption of full convergence between cash and futures prices at contract expiration. The Appendix shows how the derived formulations are affected for the case where the commodity hedged is not deliverable on the futures contract.

Assume the following processes for generating spot and futures prices:

$$\tilde{P}_c(1) = P_c(0) + (1/T)[P_f(0,T) - P_c(0)] + \tilde{x}_1 \tag{9}$$

$$\tilde{P}_f(1,T) = P_f(0,T) + \tilde{y}_1 \tag{10}$$

where:

T is the number of periods to expiration of the contract.

$$\begin{aligned} E(\tilde{x}_i) &= E(\tilde{y}_i) = 0 && 1 \le i \le T \\ \tilde{x}_T &= \tilde{y}_T && \\ \text{Var}(\tilde{x}_i) &= \sigma_x^2;\ \text{Var}(\tilde{y}_i) = \sigma_y^2 && 1 \le i \le T \\ \text{Cov}(\tilde{x}_i, \tilde{y}_j) &= \rho_{xy}\sigma_x\sigma_y && i = j \\ &= 0 && i \ne j \end{aligned}$$

The spot-generating process chosen suggests that the best estimate of tomorrow's spot price is today's spot price plus a carrying charge reflected in the basis. The futures generating process assumes that today's futures price is the best estimate of tomorrow's futures price, a random walk process.

The two processes are selected for their reasonableness and simplicity. The choice of the spot-generating process simply reflects the information on which a hedger acts. The hedger must expect convergence between spot and futures prices to take place. Linear convergence is selected for expositional convenience. Where the futures price generating process is concerned, it is assumed that the hedger is, on average, not prepared to speculate on futures prices.[6] Hence, the assumption that today's futures price is the best estimate of tomorrow's futures price.

Using eqs. (9) and (10) the expression for the basis is obtained.

$$\begin{aligned} \tilde{B}(1,T) &= \tilde{P}_c(1) - \tilde{P}_f(1,T) \\ &= B(0,T)[(T-1)/T] + [\tilde{x}_1 - \tilde{y}_1] \end{aligned} \tag{11}$$

The correlation between the basis and the futures price is shown below:

$$\begin{aligned} \rho_{Bf} &= \text{Cov}[\tilde{B}(1,T), \tilde{P}_f(1,T)]/\sigma_B(1,T)\sigma_f(1,T) \\ &= [\rho_{xy}\sigma_x\sigma_y - \sigma_y^2]/\sigma_y(\sigma_x^2 + \sigma_y^2 - 2\rho_{xy}\sigma_x\sigma_y)^{1/2} \\ &= [\rho_{xy}\sigma_x - \sigma_y]/(\sigma_x^2 + \sigma_y^2 - 2\rho_{xy}\sigma_x\sigma_y)^{1/2} \end{aligned} \tag{12}$$

Utilizing eqs. (9), (10), and (11) and the technique of successive substitution of $\tilde{P}_c(1), \tilde{P}_c(2) \ldots \tilde{P}_c(t-1)$ and $\tilde{P}_f(1,T), \tilde{P}_f(2,T) \ldots \tilde{P}_f(t-1,T)$, it can be shown that $\tilde{P}_c(t)$, $\tilde{P}_f(t,T)$, and $\tilde{B}(t,T)$ "t" periods in the future are given by:

$$\begin{aligned} \tilde{P}_c(t) = {} & [(T-t)/T]P_c(0) + (t/T)P_f(0,T) \\ & + [(T-t)/(T-1)]\tilde{x}_1 + [(T-t)/(T-2)]\tilde{x}_2 + \ldots \\ & + [(T-t)/(T-t+1)]\tilde{x}_{t-1} + \tilde{x}_t \\ & + [(t-1)/(T-1)]\tilde{y}_1 + [(t-2)/(T-2)]\tilde{y}_2 + \ldots \\ & + \{[t-(t-1)]/[T-(t-1)]\}\tilde{y}_{t-1} \ldots \end{aligned} \tag{13}$$

$$\tilde{P}_f(t,T) = P_f(0) + \tilde{y}_1 + \tilde{y}_2 + \tilde{y}_3 + \ldots + \tilde{y}_t \tag{14}$$

[6]This is not to suggest that hedgers should *never* speculate on futures prices. For, even though futures prices may, on average, unbiased, there could be intervening time periods where bias exists. A hedger may choose to be a selective hedger during such periods.

$$\tilde{B}(t,T) = \tilde{P}_c(t) - \tilde{P}_f(t,T)$$
$$= [(T-t)/T]B(0,T) + (T-t)[(\tilde{x}_1 - \tilde{y}_1)/(T-1) + \ldots + (\tilde{x}_t - \tilde{y}_t)/(T-t)] \qquad (15)$$

It can be shown that the expressions for $\text{Var}[\tilde{P}_f(t,T)]$, $\text{Var}[\tilde{B}(t,T)]$, and $\text{Cov}[\tilde{B}(t,T), \tilde{P}_f(t,T)]$ are as displayed below:

$$\text{Var}[\tilde{P}_f(t,T)] = t\sigma_y^2 \qquad (16)$$

$$\text{Var}[\tilde{B}(t,T)] = [(T-t)/T]t[\sigma_x^2 + \sigma_y^2 - 2\rho_{xy}\sigma_x\sigma_y] \qquad (17)$$

$$\text{Cov}[\tilde{B}(t,T), \tilde{P}_f(t,T)] = (T-t)[\ln(T) - \ln(T-t)](\rho_{xy}\sigma_x\sigma_y - \sigma_y^2) \qquad (18)$$

The minimum-variance hedge ratio for the hedge-lifted "t" periods into the hedge can now be obtained by substituting eqs. (16) and (18) into eq. (7):

$$\text{MVHR}(t) = 1 + \text{Cov}[\tilde{B}(t,T), \tilde{P}_f(t,T)]/\text{Var}[\tilde{P}_f(t,T)]$$
$$= 1 + (T-t)[\ln(T) - \ln(T-t)](\rho_{xy}\sigma_x\sigma_y - \sigma_y^2)/t\sigma_y^2 \qquad (19)$$

The residual risk of the hedge RESID(t) is given by:

$$\text{RESID}(t) = \sigma_B^2(t,T)[1 - (\rho_{Bf})^2]$$
$$= [(T-t)/T]t[\sigma_x^2 + \sigma_y^2 - 2\rho_{xy}\sigma_x\sigma_y]$$
$$\times \{1 - [\rho_{xy}\sigma_x - \sigma_y]^2/(\sigma_x^2 + \sigma_y^2 - 2\rho_{xy}\sigma_x\sigma_y)\}$$
$$= [(T-t)/T]t\sigma_x^2[1 - (\rho_{xy})^2] \qquad (20)$$

Interpretation of MVHR(t) and RESID(t)

The importance of hedge timing on the minimum-variance hedge ratio and the measure of residual risk is clearly seen in eqs. (19) and (20). Consider, for example, a hedge to be lifted at contract expiration. Full convergence between cash and futures prices guarantees that a full-hedge lifted at contract expiration is riskless. The minimum-variance hedge ratio must be unity. Substituting $t = T$ in eqs. (19) and (20) confirms that MVHR = 1 and RESID = 0. As the hedge lifting date recedes from expiration ($t < T$), the behavior of MVHR(t) is seen to depend critically on the sign of the entire expression containing the variance and covariance terms ($\rho_{xy}\sigma_x\sigma_y - \sigma_y^2$), which is none other than the covariance between the basis and the futures price. Since the covariance should be negative, as argued earlier, MVHR(t) should monotonically decrease as the hedge-lifting date recedes from contract expiration.

Both MVHR(t) and RESID(t) reduce to the estimates of the minimum-variance hedge ratio, and its residual risk is obtained via a simple regression between cash and futures prices if the assumptions underlying the regression are made. Implicit in a simple regression of cash on futures prices is that relative price changes are invariant with time. Putting it differently, the minimum-variance hedge ratio is, in theory, the same for a one-day hedge or a ten-day hedge, no matter how far from contract expiration the hedge is initiated.

Hence, substituting the $t = 1$ and permitting "T" to approach infinity, $(T-1)[\ln(T) - \ln(T-1)]$ approaches 1.00. Making these substitutions in eq. (19), MVHR(t) reduces to the regression hedge ratio.

$$\text{MVHR}(t = 1) = \rho_{xy}\sigma_x/\sigma_y \qquad (21)$$

Similarly, for $T \gg t = 1$, $(T - 1)/T \cong 1$. Substituting it in the expression for RESID(t), the residual risk of the hedge reduces to:

$$\mathrm{RESID}(t = 1) = \sigma_x^2[1 - (\rho_{xy})^2] \tag{22}$$

This is simply the variation in cash prices that remains unexplained by regression.

EMPIRICAL RESULTS

The empirical tests are limited to cases of the hedging of commodities that underlie the futures contract to find if it is possible to reduce risk below that of basis risk. If it is possible to do so, then a minimum-variance hedge ratio makes sense because the theory has demonstrated that the hedge ratio, minimum-variance or other, cannot affect the expected return on a hedge if the current futures price is an unbiased estimate of a future futures price.

The variable of critical importance in this regard is the correlation between the basis and the futures price (ρ_{Bf}). If it can be shown that ρ_{Bf} is zero, then the minimum-variance hedge ratio is unity and, therefore, risk cannot be reduced below basis risk [see eq. (8)]. If ρ_{Bf} is less than zero, then it is possible to reduce risk below that of basis risk. Also, the hedge ratio that accomplishes this risk reduction must have a time dimension. It will be unity for a hedge lifted at contract expiration and it will monotonically decline as the hedge lifting date recedes from contract expiration.

Data

Several futures markets are tested. Among the grains, wheat and corn futures are selected. Among the financials, T-bills and Eurodollar futures are used. The data for wheat and corn extend from January 1, 1983 through December 31, 1985. For T-bills and Eurodollars the time period included extends from January 1, 1986 through December 31, 1989. Cash prices for wheat and corn are from the USDA. The wheat prices are for No. 2 Soft Red Wheat. Corn prices are for No. 2 Yellow Corn. Both prices represent bids for delivery in the Chicago area and are deliverable on the contract traded at the Chicago Board of Trade (CBT). The futures prices for the two commodities are those traded at the CBT. The futures prices are from the Statistical Abstracts Manuals put out by the CBT. Cash and futures prices for the T-bill and Eurodollar contracts are from Data Resources Incorporated (DRI). Cash prices for 90-day T-bills are derived from secondary market offerings of outstanding and when-issued T-bills adjusted for a maturity of 90 days. The T-bill contract is the one traded at the International Monetary Market (IMM). The cash market quote for the eurodollar contract is 90-day LIBOR. The futures contract is the one traded at the IMM.

The futures price series includes only prices of the nearest futures contract. For example, prices of the May, 1984 contract for wheat are used from March 1, 1984 through April 30, 1984. On May 1, 1984 a switch is made to the July 1984 contract, and so on. This procedure prevents the possibility of squeezes in the delivery month from distorting prices. Furthermore, this price series is assured to have come from a highly liquid market. Both open interest and volume are highest for the nearby futures. They also drop sharply when the contract enters its delivery month.

In correlation studies it is essential to measure unanticipated changes in the variables under consideration. Since a random walk process is assumed for futures prices, daily changes in futures prices properly measure the unanticipated changes.

However, for the basis, convergence to zero as contract expiration is approached requires a mechanism for obtaining the expected basis against which to measure the unanticipated change. The expected basis for any day is obtained from the previous day's basis less the moving average of the daily changes in the basis measured over the previous five days. There are several reasons for using the moving average technique to obtain the expected basis. The main one being that convergence to a zero basis at expiration may not take place because of the existence of several grades of the commodity that are eligible for delivery against the futures contract. If the grade being hedged is not the one that is most economically deliverable, then its price will not converge fully to the futures price (at expiration). Hence, the use of a moving average technique to reflect a gradual narrowing of the basis to its final value. A five-day moving average is chosen rather than a ten-day or 20-day moving average to economize on the data set. Since only data on the nearby contract is used, a moving average larger than five days would result in a loss of more than five observations per contract month. The change in the basis is obtained as the realized basis on any day less the expected basis. Using this procedure, a combined series of daily changes in futures prices and the basis for the entire period are obtained for each futures contract. For wheat and corn, there are five contracts per calendar year—March, May, July, September, and December. For T-bills and Eurodollars, there are four contracts per calendar year—March, June, September, and December.

RESULTS

Table I displays the correlation coefficients obtained from the sample data for wheat, corn, T-bill, and Eurodollar futures contracts. The percent of risk reduction possible with minimum-variance hedging below that of a full-hedge is reported also. Alternatively, the percent of risk reduction may be viewed as the fraction of basis risk that may be reduced through minimum-variance hedging. It is obtained as the square of the correlation coefficient between the basis and the futures price as shown below.

$$\begin{aligned} \%\text{ risk reduction} &= 1 - \mathrm{RESID}(t,T)/\sigma_B^2(t,T) \\ &= (\rho_{Bf})^2 \end{aligned} \tag{23}$$

A high value for % risk reduction indicates that risk can be reduced substantially below that of a full-hedge (basis risk) through minimum-variance hedging.

Table I
RISK-REDUCTION POTENTIAL OF MINIMUM-VARIANCE HEDGING

Futures Contract	Correlation Coefficient (ρ_{Bf})	Percent[a] Risk Reduction $(\rho_{Bf})^2$	No. of Observations
Wheat	−0.1759	3.10%	652
Corn	−0.1891	3.58%	651
T-bills	−0.5605	31.42%	866
Eurodollars	−0.4658	21.70%	866

Note: All correlation coefficients are different from zero at the 1% level of significance.
[a]Percent risk reduction possible below that of basis risk.

Note that the correlation coefficients for all contracts are negative which confirms that the minimum-variance hedge ratio should, at most, equal 1.00 [see equation (7)].[7] The relative size of the correlation coefficients are more interesting. Their absolute values are much higher for both financial futures contracts than for the grain futures which indicates a weaker bond between cash and futures prices for financial futures than for grain futures. The % risk reduction column indicates that substantial risk reduction is possible through minimum-variance hedging for both T-bills and Eurodollar futures. For example, for the T-bill contract, minimum-variance hedging can reduce risk by as much as 31.42% of that of a full-hedge. For the Eurodollar contract, the equivalent risk reduction is 21.70%. This is consistent with the results obtained by Castelino (1990a) for hedging 90-day T-bills and 90-day LIBOR with the T-bill and the Eurodollar futures contracts, respectively. That study shows that time-dependent minimum-variance hedge ratios substantially outperform hedges based either on regression analysis or full-hedges on a pure risk-reduction basis. For wheat and corn, the risk-reduction potential of minimum-variance hedging is almost nonexistent. In both cases, minimum-variance hedging can reduce risk by only 3% to 4% below that of a full hedge.

The existence of negative correlation between the basis and the futures price indicates the existence of a time dimension to the minimum-variance hedge ratio. Table II shows how significantly the hedge ratios depart from unity[8] as the hedge-lifting date recedes from contract expiration. Using the estimates of the single-period covariance between the basis and the futures price and the variance of the futures price, eq. (19) is used to obtain values for MVHR(t). "T" is fixed at 90 days, i.e., all hedges are initiated 90 days from expiration. The time of lifting of the hedge, t, is allowed to vary from 1 to 90, i.e., a one-day hedge to a 90-day hedge.

Note that for all contracts, the hedge ratios start from their lowest values for hedges lifted far from expiration and monotonically increase to unity as the hedge-lifting date approaches contract expiration. This is consistent with the need to adjust the hedge ratio in response to basis risk. Note also, the significantly lower minimum-variance hedge ratios for both the T-bill and Eurodollar futures contracts compared to either wheat or corn for any prespecified hedge-lifting date, an indication of much higher basis risk in both interest rate futures compared to the grain futures. For example, for the T-bill contract, the hedge ratios for a hedge lifted 30 days from contract expiration if 0.7137. It is 0.9587 for wheat. In fact, the departures of the hedge ratios away from 1.00 are relatively small for both wheat or corn. This, combined with the low-risk reduction potential suggested in Table I, makes one wonder if it is worthwhile placing a minimum-variance hedge in either of these contracts. For both T-bills and Eurodollars minimum-variance hedging appears to make sense because substantial risk reduction is possible.

[7]This conclusion needs to be qualified, particularly with respect to wheat and corn. In both cases, the quality of the commodity hedged, even though deliverable on the contract, may turn out not to be the cheapest deliverable grade. In such instances, the minimum-variance hedge ratio may not, in general, be 1.00 for a hedge lifted at contract expiration, though it will be very close to 1.00 for deliverable grades at the delivery location. For nondeliverable grades at varied locations the expiration date hedge ratios may depart significantly from 1.00. The extent of the departure from 1.00 depends on the covariance of the spread between the commodity hedged and the cheapest-to-deliver grade, and the futures price (see Appendix). The adjustment to the expiration date hedge ratio for such a grade can be estimated as the slope coefficient of the regression between the aforementioned spread and the futures price (see Appendix). Given this expiration date hedge ratio, the hedge ratio should decline from its expiration date value as the hedge-lifting date recedes from contract expiration, as suggested in this article.

[8]See footnote 7.

Table II
THE TIME DIMENSION TO THE MINIMUM-VARIANCE HEDGE RATIO

Hedge-Lifting Date (Days From Expiration)	Minimum-Variance Hedge Ratio			
	Wheat	Corn	T-bills	Euro$
85	0.9269	0.8576	0.4936	0.4642
80	0.9291	0.8619	0.5090	0.4804
75	0.9314	0.8664	0.5249	0.4973
70	0.9338	0.8711	0.5416	0.5150
65	0.9363	0.8760	0.5591	0.5335
60	0.9390	0.8812	0.5774	0.5528
55	0.9418	0.8866	0.5967	0.5733
50	0.9447	0.8923	0.6171	0.5949
45	0.9478	0.8984	0.6388	0.6178
40	0.9512	0.9049	0.6619	0.6423
35	0.9548	0.9119	0.6868	0.6686
30	0.9587	0.9195	0.7137	0.6971
25	0.9629	0.9278	0.7433	0.7283
20	0.9677	0.9370	0.7761	0.7630
15	0.9730	0.9475	0.8133	0.8024
10	0.9793	0.9598	0.8569	0.8486
5	0.9872	0.9751	0.9114	0.9062
0	1.0000	1.0000	1.0000	1.0000

Note: The hedge initiation date is 90 days from contract expiration.

Basis Risk and the Need for Minimum-Variance Hedging

Why is the potential for risk reduction through minimum-variance hedging (relative to a full-hedge) so much greater in both financial futures contracts compared to the grain futures contracts? The answer lies in Working's theory of the price of storage [see Working (1948, 1949)].

In the case of wheat and corn the fact that both commodities can be carried for delivery necessarily locks their spot and futures prices into a (basis) relationship that is barely affected by events forecasted to occur within a time period spanning their different delivery dates. For example, the forecast of a drought in the following crop year causes a sharp increase in the entire spectrum of futures prices—in the current crop year as well as in the new crop year. This sharp futures price increase immediately forces a large quantity of current supplies into storage, thereby reducing the amount available for current consumption. This results in an immediate rise in spot prices leaving the basis relatively unchanged.

The same cannot be said for either T-bill or Eurodollar futures. In either case, *only* at contract expiration is the instrument *deliverable* on the contract identical to the instrument *underlying* the contract. For example, the instrument underlying the T-bill contract is always a 13-week T-bill. Thirteen weeks from contract expiration the instrument deliverable on the T-bill contract is a 26-week T-bill. Five weeks from expiration, the deliverable bill is a 18-week T-bill. This distinction is important because the futures contract may display a price behavior that is different from both—the instrument deliverable against it, as well as the instrument underlying it.

One can easily draw up several plausible events which can cause 13-week T-bills to rise (fall) in price while the T-bill contract falls (rises) in price. Leibowitz (1981a, 1981b), for example, shows how changes in the shape of the yield curve can result in price changes for a futures contract on a short-term debt instrument that is substantially different from that of the debt instrument itself which undermines the predictability of the basis. In effect, the basis for a futures contract on a debt instrument can be highly unstable. The level of instability of the basis depends on the relative maturity of the instrument underlying the futures contract to the maturity of the futures contract itself. In principle, the shorter the maturity of the instrument underlying the contract relative to the contract itself, the greater is the basis instability. This basis instability translates into poor risk-reduction potential. Hence, the need for minimum-variance hedging should be viewed as an appropriate response to the ineffectiveness rather than the effectiveness of a futures contract to reduce risk.

Finally, what does the existence of a minimum-variance hedge ratio, one that can reduce the risk of a hedge below that of basis risk, suggest to a hedger? Only that it is possible to reduce risk below that of basis risk. It says absolutely nothing about whether or not a hedge should actually be placed. That decision should depend on the return implied by the basis. If the hedger feels that the basis reflects an attractive profit (or cost) opportunity, then a hedge should be placed. The decision regarding the minimum-variance hedge comes only after the decision to hedge is made, not before, If the decision to place a minimum-variance hedge is made without any consideration to the profit implied by the basis, then the hedger could very well be locking-in a low profit or even a loss (albeit, at low risk). Castelino, Francis, and Wolf (1991) show precisely this result in a study of cross-hedging 90-day CDs with the T-bill contract versus the Eurodollar contract. A pure, risk minimizer would always select the Eurodollar contract as the hedging vehicle. However, if hedges are placed in consideration of the return implied by the basis, the T-bill contract is often the superior hedging vehicle.

CONCLUSION

If current futures prices are unbiased estimates of future futures prices, then the expected return on a hedge is unaffected by the hedge ratio. The hedge ratio, however, affects the risk of the hedge. A minimum-variance hedge ratio does exist. It is the hedge ratio that minimizes risk for the level of return implied by the basis.

Minimum-variance hedge ratios possess a time-dimension. They are low for hedges lifted far from contract expiration and monotonically increase as the hedge-lifting date approaches contract expiration. This time dimension is consistent with the notion that hedge ratios should be reduced in response to increasing basis risk. Basis risk is high far from expiration and low close to it. Empirical tests reveal a significant time dimension to the minimum-variance hedge ratio for T-bill and Eurodollar futures. An insignificant one is revealed for wheat and corn futures.

Finally, the existence of a minimum-variance hedge ratio by no means implies that a hedger should use it at all times. The return implies by the basis at the time of hedge initiation should be a major factor in the decision to hedge or not to hedge. If the return is attractive, the hedger may choose to hedge using the minimum-variance hedge ratio, otherwise, the hedger should not. This is consistent with the theory proposed by Working that hedging should take place on the anticipation of a favorable change in the basis.

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Appendix

MINIMUM-VARIANCE HEDGE RATIO FOR A NONDELIVERABLE GRADE

Let $P_{ND}(t)$ be the price at time "t" of a nondeliverable commodity that is hedged in the futures market.

The basis between the nondeliverable grade and the futures contract, $B_{ND}(t,T)$, can be decomposed into a spread between the hedged grade and the cheapest-to-deliver grade, and the basis between the cheapest to deliver grade and the futures price as follows:

$$\begin{aligned}\tilde{B}_{ND}(t,T) &= [\tilde{P}_{ND}(t) - \tilde{P}_c(t)] + [\tilde{P}_c(t) - \tilde{P}_f(t,T)] \\ &= \tilde{S}(t) + \tilde{B}(t,T) \qquad \text{(A1)}\end{aligned}$$

where $S(t)$ is the spread between the hedged grade and the cheapest-to-deliver grade.

The minimum-variance hedge ratio of eq. (7) can now be expressed as:

$$\begin{aligned}\mathrm{MVHR}(t) &= \{1 + \mathrm{Cov}[\tilde{S}(t), \tilde{P}_f(t,T)]/\mathrm{Var}[\tilde{P}_f(t,T)]\} \\ &\quad + \mathrm{Cov}[\tilde{B}(t,T), \tilde{P}_f(t,T)]/\mathrm{Var}[\tilde{P}_f(t,T)] \qquad \text{(A2)}\end{aligned}$$

Now there is no a priori reason to believe that the spread between the hedged instrument and the cheapest-to-deliver instrument should vary as a function of time to expiration. Hence, the ratio of its covariance with the futures price and the variance of the futures price could be estimated through simple regression. One would expect the slope coefficient of such a regression to be close to zero. If it is significantly different from zero, then the expiration date hedge ratio will be different from 1.00; but the hedge ratio for any other hedge lifting date should be less than its expiration date value, as predicted by this study.