Hedge Ratios under Inherent Risk Reduction in a Commodity Complex: An Interpretation

Jacques A. Schnabel

INTRODUCTION

In the usual textbook treatment of hedging decisions, as exemplified in Chance (1989), Duffie (1989), and Kolb (1988), the decision-maker projects the receipt or delivery of a single commodity at some date in the future. To hedge the uncertainty in that future commitment, the decision-maker takes short or long positions in the futures market in that commodity to optimize some criterion. In a recent article in this journal Tzang and Leuthold (1990), henceforth TL, take issue with how the hedger's situation is conventionally characterized. Arguing that, in many situations, the decision-maker faces multiple risks simultaneously, they derive a model to generate simultaneous minimum risk hedge ratios among commodities which are linked via a fixed production relationship. They develop their general model in the specific context of a soybean processor who buys soybeans to produce and sell soybean oil and soybean meal. They argue that such a decision-maker would be interested in minimizing not so much the separate variances of the soybean cost price, the soybean meal revenue price, and the soybean oil revenue price, but the variance of the margins between the prices of the soybean input and the meal and oil outputs of the production process. Clearly, the TL study represents an important breakthrough in the hedging literature because the necessity to hedge a multiplicity of risks simultaneously is ubiquitous.

The aims of this addendum to the TL article are to review the theoretical superiority of their hedging approach to the traditional approach, and to provide a multiple regression interpretation of their model. Whereas TL focused on a situation of one input (soybeans) and two outputs (soybean meal and soybean oil), this article considers the simpler case of a production process with one input and only one output. This simplification results in a clearer exposition without loss of generality.

A COMPARISON OF THE TRADITIONAL AND TZANG–LEUTHOLD APPROACHES

Consider a decision-maker who, at time 0, projects the need to purchase Q_I units of an input into a production process at time 1. The uncertain time 1 price per unit of

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Jacques A. Schnabel is a Professor of Business at Wilfrid Laurier University.

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the input is denoted $\tilde{P}_I$.[1] To hedge the uncertainty in this input cost, X_I futures contracts would be purchased at time 0 at a known futures price of F_I, and sold at time 1 at an unknown futures price of $\tilde{F}_I$. The input will be processed and at time 2, the output amounting to Q_O units will be sold at an uncertain per unit price of $\tilde{P}_O$. To hedge the uncertainty in the output revenue, X_O futures contracts would be sold at time 0 at a known futures price of F_O, and purchased at time 2 at an unknown futures price of $\tilde{F}_O$. Clearly, Q_I and Q_O are linked by the production function which the decision-maker employs. Hedged input cost and output revenue, $\tilde{H}_I$ and $\tilde{H}_O$ respectively, are given by the following equations.

$$\tilde{H}_I = \tilde{P}_I Q_I - (\tilde{F}_I - F_I)X_I \tag{1}$$

and

$$\tilde{H}_O = \tilde{P}_O Q_O - (\tilde{F}_O - F_O)X_O \tag{2}$$

If one were to adopt the traditional approach to hedging these input and output exposures, the following two problems would be separately solved: minimize $\sigma^2(\tilde{H}_I)$ with respect to X_I, resulting in X_I' and minimize $\sigma^2(\tilde{H}_O)$ with respect to X_O, resulting in X_O', where $\sigma^2(\cdot)$ denotes the variance operator. It is well known that

$$X_I' = \beta_{I,FI} Q_I \tag{3}$$

and

$$X_O' = \beta_{O,FO} Q_O \tag{4}$$

where $\beta_{I,FI} = \text{cov}(\tilde{P}_I, \tilde{F}_I)/\sigma^2(\tilde{F}_I)$ and $\beta_{O,FO} = \text{cov}(\tilde{P}_O, \tilde{F}_O)/\sigma^2(\tilde{F}_O)$ are the population values of slope coefficients obtained in two separate simple regressions of $\tilde{P}_I$ on $\tilde{F}_I$ and $\tilde{P}_O$ on $\tilde{F}_O$, respectively, and $\text{cov}(\cdot, \cdot)$ denotes the covariance operator.

However, TL argue that it is more appropriate to depict the uncertainty in the unhedged margin, $\tilde{P}_O Q_O - \tilde{P}_I Q_I$, as the ultimate object of concern and the variance in the hedged margin, $\tilde{M} = \tilde{H}_O - \tilde{H}_I$, as the criterion to minimize.[2] Thus, TL seek to minimize $\sigma^2(\tilde{M})$ with respect to X_I and X_O. Denote the optimal values of these decision variables obtained by solving this problem as X_I^* and X_O^*, respectively. Since,

$$\begin{aligned}\sigma^2(\tilde{M}) = {} & \sigma^2(\tilde{P}_O)Q_O^2 + \sigma^2(\tilde{P}_I)Q_I^2 + \sigma^2(\tilde{F}_O)X_O^2 \\ & + \sigma^2(\tilde{F}_I)X_I^2 - 2\text{cov}(\tilde{P}_O, \tilde{P}_I)Q_O Q_I \\ & - 2\text{cov}(\tilde{P}_O, \tilde{F}_O)Q_O X_O + 2\text{cov}(\tilde{P}_O, \tilde{F}_I)Q_O X_I \\ & + 2\text{cov}(\tilde{P}_I, \tilde{F}_O)Q_I X_O - 2\text{cov}(\tilde{P}_I, \tilde{F}_I)Q_I X_I \\ & - 2\text{cov}(\tilde{F}_O, \tilde{F}_I)X_O X_I\end{aligned} \tag{5}$$

taking the derivatives of $\sigma^2(\tilde{M})$ with respect to X_I and X_O, and equating these derivatives to zero, result in the following implicit optimality conditions.

$$X_I^* = \beta_{I,FI} Q_I - \beta_{O,FI} Q_O + \beta_{FO,FI} X_O^* \tag{6}$$

and

$$X_O^* = \beta_{O,FO} Q_O - \beta_{I,FO} Q_I + \beta_{FI,FO} X_I^* \tag{7}$$

[1]The convention of using tildes (˜) to denote random variables is followed here. The notation employed here differs from that used in the TL article, as it is felt that the revised notation results in a clearer presentation.

[2]To simplify the presentation, time value of money considerations are ignored for the time 1 cash flow as in the TL article.

In these equations, $\beta_{O,FI} = \text{cov}(\tilde{P}_O, \tilde{F}_I)/\sigma^2(\tilde{F}_I)$ is the population value slope coefficient in a simple regression of $\tilde{P}_O$ on $\tilde{F}_I$. $\beta_{FO,FI} = \text{cov}(\tilde{F}_O, \tilde{F}_I)/\sigma^2(\tilde{F}_I)$, $\beta_{I,FO} = \text{cov}(\tilde{P}_I, \tilde{F}_O)/\sigma^2(\tilde{F}_O)$, and $\beta_{FI,FO} = \text{cov}(\tilde{F}_I, \tilde{F}_O)/\sigma^2(\tilde{F}_O)$ may be similarly interpreted as population value simple regression slope coefficients of $\tilde{F}_O$ on $\tilde{F}_I$, $\tilde{P}_I$ on $\tilde{F}_O$, and $\tilde{F}_I$ on $\tilde{F}_O$, respectively.

Equations (6) and (7) may be solved to yield the following explicit formulas.

$$X_I^* = \left[\frac{\beta_{I,FI} - \beta_{FO,FI}\beta_{I,FO}}{1 - \beta_{FO,FI}\beta_{FI,FO}}\right]Q_I - \left[\frac{\beta_{O,FI} - \beta_{FO,FI}\beta_{O,FO}}{1 - \beta_{FO,FI}\beta_{FI,FO}}\right]Q_O \tag{8}$$

and

$$X_O^* = \left[\frac{\beta_{O,FO} - \beta_{FI,FO}\beta_{O,FI}}{1 - \beta_{FI,FO}\beta_{FO,FI}}\right]Q_O - \left[\frac{\beta_{I,FO} - \beta_{FI,FO}\beta_{I,FI}}{1 - \beta_{FI,FO}\beta_{FO,FI}}\right]Q_I \tag{9}$$

The differences between X_I^* and X_O^* on the one hand and X_I' and X_O' on the other, may be noted by comparing eqs. (8) and (9) with eqs. (3) and (4), respectively. There are no differences if the following prices are uncorrelated: $\tilde{F}_O$ with $\tilde{F}_I$, $\tilde{P}_I$ with $\tilde{F}_O$, and $\tilde{P}_O$ with $\tilde{F}_I$. If any of these correlations are nonzero, then in general X_I' is not equal to X_I^* and X_O' is not equal to X_O^*. The theoretical superiority of the TL procedure lies in its correctly taking account of the potential nonzero correlations among these sets of variables.

A MULTIPLE REGRESSION INTERPRETATION OF THE TZANG–LEUTHOLD MODEL

The TL model reviewed in the preceding section may be given a multiple regression interpretation. Consider the following model where the unhedged margin is defined as the regressand, and the two futures prices are defined as the regressors.

$$\tilde{P}_O Q_O - \tilde{P}_I Q_I = \alpha + \alpha_O \tilde{F}_O + \alpha_I \tilde{F}_I + \tilde{\varepsilon} \tag{10}$$

The usual OLS assumptions that $\text{cov}(\tilde{\varepsilon}, \tilde{F}_O) = \text{cov}(\tilde{\varepsilon}, \tilde{F}_I) = 0$ are made. The population values of the slope coefficients of this regression are given by the following.

$$\alpha_I = \frac{\text{cov}(\tilde{P}_O Q_O - \tilde{P}_I Q_I - \alpha_O \tilde{F}_O, \tilde{F}_I)}{\sigma^2(\tilde{F}_I)} \tag{11}$$

and

$$\alpha_O = \frac{\text{cov}(\tilde{P}_O Q_O - \tilde{P}_I Q_I - \alpha_I \tilde{F}_I, \tilde{F}_O)}{\sigma^2(\tilde{F}_O)} \tag{12}$$

These simplify to the following

$$\alpha_I = \beta_{O,FI} Q_O - \beta_{I,FI} Q_I - \beta_{FO,FI}\alpha_O \tag{13}$$

and

$$\alpha_O = \beta_{O,FO} Q_O - \beta_{I,FO} Q_I - \beta_{FI,FO}\alpha_I \tag{14}$$

Observe that eqs. (6) and (13) are negatives of each other and eqs. (7) and (14) are identical to each other. Thus, $\alpha_I = -X_I^*$ and $\alpha_O = X_O^*$. To minimize the variance in the hedged margin, positions in the two futures markets should be calculated by regressing the margin onto the two futures prices.

This multiple regression interpretation yields an approach to solving the TL model which is computationally superior to the procedure TL employ. TL run a number of simple regressions, and then substitute the estimated regression coefficients into a series of formulas for the optimal hedge ratios. Their approach essentially entails first calculating the β terms in eqs. (8) and (9), i.e., the simple regression coefficients, and then substituting these into the two equations. On the other hand, the approach explained in this section entails running a single multiple regression. The coefficients of this regression yield the same values of the optimal hedge ratios in the TL model.[3]

Consider next the question of gauging hedging effectiveness. Ederington (1979) has suggested the use of the R^2 of the regression of the spot price onto the futures price as such a measure. He has shown that this R^2 indicates the percentage reduction in the variance of the decision-maker's position when the variance-minimizing hedge is employed, in contrast to the variance attained when no hedging is involved. It is shown here that the R^2 in eq. (10) may be similarly interpreted.

Define $\tilde{M}^0$ as the unhedged margin, i.e.,

$$\tilde{M}^0 = \tilde{P}_O Q_O - \tilde{P}_I Q_I \tag{15}$$

and $\tilde{M}^*$ as the hedged margin with the values X_I^* and X_O^* employed, i.e.,

$$\tilde{M}^* = \tilde{M}^0 - (\tilde{F}_O - F_O)X_O^* + (\tilde{F}_I - F_I)X_I^* \tag{16}$$

Then, following Ederington, hedging effectiveness, denoted HE, may be measured as follows,

$$\text{HE} = \frac{\sigma^2(\tilde{M}^0) - \sigma^2(\tilde{M}^*)}{\sigma^2(\tilde{M}^0)} \tag{17}$$

Observe that eq. (10) may be rewritten as follows

$$\tilde{\varepsilon} = \tilde{M}^0 - (\alpha + \alpha_O \tilde{F}_O + \alpha_I \tilde{F}_I) \tag{18}$$

Given the previous results regarding the slope coefficients in that equation, eq. (10) may also be written as

$$\tilde{M}^* = \tilde{M}^0 - (\alpha + \alpha_O \tilde{F}_O + \alpha_I \tilde{F}_I) \tag{19}$$

where $(X_I^* F_I - X_O^* F_O)$ is impounded in the intercept term, α. Thus, $\sigma^2(\tilde{M}^*) = \sigma^2(\tilde{\varepsilon})$. Since the R^2 in eq. (10) is given by the following

$$R^2 = 1 - \frac{\sigma^2(\tilde{\varepsilon})}{\sigma^2(\tilde{M}^0)} \tag{20}$$

clearly $R^2 = \text{HE}$. Thus, in contrast to TL, who reject the use of R^2 to gauge hedging effectiveness, this section makes a case for the use of R^2 in the context of the TL model.

CONCLUSION

This communication interprets the TL approach to simultaneous hedging of multiple risks in a multiple regression context. The theoretical superiority of the TL approach compared to the traditional approach is reviewed. The multiple regression interpretation developed here yields a computationally superior approach in implementing the TL model and a hedging effectiveness measure consistent with the existing hedging literature.

[3]A similar multiple regression approach to hedge ratio determination has been applied by Poitras (1988) in his model of foreign currency and interest rate hedging.

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