Application of Mean-Variance Analysis to Broad-Based Futures Contracts

Da-Hsiang Donald Lien

INTRODUCTION

In the futures market literature, the most popular methodology for analyzing optimal trading behavior, especially hedging, is the mean-variance (MV) framework.[1] A considerable amount of research has gone into identifying those conditions under which the MV analysis is consistent with the von Neumann–Morgenstern axioms of rational behavior [hereafter, referred to as the expected utility (EU) approach]. Several answers are provided [see Tobin (1958); Feldstein (1969); Baron (1977); Chamberlin (1983); Meyer (1987)]. Conformity is obtained when the mean-variance expected utility function is specified as a quadratic function over wealth (Baron, 1977). However, by taking this specification, an individual is assumed to exhibit increasing risk aversion,[2] a property that is rebuted by empirical evidence. For this reason and others, being restsricted to the use of quadratic utility functions is rather undesirable. Feldstein (1969) shows that the MV approach fits within the EU framework only if each asset has a distribution such that any linear combination of these assets has a distribution which can be completely characterized by only two independent parameters. Multivariate normal distributions satisfy the requisite property, but the other two parameter families, such as the lognormal, fail to do so [Tobin (1958); Feldstein (1969)]. Nonetheless, Meyer (1987) argues that, for many economic models, all possible decision variables generate random outcomes that differ from each other by two parameters, mean and variance. Hence, the MV analysis agrees with the EU approach under quite tenable restrictions.[3]

The author acknowledges John Cita, James Quirk, Joseph Sicilian, De-Min Wu, the editor, and an anonymous referee for helpful comments. Norman Clifford assisted in computer programming. John Cita provided editorial assistance. The author is solely responsible for any remaining errors.

[1]In this context, of course, the choice set consists of various hedging alternatives, each characterized by the mean and variance of their returns.

[2]Simply meaning, as individuals gain wealth, they become more averse to risk taking.

[3]While Meyer argues that expected utility analysis to several economic models can be reduced to a two-moment decision problem, the resulting mean-variance function may be rather sophisticated. Here the concern is with MV analysis in the narrow sense. That is, the mean-variance function is linear in both of its arguments.

Da-Hsiang Donald Lien is an Associate Professor in the Department of Economics at the University of Kansas.

The Journal of Futures Markets, Vol. 12, No. 1, 19–32 (1992)

CCC 0270-7314/92/010019-14$04.00

The above discussion centers on the consistency between EU and MV as preference orderings. Yet, a more important question is whether MV analysis provides the same solution to a decision problem as the EU approach. Some positive answers are provided in Baron (1977) and Benniga, Elder, and Zilcha (1983). For problems where the two approaches yield solutions that differ by a small amount, the solution obtained by MV analysis may be treated as an approximation to the one determined by the more complicated EU approach [Anderson and Danthine (1981); Newbery (1987, 1988)]. Samuelson (1970) and Tsiang (1972) indicate MV analysis is a good approximation when the absolute or relative risk is small. This article investigates whether the mean-variance analysis provides a good approximation in the context of hedging with broad-based futures contracts.

It is assumed here that the MV analysis agrees with the EU approach in the case of narrow-based futures contracts (or, alternatively, forward contracts). The hedger is assumed to have an exponential utility function.[4] The hedger incurs no quantity risk, and there is a forward market for the underlying commodity. It is assumed also that the spot forward price for the commodity is normally distributed. Consequently, the hedger's future income is normally distributed. The assumption of an exponential utility function reduces the EU approach to the MV analysis with mean and variance entering the objective function in a linear, additive fashion. By contrast, Newbery (1988) considers the existence of quantity risk that is represented by a normal random variable. Since the product of two normal variables (in this case, the future income) is no longer normally distributed, the MV analysis does not agree with the EU approach. Nevertheless, Newbery (1988) shows it provides good approximations to both the equilibrium futures price and welfare gains, but errors in trade volume can be large. The original set of assumptions is modified by allowing the hedger to trade in a broad-based futures contract instead of a forward contract. While the latter deals exclusively with a single grade of a commodity, the former provides for multiple deliverable grades at maturity. The futures price is then equal to the spot price of the cheapest-to-deliver grade,[5] which implies the futures price is an order statistic. It is assumed that spot prices for various deliverable grades follow a multivariate normal distribution (to ensure future income from spot markets is normally distributed). Thus, the futures price is not normally distributed, nor is the future income for hedgers in the futures market [like the

[4]To argue that the MV analysis provides an approximation to the EU approach, one generally starts with the latter and takes the Taylor expansion to the second order which, as a result, achieves equivalence with the former approach [Anderson and Danthine (1981); Newbery (1987)]. In this case, the MV analysis is restricted to a linear, additive function of mean and variance. A multivariate normal distribution reduces the EU approach to the MV analysis, but not necessarily with a linear, additive function. To serve the latter purpose, an additional assumption needs to be imposed on the form of utility function, such as supposing an exponential form.

[5]The futures price of a broad-based futures contract is not necessarily equal to the cash price of the cheapest-to-deliver grade. Kuhn (1988) shows that the futures price in the coffee C market primarily reflects the average cash price of the most available grades. The conclusion is likely to apply to several commodity futures once transaction costs are taken into account. A possibly better alternative, which is unexploited, may assume convergence to the minimum cash price of the most available grades. For example, CBOT wheat futures allows for 11 deliverable grades, but only the two most available grades (no. 2 red hard and no. 2 red soft) are considered in Kamara and Siegel (1987). The futures price is then modeled as the minimum of the two cash prices. Similar arguments apply to the CME T-bill futures. In any case, the analysis provided here can be easily adapted to other assumptions concerning the futures price at maturity. Certainly, some complication will arise if the transaction cost is now added to the framework.

problem considered by Newbery (1988)]. Using broad-based contracts implies that the results determined by the MV analysis will differ from those obtained with the EU approach. The two approaches are compared in terms of optimal hedging policies. Since Newbery's model focuses upon narrow-based contracts, it does not fit the purposes of this study. Instead, the model of Lien (1988) is used.

A recent justification for the MV analysis along the line of nonexpected utility (NEU) analysis should be noted. This analysis is motivated, in part, by objections to the axiom of independence of irrelevant alternatives which underlies the von Neumann–Morgenstern EU framework (Machina, 1982a). Epstein (1985) shows that the MV analysis is the only acceptable approach if an individual's preference exhibits systematically changing risk aversions (such as increasing or decreasing risk aversion). Since the importance and relevance of decreasing risk aversion are well recognized in the literature [Pratt (1964); Ross (1981); Machina (1982b); Machina and Neilson (1987)], Epstein then uses it as the main argument for applying the MV analysis. Nonetheless, the definition of a systematically changing risk aversion is very stringent because it requires the same changing pattern for any random initial wealth, and any random increment. One can argue that the conclusion of Epstein (1985) is simply an implication of this severe requirement and, therefore, it should be discounted. To determine the theoretically correct benchmark, the model is solved first by using the EU framework. Then one can proceed to determine how well the MV solution approximates the correct one, i.e., the solution obtained via the EU approach.

THE BASIC MODEL

The model developed in this section is an extension of Lien (1988). First, a solution is obtained using the EU analysis. Then, similar to the approach used by Kamara and Siegel (1987), results are derived by using the MV framework. A two-period model is considered. Let $t = 0$ denote the current time, and let $t = 1$ denote the maturity date of the futures contract. For mathematical simplicity, it is assumed that the contract allows only two deliverable grades: 1 and 2. The spot price of the ith grade at time t is P_{it}; $i = 1, 2$; $t = 0, 1$. The futures price at time t is f_t, $t = 0, 1$. The representative hedger observes P_{10}, P_{20}, and f_0. The prices in the future, P_{11}, P_{21}, and f_1, are all unknown. Assume that the hedger perceives (P_{11}, P_{21}) as a bivariate normal vector such that $E(P_{i1}) = \mu_i$, $\mathrm{Var}(P_{i1}) = \sigma_i^2$, and $\mathrm{Cov}(P_{11}, P_{21}) = \rho\sigma_1\sigma_2$ where $E(\cdot)$, $\mathrm{Var}(\cdot)$, and $\mathrm{Cov}(\cdot\cdot)$ are, respectively, expectation, variance, and convariance operators conditional on the information available at $t = 0$. Since $f_1 = \min(P_{11}, P_{21})$, the probability density function of f_1 is determined by the joint density of P_{11} and P_{21}.

The hedger attempts to maximize the expected utility of end-period wealth. Let S_i denote the spot position on the ith grade at $t = 0$, and let V denote the futures position at $t = 0$. A positive S_i (or V) implies the hedger buys from the spot (or futures) market at $t = 0$. Negative values indicate short positions. Let W_0 denote initial wealth; end-period wealth is given by:

$$W_1 = W_0 + S_1(P_{11} - P_{10}) + S_2(P_{21} - P_{20}) + V(f_1 - f_0)$$

Next, an exponential utility function is adopted: $U(W) = -\exp(\alpha W)$, $\alpha < 0$. The objective function for the hedger is:

$$EU(W_1) = -E(\exp(\alpha W_1)) = -\exp(M_0)E[\exp(\tilde{S}_1 P_{11} + \tilde{S}_2 P_{21} + \tilde{V} f_1)]$$

where $M_0 = \alpha W_0 - \tilde{S}_1 P_{10} - \tilde{S}_2 P_{20} - Vf_0$; $\tilde{S}_i = \alpha S_i$, $i = 1, 2$; $\tilde{V} = \alpha V$. Since α measures risk aversion, $\tilde{S}_i$ *and* $\tilde{V}$ are risk-adjusted spot and futures positions.

As stated above, $f_1 = P_{11}$ or P_{21}, depending on which is smaller. $\mathrm{EU}(W_1)$ is decomposed accordingly:

$$EU(W_1) = E[U(W_1)|P_{11} \leqslant P_{21}] \cdot \mathrm{Prob}(P_{11} \leqslant P_{21}) + E[U(W_1)|P_{11} \geqslant P_{21}] \cdot \mathrm{Prob}(P_{11} \geqslant P_{21}) \qquad (1)$$

where, for any event A, $\mathrm{Prob}(A)$ denotes the probability that A occurs, and $E[U(W_1)|A]$ denotes expected utility conditional upon the occurrence of A. The two terms on the RHS of eq. (1) are symmetric in the sense that, upon interchanging P_{11} with P_{21} in the first term, the second term is derived. After algebraic manipulation (see Appendix I):

$$E[U(W_1)|P_{11} \leqslant P_{21}] \cdot \mathrm{Prob}(\mathrm{P}_{11} \leqslant \mathrm{P}_{21}) = -\exp[M_0 + (\tilde{S}_1 + \tilde{V})\mu_1 + \tilde{S}_2\mu_2 + (\tilde{S}_1 + \tilde{V})^2\sigma_1^2/2 + \tilde{S}_2^2\sigma_2^2/2 + \rho\sigma_1\sigma_2(\tilde{S}_1 + \tilde{V})\tilde{S}_2] \cdot K \qquad (2)$$

where

$$K = \Phi[-\sigma_1^2(\tilde{\mathrm{S}}_1 + \tilde{\mathrm{V}}) + \sigma_2^2\tilde{\mathrm{S}}_2 + \rho\sigma_1\sigma_2(\tilde{\mathrm{S}}_1 + \tilde{\mathrm{V}} - \tilde{\mathrm{S}}_2) - (\mu_1 - \mu_2)/\sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}]$$

with $\Phi(\cdot)$ being the cumulative density function of a standard normal random variable. Similarly:

$$E[U(W_1)|P_{11} \geqslant P_{21}] \cdot \mathrm{Prob}(P_{11} \geqslant P_{21}) = -\exp[M_0 + \tilde{S}_1\mu_1 + (\tilde{S}_2 + \tilde{V})\mu_2 + \tilde{S}_1^2\sigma_1^2/2 + (\tilde{S}_2 + \tilde{V})^2\sigma_2^2/2 + \rho\sigma_1\sigma_2(\tilde{S}_2 + \tilde{V})\tilde{S}_1] \cdot H \qquad (2')$$

where

$$H = \Phi[-\sigma_2^2(\tilde{S}_2 + \tilde{V}) + \sigma_1^2\tilde{S}_1 + \rho\sigma_1\sigma_2(\tilde{S}_2 + \tilde{V} - \tilde{S}_1) - (\mu_2 - \mu_2)/\sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}].$$

Upon substituting eq. (2) and (2′) into eq. (1), $E[U(W_1)]$ is explicitly expressed as a function of $\tilde{\mathrm{S}}_1$, $\tilde{\mathrm{S}}_2$, and $\tilde{\mathrm{V}}$ with parameters μ_1, μ_2, σ_1^2, σ_2^2, ρ, and αW_0. The hedger's problem may be solved by the conventional first-order condition approach.

The framework used by Kamara and Siegel (1987) is adapted here to solve the hedger's problem using MV analysis. The intent is to provide a comparison of the two analyses. Under the current assumptions, making comparisons is quite complicated. To streamline the comparison, two additional assumptions are made[6]: $\mu_1 = \mu_2 = \mu$ and $\sigma_1^2 = \sigma_2^2 = \sigma^2$. Since both grades are deliverable at par price, and since relative market conditions also influence relative spot prices, it seems reasonable to assume the two spot prices at maturity have the same expectation. Otherwise, one grade is more likely to dominate the other, and the multiple delivery specification is ineffective to some extent. This assumption implies $\mu_1 = \mu_2 = \mu$. Empirical evidence, at least for CBOT wheat futures supports the second assumption. Using data from 1970–1981, Kamara and Siegel (1987) show that the two most popular delivery grades, no. 2 red hard and no. 2 red soft wheat, satisfy the equal variance assumption.

[6]Although an attempt is made to justify the use of these two assumptions, they are mainly imposed for reasons of simplifying the analysis. The methods adopted here apply to the most general cases as well.

Given these two assumptions, the mean and variance of end-period wealth are, respectively:

$$E(W_1) = (W_0 - S_1P_{10} - S_2P_{20} - Vf_0) + (S_1 + S_2 + V)\mu - V\sigma A \tag{3}$$

$$\mathrm{Var}(W_1) = \sigma^2[S_1^2 + S_2^2 + 2\rho S_1S_2 + \rho S_1V + \rho S_2V + (1 - A^2)V^2] \tag{4}$$

where $A = \sqrt{(1 - \rho)/\pi}$. Using mean-variance analysis, the hedger attempts to maximize $E(W_1) + (\alpha/2)\mathrm{Var}(W_1)$. With further assumptions of $P_{10} = P_{20} = P_0$, the optimal spot and futures positions using the MV analysis ($\tilde{S}_1^*$, $\tilde{S}_2^*$, and $\tilde{V}^*$) satisfy the following equations:

$$\tilde{S}_1^* = \tilde{S}_2^* = \tilde{S}^* \tag{5a}$$

$$(1 + \rho)\tilde{S}^* + (1 + \rho)\tilde{V}^*/2 = (P_0 - \mu)/\sigma^2 \tag{5b}$$

$$(1 + \rho)\tilde{S}^* + (1 - (1 - \rho)/\pi)\tilde{V}^* = (f_0 - \mu + \sigma A)/\sigma^2 \tag{5c}$$

On the other hand, let $\tilde{S}_1^{**}$, $\tilde{S}_2^{**}$, and $\tilde{V}^{**}$ denote the optimal spot and futures positions resulting from the expected utility approach,[7] then one obtains (see Appendix II):

$$\tilde{S}_1^{**} = \tilde{S}_2^{**} = \tilde{S}^{**} \tag{6a}$$

$$(1 + \rho)\tilde{S}^{**} + (1 + \rho)\tilde{V}^{**}/2 = (P_0 - \mu)/\sigma^2 \tag{6b}$$

$$(1 + \rho)\tilde{S}^{**} + \tilde{V}^{**} = (f_0 - \mu)/\sigma^2 + \sqrt{(1 - \rho)/2\sigma^2} \cdot \phi(\theta\tilde{V}^{**})/\Phi(\theta\tilde{V}^{**}) \tag{6c}$$

where $\theta = -\sqrt{(1 - \rho)/2} \cdot \sigma$, and $\phi(\cdot)$ is the probability density function of a standard normal random variable.

COMPARISON OF OPTIMAL HEDGING POLICIES

From eqs. (5a)–(6c), one can ascertain that the difference between the solutions, ($\tilde{S}^*$, $\tilde{V}^*$) and ($\tilde{S}^{**}$, $\tilde{V}^{**}$) is due to the difference between eqs. (5c) and (6c). However, if one takes Taylor's expansion of $\phi(\theta\tilde{V}^{**})/\Phi(\theta\tilde{V}^{**})$ around the origin and truncates it at the first order term, then eq. (6c) may be rewritten as:

$$(1 + \rho)\tilde{S}^{**} + (1 - (1 - \rho)/\pi)\tilde{V}^{**} = (f_0 - \mu + \sigma\sqrt{(1 - \rho)/\pi})/\sigma^2$$

which is the same as eq. (5c).

Since eqs. (5b) and (5c) form a system of linear equations in two unknowns, the solution is uniquely determined and using the Taylor approximation, $\tilde{S}^* = \tilde{S}^{**}$, $\tilde{V}^* = \tilde{V}^{**}$. In other words, when the optimal futures position resulting from the EU approach (i.e., $|\tilde{V}^{**}|$) is small, such that higher order terms of Taylor's expansion are negligible, the MV analysis provides a good approximation of the optimal hedging policies.

Of course, whenever $|\tilde{V}^{**}|$ is sufficiently large, the above conclusion is invalid. To further investigate the relationship between eqs. (5c) and (6c), $\tilde{S}^*$ and $\tilde{S}^{**}$ are treated as parameters. Consequently, eq. (5c) defines a function, $h(\cdot)$, such that $\tilde{V}^* = h(\tilde{S}^*)$; whereas, eq. (6c) defines another function $k(\cdot)$, with $\tilde{V}^{**} = k(\tilde{S}^{**})$. Ignoring the superscripts, one can graph $h(\cdot)$ and $k(\cdot)$ as two curves in $(\tilde{S}, \tilde{V})$ space.

[7]Note that W_0 does not appear in eqs. (5) and (6), implying the initial wealth level does not affect the comparison of the two solutions. Thus, unlike Tsiang (1972), smaller relative risk does not suffice to support the MV analysis as an approximation to the EU approach.

Assuming that the futures market is established at a martingale equilibrium,[8] it holds that $f_0 = E(f_1) = \mu - \sigma A$. In a martingale equilibrium, $h(\cdot)$ is a straight line passing through the origin with the slope,

$$h'(\tilde{S}) = dh(\tilde{S})/d\tilde{S} = -(1 + \rho)(1 - (1 - \rho)/\pi)^{-1} \tag{7}$$

The line is shown in Figure 1 by L_2. The graph of $k(\cdot)$ is denoted L_1, its slope is given by:

$$k'(\tilde{S}) = dk(\tilde{S})/d\tilde{S} = -(1 + \rho)\{1 + (1 - \rho)/2 \cdot d(\phi(x)/\Phi(x))/dx\}^{-1}$$

where $x = -\sqrt{(1 - \rho)/2}\,\sigma k(\tilde{S})$. By comparing $h(\cdot)$ with $k(\cdot)$ affords yet another means to compare the MV and EU methods. This comparison is facilitated by three results derived in Appendix III. One result is (1) $k(0) = 0 = h(0)$. Intuitively, this result shows a risk-averse hedger will not participate in an unbiased futures

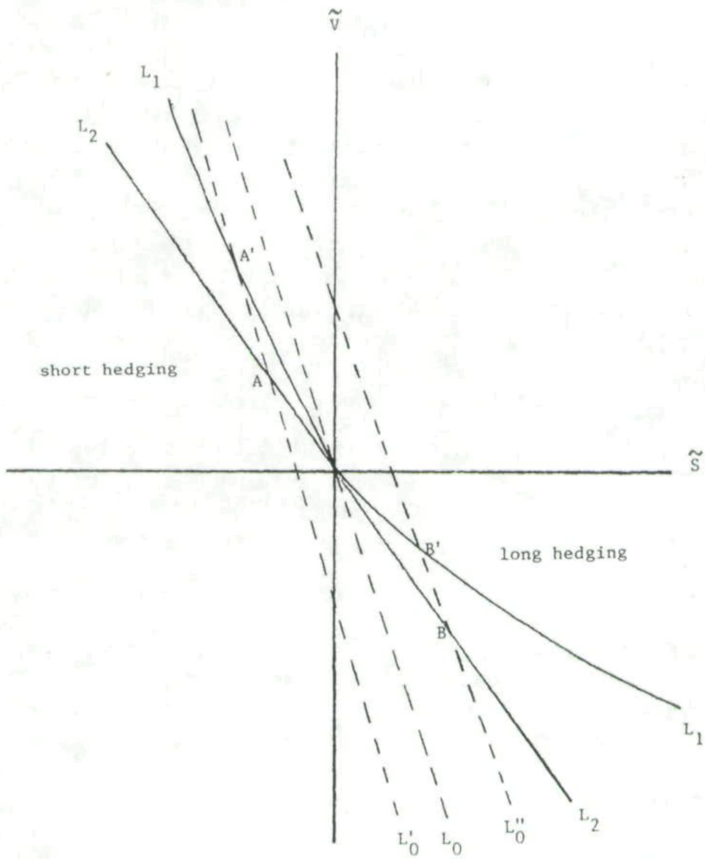

Figure 1
Optimal spot and futures positions.

[8]In a forward market with imperfect hedging, Benniga, Eldor, and Zilcha (1983) show the MV analysis and EU approach provide the same optimal futures position, provided the following conditions are satisfied: (1) the forward market is unbiased and (2) the joint probability distribution of f_t and b_t is separable; where f_t is the price of the forward contract and $b_t = P_t - \alpha_t - \beta_t f_t$ with P_t being the cash price of the underlying commodity. The result, however, does not apply to this case. First, a broad-based futures contract is referred to instead of a forward contract. Second, the separability assumption does not hold here for either grade.

market whenever there is no exposure to risk on spot positions. Both MV and EU analyses verify this conjecture. Another result is (2) $k'(\cdot) < 0$. That is, similar to the MV analysis, the EU approach chooses a larger optimal futures position for larger spot positions in both grades. Finally, (3) $k'(0) = h'(0)$ and (4) $k''(\cdot) > 0$. With these results, it is easy to see that L_1 is tangent to L_2 at the origin point, and lies above L_2 everywhere (consult Fig. 1).

Several implications result. As the spot position increases (for either a short or a long hedger) the gap between the two approaches widens in the sense that optimal futures positions prescribed by them diverge further from each other. Since a hedger's risk is proportional to the square of the spot position, this result states, like Samuelson (1970), the MV analysis is a good approximation when the absolute risk is small. Also, while the MV analysis prescribes a constant hedge ratio for all spot positions, the EU approach dictates a larger (resp. smaller) hedger ratio for a larger short (resp. long) hedger. To see this result, note that for any point $(\tilde{S}, \tilde{V})$ on L_1 or L_2, the hedge ratio is represented by the slope of the line connecting the origin to the point $(\tilde{S}, \tilde{V})$. In addition, for a short hedger (corresponding to the case $\tilde{S} < 0$) the MV analysis is more conservative because it suggests selling a smaller amount of futures contracts compared to the EU approach. On the other hand, for a long hedger (such that $\tilde{S} > 0$) the MV analysis is more aggressive, since it recommends the hedger purchase a larger futures position.

The last result is based on a nontradable spot position, S (or, alternatively, $\tilde{S}$). Consider $\tilde{S}$ to be a choice variable. To proceed, we graph eq. (5b) in $(\tilde{S}, \tilde{V})$ space. Corresponding to different signs of $P_0 - \mu$, the equation is shown as one of the lines[9] L_0, L_0', and L_0'' in Figure 1. For example, where $P_0 - \mu < 0$, the line L_0' results. In this case, the MV analysis shows the optimal spot and futures positions represented by point A in Figure 1, whereas the EU approach supports point A', which is northwest of A. Note that when $P_0 - \mu < 0$ the current spot price is lower than the expected futures price. This implies the hedger should choose a long spot position and a short futures position. Both analyses validate this result; although the MV analyses chooses a smaller spot position and, consequently, a significantly smaller futures position than the EU approach. In case $P_0 - \mu > 0$, similar analysis shows the MV solution dictates a larger spot position and a significantly larger futures position than the EU approach (upon comparing B to B'). The fact that the spot position is a choice variable widens the gap between the two approaches for both short and long hedgers.

SOME NUMERICAL RESULTS

To illustrate, in greater detail, the above qualitative conclusions, a simulation study is performed. First, assume that spot positions are nontradable, varying from -10, -2, -0.5 to 0.5, 2, and 10 (positive and negative positions correspond to being long and short, respectively). Assume, also, that $\sigma = 1$. The results are summarized in Table I. For example, when $\rho = 0.1$ (indicating the spot prices of the two grades are only slightly correlated), the EU approach prescribes the optimal (adjusted) futures positions of 19.1208, 3.4422, and 0.7974 when $\tilde{S} = -10, -2$, and -0.5, respectively. Alternatively, the MV analysis suggests positions of 15.4165, 3.0833, and 0.7708; obviously these figures are lower than those given by the EU approach. The MV

[9]The slope of lines L_0, L_0', and L_0'' is (-2). It can be shown that the slope of L_1 is smaller than 2 in absolute value. Thus, these lines intersect L_1 from below, as shown in Figure 1.

Table I
THE DIFFERENCE BETWEEN THE TWO APPROACHES
(WITH FIXED SPOT POSITIONS)

Correlation Coefficient		Spot Position					
		−10.0	−2.0	−0.5	0.5	2.0	10.0
0.1	(a)	19.1208	3.4422	0.7974	−0.7432	−2.6801	−11.5352
	(b)	15.4165	3.0833	0.7708	−0.7708	−3.0833	−15.4165
0.2	(a)	19.2444	3.5262	0.8281	−0.7805	−2.8533	−12.5046
	(b)	16.0998	3.2200	0.8050	−0.8050	−3.2200	−16.0998
0.3	(a)	19.3521	3.6004	0.8558	−0.8154	−3.0225	−13.4720
	(b)	16.7271	3.3454	0.8364	−0.8364	−3.3454	−16.7271
0.4	(a)	19.4479	3.6670	0.8810	−0.8481	−3.1874	−14.4370
	(b)	17.3050	3.4610	0.8653	−0.8653	−3.4610	−17.3050
0.5	(a)	19.5350	3.7276	0.9042	−0.8785	−3.3474	−15.3989
	(b)	17.8392	3.5678	0.8920	−0.8920	−3.5678	−17.8392
0.6	(a)	19.6161	3.7839	0.9256	−0.9069	−3.5012	−16.3568
	(b)	18.3344	3.6669	0.9167	−0.9167	−3.6669	−18.3344
0.7	(a)	19.6943	3.8373	0.9456	−0.9333	−3.6472	−17.3090
	(b)	18.7948	3.7590	0.9397	−0.9397	−3.7590	−18.7948
0.8	(a)	19.7733	3.8891	0.9644	−0.9576	−3.7825	−18.2523
	(b)	19.2238	3.8448	0.9612	−0.9612	−3.8448	−19.2238
0.9	(a)	19.8608	3.9416	0.9824	−0.9800	−3.9030	−19.1784
	(b)	19.6247	3.9249	0.9812	−0.9812	−3.9249	−19.6247

(a): Expected utility approach; (b): mean-variance analysis.

analysis leads to fewer contracts being sold. The difference, in absolute value and in proportion, shrinks when $|\tilde{S}|$ decreases. Similarly, when $\tilde{S} = 0.5$, 2, and 10, the futures positions prescribed by EU approach are −0.7432, −2.6801, and −11.5352, respectively. The MV analysis suggests −0.7708, −3 0833, and −15.4165. In other words, the hedger buys more futures contracts than necessary if MV analysis is used. Again, the difference shrinks when $|\tilde{S}|$ decreases.

Table I also shows, for any fixed $\tilde{S}$ value, the difference between the two approaches decreases as ρ increases. In fact, when $\rho = 1$, eq. (6c) and (5c) both reduce to eq. (5b), which leads to a "perfect hedge" solution: $\tilde{V}^* = \tilde{V}^{**} = 0.5\,\tilde{S}$. Moreover, when $|\tilde{S}|$ is large, $\tilde{V}^{**}$ is approximately $-(1+\rho)\tilde{S}$ and $\tilde{V}^*$ are $-(1+\rho)\cdot(1-(1-\rho)/\pi)^{-1}\tilde{S}$; thus, approximately

$$(|\tilde{V}^*| - |\tilde{V}^{**}|)/|\tilde{V}^{**}| = (1-\rho)/(\pi - 1 + \rho)$$

which is a decreasing function of ρ. This result implies the optimal futures position for a short (resp. long) hedger is more (resp. less) sensitive to changes in the correlation of spot prices under the MV analysis compared to the EU analysis. That is, not only does the MV analysis provide biased futures positions, but the comparative statics results are also unreliable.

This result can be given a different interpretation in terms of hedge ratio. Hedge ratio is defined as (absolute value of) the futures position divided by the total spot position. The MV analysis leads to a constant hedge ratio for any short or long hedger. (In fact, the simplicity of this result is the very reason the hedge ratio plays an important role in the literature). Using Table I, one can show that the hedge ratio increases with an increasing correlation coefficient; for example, the hedge ratio changes from 0.771 when $\rho = 0.1$ to 0.981 when $\rho = 0.9$ (see Table II). This is not surprising, since a larger correlation coefficient implies that the futures contract is a better hedging instrument. On the other hand, the EU approach does not establish a constant hedge ratio for any spot positions specified a priori. For example, a short hedger with 10 units of either grade has optimal hedge ratios 0.956, 0.977, and 0.993 as $\rho = 0.1$, 0.5, and 0.9, respectively. However, given the correlation coefficient, the MV analysis always chooses a smaller hedge ratio for short hedgers as compared to the EU approach. Furthermore, the gap increases as the spot position increases. Suppose $\rho = 0.4$, the MV analysis prescribes the ratio of 0.865; the EU approach chooses the ratio to be 0.881, 0.917, and 0.972 for a short hedge with 0.5, 2, and 10 units of either grade, respectively. On the other hand, long hedgers that follow the MV analysis choose a larger hedge ratio with the gap (between the MV and EU ratios) increasing with the size of the long spot position. For example, when $\rho = 0.7$, the hedge ratio following MV analysis is 0.938, whereas it is 0.933, 0.912,

Table II
HEDGE RATIOS (WITH FIXED SPOT POSITIONS)

Correlation Coefficient		**Spot Position**					
		−10.0	**−2.0**	**−0.5**	**0.5**	**2.0**	**10.0**
0.1	(a)	0.956	0.861	0.797	0.743	0.670	0.577
	(b)	0.771	0.771	0.771	0.771	0.771	0.771
0.2	(a)	0.962	0.882	0.828	0.781	0.713	0.625
	(b)	0.805	0.805	0.805	0.805	0.805	0.805
0.3	(a)	0.968	0.900	0.856	0.815	0.756	0.674
	(b)	0.836	0.836	0.836	0.836	0.836	0.836
0.4	(a)	0.972	0.917	0.881	0.848	0.797	0.722
	(b)	0.865	0.865	0.865	0.865	0.865	0.865
0.5	(a)	0.977	0.932	0.904	0.879	0.837	0.770
	(b)	0.892	0.892	0.892	0.892	0.892	0.892
0.6	(a)	0.981	0.946	0.926	0.907	0.875	0.818
	(b)	0.917	0.917	0.917	0.917	0.917	0.917
0.7	(a)	0.985	0.959	0.946	0.933	0.912	0.865
	(b)	0.938	0.938	0.938	0.938	0.938	0.938
0.8	(a)	0.989	0.972	0.964	0.958	0.946	0.913
	(b)	0.961	0.961	0.961	0.961	0.961	0.961
0.9	(a)	0.993	0.985	0.982	0.980	0.976	0.959
	(b)	0.981	0.981	0.981	0.981	0.981	0.981

(a): Expected utility approach; (b): mean-variance analysis.

and 0.865, respectively, for a long hedger with 0.5, 2, and 10 units of long spot position on either grade.

Turning to the case of tradable spot positions, there are two possibilities: (1) the spot market is biased downward such that $(P_0 - \mu)/\sigma^2 = -0.25$; (2) the spot market is biased upward such that $(P_0 - \mu)/\sigma^2 = 0.25$. The two possibilities correspond to short and long hedgers, respectively. Numerical results are provided in Table III. For the case of long hedgers, Table III shows that the extra freedom of choosing spot positions enlarges the difference between the two approaches. For example, when $\rho = 0.9$, the MV analysis suggests an optimal futures position of -4.9522, much larger than the counterpart solution determined by the EU approach, which was -13.7597. Also note that the difference actually increases as ρ increases; this is in contrast to the case of nontradable spot positions. Similar conclusions apply to short hedging, although the difference between the two approaches is much smaller than in the case of long hedging. For example, when $\rho = 0.9$, $|\tilde{V}^{**}| - |\tilde{V}^{*}|$ is 2.9236 for short hedging, as compared to -8.8075 for long hedging.

Table III
THE DIFFERENCE BETWEEN THE TWO APPROACHES

		Bias in Spot Market			
		−0.25		0.25	
Correlation Coefficient		Spot Position	Futures Position	Spot Position	Futures Position
0.1	(a)	−0.5712	0.9150	0.4475	−0.6678
	(b)	−0.9917	1.5288	0.9917	−1.5288
0.2	(a)	−0.6256	1.0428	0.4771	−0.7459
	(b)	−1.0683	1.7199	1.0683	−1.7199
0.3	(a)	−0.7017	1.2111	0.5188	−0.8452
	(b)	−1.1751	1.9656	1.1751	−1.9656
0.4	(a)	−0.8104	1.4422	0.5772	−0.9759
	(b)	−1.3252	2.2932	1.3252	−2.2932
0.5	(a)	−0.9728	1.7788	0.6613	−1.1559
	(b)	−1.5426	2.7519	1.5426	−2.7519
0.6	(a)	−1.2345	2.3128	0.7883	−1.4203
	(b)	−1.8762	3.4399	1.8762	−3.4399
0.7	(a)	−1.7150	3.2830	0.9979	−1.8487
	(b)	−2.4403	4.5865	2.4403	−4.5865
0.8	(a)	−2.8504	5.5619	1.4045	−2.6701
	(b)	−3.5788	6.8798	3.5788	−6.8798
0.9	(a)	−8.4074	16.6833	2.5419	−4.9522
	(b)	−7.0114	13.7597	7.0114	−13.7597

(a): Expected utility approach; (b): mean-variance analysis.

Optimal hedge ratios resulting from the different analytical frameworks are shown in Table IV. Again, the MV analysis generates a smaller hedge ratio for short hedgers, and a larger hedge ratio for long hedgers. Suppose that $\rho = 0.6$, the EU approach recommends a hedge ratio of 0.937 and 0.901 for short and long hedgers respectively; whereas MV analysis suggests a constant ratio of 0.917.

CONCLUSIONS

In this article, a basic model of broad-based futures contracts is used to evaluate mean-variance analysis as an approximation to the more general expected utility approach. It is shown that the MV analysis provides a good approximation only when the hedger is not active in spot markets. More importantly, the bias produced by this analysis for either short or long hedgers is always unidirectional: upward bias for long hedgers and downward for short hedgers. The degree of risk aversion accentuates the bias in terms of futures positions but has no effects on the bias in hedge ratios.

The conclusions are derived assuming: (1) There are only two deliverable grades and each has the same probability density function over the price at maturity. (2) The futures market is established at a martingale equilibrium. (3) The assumptions

Table IV
HEDGE RATIOS

Correlation Coefficient		Bias in Spot Market −0.25	Bias in Spot Market 0.25
0.1	(a)	0.801	0.746
	(b)	0.771	0.771
0.2	(a)	0.833	0.782
	(b)	0.805	0.805
0.3	(a)	0.863	0.815
	(b)	0.836	0.836
0.4	(a)	0.890	0.845
	(b)	0.865	0.865
0.5	(a)	0.914	0.874
	(b)	0.892	0.892
0.6	(a)	0.937	0.901
	(b)	0.917	0.917
0.7	(a)	0.957	0.926
	(b)	0.940	0.940
0.8	(a)	0.976	0.951
	(b)	0.961	0.961
0.9	(a)	0.992	0.974
	(b)	0.981	0.981

(a): Expected utility approach; (b): mean-variance analysis.

of Kuhn (1988) are implicitly assumed to ensure that the futures price equals the minimum of the two spot prices at maturity. Some of these assumptions can be easily eliminated without generating any complication. For example, a backwardation equilibrium for the futures market can be allowed. In this case, L_1 and L_2 in Figure 1 will simultaneously move downward along the vertical axis. The conclusions established for the martingale equilibrium remain intact. Similar observations apply to a futures market with contango. Note that this study's model does not endogenize the determination of the initial futures price; it compares MV and EU approaches after the initial price is determined.

To apply this model to the case of more than two grades, or to allow for asymmetric density functions, necessitates complicated computation. Although it is believed that the results apply to the general setting, these extensions are left for future research. Finally, without making the assumptions of Kuhn (1988), one would need to consider incorporating inspection and transportation costs which would make the model more complicated.

The results of this study are not meant to be directly applicable to futures markets. Nevertheless, some insight is provided about the application of mean-variance analysis to a broad-based futures contract. For any specific futures market, the model needs to be modified to incorporate its specific characteristics.

Appendix I

To derive eq. (1), note that $f_1 = P_{11}$ whenever $P_{11} \leqslant P_{21}$. Thus,

$E[U(W_1)|P_{11} \leqslant P_{21}] \cdot \text{Prob}(P_{11} \leqslant P_{21})$

$$= -\int_{-\infty}^{\infty}\int_{P_{11}}^{\infty} \exp((\tilde{S}_1 + \tilde{V})P_{11} + \tilde{S}_2P_{21})g(P_{11}, P_{21})dP_{21}dP_{11}\exp(\mathrm{M}_0) \qquad \text{(A1)}$$

where $g(\cdot,\cdot)$ denotes the joint density function of P_{11} and P_{21}.
Let $\epsilon_1 = P_{11} - \mu_1$ and $\epsilon_2 = P_{21} - \mu_2$. Then, (ϵ_1, ϵ_2) is a bivariate normal vector with zero mean. The RHS of eq. (A1) may be rewritten as:

$$-\exp(M_0 + (\tilde{S}_1 + \tilde{V})\mu_1 + \tilde{S}\mu_2) \cdot \int_{-\infty}^{\infty}\int_{\epsilon_1+\mu_1-\mu_2}^{\infty} \exp((\tilde{S}_1 + \tilde{V})\epsilon_1 + \tilde{S}_2\epsilon_2)$$

$$b(\epsilon_1, \epsilon_2)d\epsilon_2 d\epsilon_1$$

where $b(\epsilon_1, \epsilon_2)$ is the joint density of ϵ_1 and ϵ_2. Lemma 2 of Lien (1986) is now applied directly to the integral to achieve eq. (1).

Appendix II

Let H_1 and H_2 denote, respectively, the first and second terms on the RHS of eq. (1). The optimal spot and futures positions ($\tilde{S}_1^{**}$, $\tilde{S}_2^{**}$, and $\tilde{V}^{**}$) may be derived by setting the partial derivatives of $(H_1 + \mathrm{H}_2)$ to zero. Since the two grades are symmetric in every aspect (i.e., $\mu_1 = \mu_2 = \mu$, $\sigma_1 = \sigma_2 = \sigma$, and $P_{10} = P_{20} = P_0$) one obtains $\tilde{S}_1^{**} = \tilde{S}_2^{**} = \tilde{S}^{**}$. Consequently, $H_1 = H_2$ when evaluated at optimal positions. Furthermore, when $\tilde{S}_1 = \tilde{S}_2$:

$$\partial H_1/\partial\tilde{V} = \{(1 + \rho)\sigma^2\tilde{S}_1 + \sigma^2\tilde{V} + (\mu_1 - f_0) - \sqrt{(1 - \rho)/2}\sigma\phi(\beta)/\Phi(\beta)\}H_1 \quad (A2)$$

where $\beta = -(1 - \rho)/2\sigma\tilde{V}$. Similarly,

$$\partial H_2/\partial\tilde{V} = \{(1 + \rho)\sigma^2\tilde{S}_2 + \sigma^2\tilde{V} + (\mu_2 - f_0) - \sqrt{(1 - \rho)/2}\sigma\phi(\beta)/\Phi(\beta)\}H_2 \quad (A3)$$

Thus, when evaluated at optimal positions, $\partial EU(W_1)/\partial\tilde{V} = 0$ implies

$$(1 + \rho)\tilde{S}^{**} + \tilde{V}^{**} = (f_0 - \mu)/\sigma^2 + \sqrt{(1 - \rho)/2\sigma^2}\phi(\theta\tilde{V}^{**})/\Phi(\theta\tilde{V}^{**}) \quad (6c)$$

where $\theta = -\sqrt{(1 - \rho)/2} \cdot \sigma$.

To determine the optimal spot position, we consider $\partial H_1/\partial\tilde{S}_1$ and evaluate at $\tilde{S}_2 = \tilde{S}_1$. The result is:

$$\partial H_1/\partial\tilde{S}_1 = H_1((1 + \rho)\sigma^2\tilde{S}_1 + \sigma^2\tilde{V} + (\mu_1 - P_0) + \theta\phi(\beta)/\Phi(\beta)) \quad (A4)$$

Similarly, we have

$$\partial H_2/\partial\tilde{S}_1 = H_2(\mu_1 - P_0 + (1 + \rho)\sigma^2\tilde{S}_1 + \sigma^2\rho\tilde{V} - \theta\phi(\beta)/\Phi(\beta)) \quad (A5)$$

Thus, the first order condition $\partial EU(W_1)/\partial\tilde{S}_I = 0$ implies $(\tilde{S}^{**}, \tilde{V}^{**})$ satisfies the following condition:

$$(1 + \rho)\tilde{S}^{**} + (1 + \rho)\tilde{V}^{**}/2 = (P_0 - \mu)/\sigma^2 \quad (6b)$$

Appendix III

From eqs. (6c) and (8), the properties of $k(\cdot)$ and $k'(\cdot)$ clearly depend upon the function $\phi(x)/\Phi(x)$ and its derivatives. Let $m(x)$ denote $\phi(x)/\Phi(x)$, Heckman and Honoré (1990) show that $-1 < m'(x) < 0$ and $m''(x) > 0$ for every x. Upon applying the first relationship to eq. (8), one obtains $k'(\cdot) < 0$. Furthermore,

$$k''(\tilde{S}) = (-\sqrt{(1 - \rho)/2})\sigma k'(\tilde{S}) \cdot (1 - \rho^2)/2 \cdot m''(x) \cdot (1 + (1 - \rho)m'(x)/2)^{-2} > 0$$

since $m''(\cdot) > 0$ and $k'(\cdot) < 0$. To evaluate $k(\cdot)$ at $\tilde{S} = 0$, note that $(\tilde{S}, \tilde{V}) = (0, 0)$ satisfies eq. (6c). Because $k'(\cdot) < 0$, $k(\cdot)$ intersects the horizontal axis once at the most, and also, $k(0) = 0$. On the other hand, by inspecting eq. (5c), it is apparent that $h(0) = 0$. Finally, notice that

$$m'(x) = -m(x)(x + m(x))$$

implies $m'(0) = -m^2(0) = -2/\pi$. If this result is applied to eq. (8):

$$k'(0) = -(1 + \rho)(1 - (1 - \rho)/\pi)^{-1} = h'(0)$$

Bibliography

Anderson, R.W., and Danthine, J.-P. (1981): "Cross Hedging," *Journal of Political Economy,* 89:1182–1196.

Baron, D. P. (1977): "On the Utility Theoretical Foundations of Mean-Variance Analysis," *Journal of Finance,* 32:1683–1697.

Benniga, S., Eldor, R., and Zilcha, I. (1983): "Optimal Hedging in the Futures Markets under Price Uncertainty," *Economics Letters,* 13:141–145.

Chamberlin, G. (1983): "A Characterization of the Distributions that Imply Mean-Variance Utility Functions," *Journal of Economic Theory,* 29:185–201.

Epstein, L. G. (1985): "Decreasing Risk Aversion and Mean-Variance Analysis," *Econometrica,* 53:945–961.

Feldstein, M. S. (1969): "Mean-Variance Analysis in the Theory of Liquidity Preference and Portfolio Selection," *Review of Economic Studies,* 36:5–12.

Heckman, J. J., and Honoré, B. E. (1990): "The Empirical Content of the Roy Model," *Econometrica,* 58:1121–1149.

Kamara, A., and Siegel, A. (1987): "Optimal Hedging in Futures Markets with Multiple Delivery Specifications," *Journal of Finance,* 42:1007–1021.

Kuhn, B. A. (1988): "A Note: Do Futures Prices Always Reflect the Cheapest Deliverable Grade of the Commodity?" *The Journal of Futures Markets,* 8:99–102.

Lien, D. (1986): "Moments of Ordered Bivariate Log-Normal Distributions," *Economics Letters,* 20:45–47.

Lien, D. (1988): "Hedger Response to Multiple Grades of Delivery on Futures Markets," *The Journal of Futures Markets,* 8:687–702.

Machina, M. J. (1982a): "'Expected Utility' Analysis without the Independence Axiom," *Econometrica,* 50:277–323.

Machina, M. J. (1982b): "A Stronger Characterization of Declining Risk Aversion," *Econometrica,* 50:1069–1079.

Machina, M. J., and Neilson, W. S. (1987): "The Ross Characterization of Risk Aversion: Strengthening and Extension," *Econometrica,* 55:1139–1149.

Meyer, J. (1987): "Two-Moment Decision Models and Expected Utility Maximization," *American Economic Review,* 77:421–430.

Newbery, D. M. (1987): "When Do Futures Destabilize Spot Prices?" *International Economic Review,* 28:291–297.

Newbery, D. M. (1988): "On the Accuracy of the Mean-Variance Approximation for Futures Markets," *Economics Letters,* 28:63–68.

Pratt, J.W. (1964): "Risk Aversion in the Small and in the Large," *Econometrica,* 32:122–136.

Ross, S. A. (1981): "Some Stronger Measure of Risk Aversion in the Small and the Large with Applications," *Econometrica,* 49:621–638.

Samuelson, P. A. (1970): "The Fundamental Approximation Theorem of Portfolio Analysis in Terms of Means, Variances and Higher Moments," *Review of Economic Studies,* 37:537–542.

Tobin, J. (1958): "Liquidity Preferences as Behavior toward Risk," *Review of Economic Studies,* 25:65–86.

Tsiang, S. C. (1972): "The Rationale of the Mean Standard Deviation Analysis, Skewness Preference, and the Demand for Money," *American Economic Review,* 62:354–371.