"Chaos" in Futures Markets? A Nonlinear Dynamical Analysis

Steven C. Blank

INTRODUCTION AND OBJECTIVES

Commodity market analysts constantly seek better explanations of price behavior in the form of economic models. Such models are used for many types of forecasting. However, the various linear models developed to date do not always work very well [Just and Rausser (1981)]. Some cases of short-term forecasting success of time series and econometric models imply that futures and spot prices are not always generated by a "random" process, but the long-term failure of these models to explain price behavior indicates they have not captured the true nature of the underlying generating process. For example, residuals in commodity price models may not be random, as assumed, but the result of a nonlinear process. It may be time to change existing theoretical assumptions and/or empirical approaches.

A new methodology, called nonlinear dynamics (or "chaos"), evolving recently in physics and other natural sciences, may offer an alternative explanation for behavior of economic phenomena [Brock (1988a)]. Based on the assumption that at least part of the underlying process is nonlinear, chaos analysis evaluates whether that process is a deterministic system [Jensen (1987)].[1] Deterministic processes that look random under statistical tests such as spectral analysis and the autocovariance function are called "deterministic chaos" in nonlinear science. In practice, some economists argue that separating deterministic and stochastic processes may be very difficult due to the "noisy" nature of economic data [Mirowski (1990)]. Although applications of chaos analysis in economics are still very rare [examples include Baumol and Benhabib; (1989) Brock (1986); Brock and Sayers (1988); Candela and Gardini (1986); Day (1983); Goodwin (1990); Lorenz (1987); Melese and Transue (1986)], there may be potential for its use in some markets.

In markets for undifferentiated commodities, intuition leads to expectations of multiple sources for chaos or nonlinear feedback [Savit (1988)]. Cyclical and seasonal patterns are often visible in charts of both prices and trade volumes. The futures price-setting process is similar to that in many financial spot markets where

This study was partially funded by a grant from the Center for the Study of Futures Markets, Columbia University. This is Giannini Foundation Research Paper No. 976.

[1]Compared to linear methods, nonlinear dynamics may be a less restrictive approach to modeling economic systems simply because of the assumptions it does *not* make, concerning the distribution, for example, as discussed later.

Steven C. Blank is an extension economist, Department of Agricultural Economics, University of California, Davis.

The Journal of Futures Markets, Vol. 11, No. 6, 711–728 (1991)

CCC 0270-7314/91/060711-18$04.00

evidence of chaos has been found [see Brock (1988b); Frank and Stengos (1989); Hinich and Patterson (1985); Scheinkman and LeBaron (1989); van der Ploeg, (1986]. Also, there are a number of marketing strategies based, at least implicitly, upon behavioral and structural aspects of commodity markets. In sum, futures markets appear to be a logical place to expand the analysis of chaos hypotheses.

Applying a new methodology to any problem raises many questions. The first question is usually "why bother?" It is sometimes claimed that if a model gives a "reasonable approximation" of reality, it is good enough. This implies that the cost of developing a more complex model may not be well spent. Considering the adequate performance of some linear economic models, some analysts may be satisfied to continue using them. But are those models "good enough" only because methods are not yet advanced enough to know that the world is round (nonlinear), not flat (linear)? It may be that BLUE (Best Linear Unbiased Estimate) is not always the relevant "color" for economic models. If the real structure of prices is nonlinear, using linear models may give noisy results; however, a completely deterministic nonlinear function may explain prices with no noise, meaning that forecasting may be possible. While the goal of improving forecasting models through the use of chaotic parameters may seem out of reach at present, the possibility makes studies such as this necessary.

The general objectives of this study are to (i) evaluate commodity futures markets using the methodology of nonlinear dynamics to detect whether there are any signs of a deterministic system underlying prices over time; and (ii) while doing so, to illustrate the empirical procedures and their limitations. Specific objectives of this study are to determine whether (a) there is a difference between chaotic analysis results for cash and futures markets of financials; (b) there is a difference between results for futures markets of financials and agricultural products; and (c) chaos is detectable over a period lengthy enough to justify development of forecasting models. The presentation begins with a summary of related studies. That is followed by a section introducing the basic concepts in chaos analysis. Empirical procedures are outlined then and the results from analyses of two heavily traded futures markets, the S&P 500 index and soybeans, are presented as examples.

SUMMARY OF PREVIOUS WORK

Applying chaos analysis methods to commodity futures prices raises questions ranging from behavioral to structural issues.[2] One motivation for considering chaos as a possible explanation for commodity spot and futures price behavior is the poor results produced by studies using traditional methods. For example, using regression analysis, Roll (1984) explains only 1–3% of the daily variation in orange juice futures prices. If deterministic models can be developed with the assistance of chaos methods, forecasting of futures prices could be improved.

Empirical applications of nonlinear dynamical analysis techniques in economics are few in number. They are concentrated in two areas: assessments of the business cycle [Brock and Sayers (1988); Day (1983); Lorenz (1987)] and financial markets [Hinich and Patterson (1985); Savit (1989); Scheinkman and LeBaron (1989); van

[2]For example, "Can nonlinear dynamics explain speculative bubbles?" or "Do trading rules, such as those imposed on futures markets, create price patterns similar to patterns generated by 'strange attractors' (defined in the next section)?" Intuitively, there is reason to believe chaos may contribute to the debate over "bubbles" and structural aspects of commodity trading may provide some of the explanation.

der Ploeg (1986)]. The results are somewhat mixed, possibly due to the nature of data used. Studies of macroeconomic variables, such as the business cycle, do not generate results as strong as micro-level analyses of some specific markets [Barnett and Chen (1988)]. Therefore, evaluating data characteristics should be an early stage of any empirical study.

The first indication that relying on traditional analysis methods alone may be inadequate come from studies evaluating the characteristics of futures market data. Cornew et al. (1984) find that futures prices are not normally distributed, creating error in traditional trading performance measures, all of which are based on the normality assumption. Helms and Martell (1985) also reject the normality assumption concerning the distribution of futures price changes. However, they find that the normal distribution fit their data better than other members of the stable Paretian class of distribution. They note that the underlying generating process does *not* appear to be stationary, possibly being a subordinated stochastic or compound process requiring a more sophisticated approach to determine its exact nature. Garcia et al. (1988) use nonparametric tests and find more "nonrandomness" in livestock futures prices than are found using traditional tests. They suggest that the nonrandomness may be nonlinear in nature. These conclusions are similar to those reached by Hinich and Patterson (1985) regarding stock market returns.

These results are significant because traditional, linear models assume the data are normally distributed [Stevenson and Bear (1970); Kenyon et al. (1987)], while nonlinear dynamics makes no a priori assumptions concerning data distributions. In fact, distributions are hypothesized to change across nonlinear systems of differeing degrees of chaotic behavior.

The usual way of looking for order in a time series entails spectral analysis. [Economic applications are illustrated by Shumway (1988) and Talpaz (1974).] The process involves transforming a series into a number of independent components associated with different frequencies. If the series includes periodic motion, the resulting power spectrum corresponds to the fundamental frequency. At the opposite extreme is white noise; all frequencies contribute equally. Unfortunately, the presence of sharp spectral peaks in a time series does not necessarily indicate a periodic attractor, nor does the absence of such peaks exclude the possibility of deterministic dynamics [Barnett and Hinich (1991)]. Therefore, other methods of analysis must be used to identify chaos in economic time series data.

An intermediate step is illustrated by Ashley and Patterson's (1989) use of the bispectral nonlinearity test. They test for a linear generating mechanism for both an aggregate stock market index and an aggregate industrial production index, rejecting the hypothesis in both cases. They conclude that their results strongly suggest that nonlinear dynamics should be an important feature of any macroeconomic model. The general state-dependent models outlined by Priestly (1988) could be one of the practical ways of pursuing nonlinear modeling in the future.

In microeconomic analysis, one method used to test the speculative efficiency hypothesis is called rescaled range analysis, which is capable of identifying persistent or irregular cyclic dependence in time series data. The hypothesis that respective price changes are independent of previous price changes is rejected in applications of rescaled range analysis to the foreign exchange market [Booth et al. (1982a)], the gold market [Booth et al. (1982b)], and the stock market [Greene and Fielitz (1977)]. Long-term persistent dependence is found in each study. A similar study of soybean futures markets finds nonperiodic cycles (persistent dependence) in both daily and intraday futures prices [Helms et al. (1984)].

Mandelbrot (1977) evaluates cotton prices and finds patterns which match across *scales* of time (daily, monthly). This raises the question, "is scaling a problem or an answer?" [Feigenbaum (1983)]. Economists use different data aggregations in their analyses (annual, quarterly, monthly, weekly, daily, and for futures—intraday). In particular, futures and spot price data for commodities give the appearance of having what Mandelbrot (1977) called *fractal dimensions.* Empirical studies for gold and silver [Frank and Stengos (1989); Scheinkman and LeBaron (1989)] and T-bills [Brock (1988b)] consider only the spot market but the fact that each undifferentiated product tested produces positive results indicates that further investigation is warranted.

CONCEPTS IN CHAOS ANALYSIS

As explained by Brock and Sayers (1988), a data time series (a_t) can be characterized as deterministically chaotic if there exists a system, (h, F, x_0), such that h maps R^n to R, F maps R^n to R^n, $a_t = h(x_t), x_{t+1} = F(x_t)$, and x_0 is the initial condition at time $t = 0$. In this case, the map F is deterministic, the state space is n-dimensional, all trajectories (x_t) lie on an attractor A, and two nearby trajectories on A locally diverge. The variable h is a general function of the unknown state vector x, and F is an unknown dynamic that governs the evolution of the state. Also, x_0 is, in general, unknown. The goal of analysis is to discover information about the system (h, F, x_t) from observations (a_t).

An *attractor, A,* is a subset of the n-dimensional phase space, R^n. Attractors, i.e., collections of points to which initial conditions tend, can be loosely defined as having the following property. Solutions (trajectories) originating at points on the attractor remain there forever; solutions based at points *not* on the attractor, but within a region called the attractor's "basin of attraction," approach the attractor to an arbitrary degree of closeness. In the case of price data, an attractor might be particular (equilibrium?) price levels or patterns.

Chaos theory deals with deterministic processes which appear to be random (stochastic), but whose dimension is finite. Specifically, a "random process" is defined to have a "high" dimension, while a "deterministic process" has a "low" dimension. The object of analysis is to distinguish between the two. To do so, "dimension" must be defined and measured. In chaos analysis, the focus is on *fractal dimensions:* similar patterns across different (time) scalings.

One of the three types of probabilistic fractal dimension[3] is the "correlation dimension" (hereafter CD). It and the Lyapunov exponent (LE) (defined below) have become the most popular measures of nonlinear systems [Abraham et al. (1984); Grassberger and Procaccia (1983); Lorenz (1987)]. It is a measure of the minimal number of nonlinear "factors" needed to describe the data [Brock and Sayers (1988)]. To estimate the correlation dimension it is first necessary to compute the distances between all the points of a time series and then to determine what fraction of those distances are less than a series of predetermined length scales. This gives a measure called the "correlation integral," which is defined for different scale lengths, g, by the equation

$$C(g) = \lim_{N\to\infty}\left[\frac{1}{N(N-1)}\right]\sum_{i\neq j}^{N}(g, X_i, X_j) \tag{1}$$

[3]Probabilistic dimensions explicitly consider the frequency distribution with which points on the attractor are visited.

where N is the sample size, X_i and X_j are (vector-valued) observations in the time series and

$$(g, X_i, X_j) = \begin{cases} 1 \text{ if } |X_i - X_j| < g \\ 0 \text{ if } |X_i - X_j| > g \end{cases}$$

Grassberger and Procaccia (1983) argue that for small g

$$C(g) = \text{Constant} \cdot g^n \tag{2}$$

where the exponent n is the CD. Empirical procedures used to estimate the CD are discussed in the methodology section later in this paper.

Low-dimensional chaos involves instability and "overshooting," while stochastic processes are infinite dimensional [Brock and Dechert (1991); van der Ploeg (1986)]. The few economic applications of this measure involve a stock returns index and gold and silver spot prices, all of which have CD estimates of about 6 (considered to be low) using weekly data [Frank and Stengos (1989); Scheinkman and LeBaron (1989)], and Treasury bill returns which have a dimension around 2 [Brock (1988b)].

Lyapunov exponents are simply generalized eigenvalues averaged over the entire attractor. They measure the average rate of contraction (when negative) or expansion (when positive) on an entire attractor. They can be positive or negative, but at least one must be positive for an attractor to be classified as chaotic or "strange" [Wolf (1986)]. In the one-dimensional case, where $x_{i+1} = F(x_i)$, the Lyapunov exponent, λ, is defined as

$$\lambda = \lim_{N\to\infty} \left(\frac{1}{N}\right) \sum \log_2 \left|\frac{dF}{dx}\right| \tag{3}$$

where the derivative is evaluated at each point on the trajectory and logarithms are taken to the base 2. Usually, LEs are reported in units of bits per observation. Positive exponents can be viewed as measuring the rate at which new information is created; specifically, the rate at which unmeasurable (because they are too small) variations are magnified to the point where they can be observed.

In summary, Savit (1988) lists three features a deterministic sequence should have to be chaotic:

1. It should sample all regions of its domain (eventually).
2. It is practically unpredictable in the long term (if there is any indeterminacy in one's knowledge or ability to compute.[4]
3. There are many initial prices, P_o, called periodic points,[5] and the sequence of prices generated by the chaotic map is periodic, i.e., over time the sequence of prices P_t eventually repeats itself.

This list makes it clear that chaos analysis must begin with a detailed description of the data involving a set of statistics new to most market analysts.

GENERAL METHODOLOGY

Empirical methods derived from those applied in previous economic studies of chaotic systems [Brock (1986); Deneckere and Pelikan (1986)] are used here. To ac-

[4]For example, rounding errors accumulate rapidly in the calculations involved in estimating fractal dimensions, thus reducing precision in distant forecasts.

[5]The existence of a large number of periodic points is just one example of hidden regularities in a chaotic system.

complish this study's first general objective, this three-step process of dynamical analysis is followed:

1. Calculate the Grassberger-Procaccia correlation dimension for various embedding dimensions [Abraham et al. (1984); Swinney (1983)].
2. Apply a residual test for the presence of deterministic chaos as an alternate for the best-fitting linear model of prices [Brock and Sayers (1988)].
3. Estimate the largest Lyapunov exponent for various embedding dimensions [Abraham et al. (1984); Swinney (1983); Wolf (1986); Wolf and Swift (1984); Wolf et al. (1985)].

Operationally, the correlation dimension [n in eq. (2)] is the slope, k, of the regression: log $C(g)$ versus log g for small values of g. To estimate the CD from a single variable time series, the procedure is as follows:

1. The data are embedded in successively higher dimensions as prescribed by Takens (1985).
2. For each embedding, $C(g)$ is computed and the scaling factor, k, is estimated.
3. The procedure is repeated until the estimates of k converge.

The correlation dimension is therefore expected to equal whatever value at which k remains stable over a number of embedding dimensions.

Lyapunov exponents are local quantities which are averaged over the attractor. To compute the largest LE from a time series, the program by Wolf et al. (1985) is used which suggests choosing a so-called "fiducial" trajectory and estimating the rate at which it and a nearby test trajectory diverge. When the distance between the two points becomes large, a new point is chosen near the fiducial trajectory and the procedure is repeated. The entire time series is stepped through in this way and the stretchings and contractions are averaged.

In practice, estimating Lyapunov exponents in this way requires experimentation on the part of an analyst. Wolf's algorithm requires that the following information be provided:

- an embedding dimension and time lag for the reconstruction of the attractor;
- the time interval over which the two pieces of the trajectory are followed (called the "evolution");
- the minimum acceptable separation between points that are to be followed; and
- the maximum acceptable separation.

As a result of this flexibility in inputs, an analyst must simply experiment. The goal of the experimentation is to find a region in parameter space over which the estimate is approximately constant. Clearly, this portion of the analysis is rough, hence the reluctance of previous authors to report exact LE values for economic data series.

The data used in this study are futures prices for the S&P 500 index and soybeans. These products are selected as examples to represent, respectively, financial and agricultural futures markets because they are traded heavily and each market has been studied previously by analysts using a variety of methods, which provides a base for comparison. The S&P index, in particular, is selected because it is the product traded on futures markets which is closest to the variables studied in earlier chaotic analyses of the cash stock market. This enables direct comparison between results from previous studies of the cash stock market and the results produced here for the futures index. For each product, daily closing price data for recent individual futures contracts and nearby contracts are evaluated. For soybeans, the November 1986 and November 1987 contracts are used. The data for each contract begins during July of the previous calendar year, giving 337 and 335 observations, respectively. The nearby futures price series, constructed from the closing prices of the futures contract closest to its maturity date at each point in time, covers the period from March 1966 through June 1988, with 5823 observations. The December 1986 and December 1987 S&P 500 contracts are used. Each contract has 250 observations covering the previous calendar year. The S&P 500 nearby series begins in May 1982, ends in December 1987, and has 1420 observations.

Before conducting nonlinear dynamical analysis, the data are detrended using ordinary least squares and autoregressive methods, deseasonalized and transformed (if necessary) as described by Baumol and Benhabib (1989). If a time series has a deterministic explanation, fitting a smooth time series model with a finite number of leads and lags will generate a residual series with the same CD and largest LE as the original series. The key is to first make the residuals as close to "white noise" as possible with traditional linear methods. In this way, if nonlinear models find significant traces of a deterministic system in the residuals, the hypothesis of a linear generating system can be rejected.

Brook and Sayers (1988) evaluate the H_o: whiteness tests and diagnostics do not alter chaotic systems and can be used to test residuals from the best linear model. They use a new "W" statistic to test for independence. Based on the correlation integral, the W statistic for embedding dimension m is defined as

$$W_m(g, N) \equiv \sqrt{N}\left(\frac{D_m(g, N)}{b_m(g, N)}\right) \tag{4}$$

where $D_m = C_m - (C_1)^m$, and b_m is an estimate of the standard deviation

$$b_m = (1, -mC_{m-1})' \sum (1, -mC_{m-1}) \tag{5}$$

as discussed by Brock (1988a). The W test detects misspecification caused by a linear model fit to nonlinear data and indicates such by giving a nonzero value. Brock and Dechert (1991) show that $D_m(g, N)$ converges to a nonzero number if evaluated using the residuals from a misspecified model, while the residuals from a correctly fitted linear model are independent and identically distributed and are characterized by $D_m(g, \infty) \equiv D_m(g) = 0$ for all m and all $g > 0$. This means that a W value of 0 indicates a stochastic process and a large W is evidence of a misspecified (nonlinear) model. Statistically significant nonlinearity appears to exist at the 5% level when the W value is greater than 1.96 [Brock and Dechert (1991)]. W statistics are reported below for the data series evaluated.

Due to the weakness of the standard CD empirical processes in some cases, it is desirable to calculate the correlation dimension for "random" numbers generated to provide a basis for comparison and to report the W statistic for those values [Brock (1986)]. The random numbers used are the residuals "scrambled" as described by Scheinkman and LeBaron (1989). If the data are stochastic, the CD estimates for the residuals and the "scrambled" residuals will be identical; if the scrambled data generate higher CD estimates, this is evidence of hidden (nonlinear) structure in the residuals. The scrambled residuals test is conducted in this study.

Also, the small sample distribution of the CD and LE statistics are illustrated here by presenting values calculated from 100 series of random numbers. The series are drawn using random number generators based on the following distributions: normal, log, chi-square, exponential, double exponential, geometric, poisson, *F*-distribution, *T*-distribution, and beta. Ten series are drawn from each distribution since no a priori assumption is made concerning which distribution is "best". Each series includes 250 observations. The 100 CD values estimated are averaged to provide a benchmark representing stochastic processes. As in the scrambled residual test, deterministic CD values must be below those estimated from the random numbers. For LEs, two sets of estimates are reported: average values for the 100 series and the highest of the 100 LEs observed for each group of experiments. The average values for the random series can be compared against the values estimated from the actual futures price data to establish the general case for evidence of deterministic processes in the LE values. A qualitative assessment of the degree of significance of actual LE values can be made by a second comparison involving the highest of the 100 random estimates. This means that an LE for futures price data must be both positive and greater than the average LE from the random data to be considered evidence of a chaotic process; but, in addition, it must be greater than the highest of the 100 random LE values to be convincing evidence.

EMPIRICAL RESULTS AND IMPLICATIONS

Estimates of correlation dimensions and Lyapunov exponents are both given below, along with a discussion of the results. First, measures used to "prewhiten" the data are noted.

A generalized autoregressive conditional heteroscedasticity (GARCH) model is used to generate the residuals used in the analysis. The GARCH procedure, developed by Bollerslev (1986), is applied in studies of exchange rates [Hsieh (1989)] and stock market returns using spot market data [Baillie and DeGennaro (1990); Scheinkman and LeBaron (1989)] and futures market data [Cheung and Ng (in press)] as well as in studies of commodity markets [Aradhyula and Holt (1988)]. GARCH methods are found to be useful in detecting nonlinear patterns in variance while not destroying any signs of deterministic structural shifts in a model [Lamoureux and Lastrapes (1990)]. Therefore, it serves as a good filter for studies of chaos.

Brock and Sayers (1988) demonstrate that the power of the W statistic is weak if an autoregressive [AR(q)] model is applied with a high order (q) and, therefore, q should be set as low as possible in GARCH whitening processes. In this study, if conventional criteria finds that a GARCH(1, 1) model produces white noise residuals, it is used. If a GARCH(1, 1) model fails this test, a GARCH(2, 2) model would be fitted next, and so forth until white noise is found.

The data for the two S&P 500 contracts cover years during which prices generally trended. Prices of the December 1986 S&P 500 contract fluctuate around an upward-sloping trendline. As a result, the GARCH(1, 1) model for the December 1986 S&P 500 contract price at time t is

$$\begin{array}{l} x_t = \underset{(0.192)}{0.207} - \underset{(0.011)}{0.024}x_{t-1} + \underset{(0.011)}{0.013}x_{t-2} + \delta_t \end{array} \quad (6a)$$

$$h_t = \underset{(0.847)}{6.592} + \underset{(0.459)}{0.633\delta_{t-1}^2} + \underset{(0.490)}{0.566h_{t-1}} \tag{6b}$$

where the x_t's are first differences, δ is a residual, h is the conditional variance of the residuals, and the standard errors are in parentheses. Stock prices are lower during 1987 due to the "Crash" in October. The uptrend-ending jolt gives the December 1987 S&P 500 contract a model of

$$x_t = \underset{(0.455)}{0.008} - \underset{(0.064)}{0.252x_{t-2}} - \underset{(0.064)}{0.120x_{t-4}} + \underset{(0.062)}{0.109x_{t-5}} + \delta_t \tag{7a}$$

$$h_t = \underset{(33.831)}{49.072} + \underset{(0.391)}{0.696\delta_{t-1}^2} + \underset{(0.359)}{0.752h_{t-1}} \tag{7b}$$

using the same notation.

The data for the two soybean contracts cover the same two years, but display trends for those years with slopes opposite to those in the S&P 500 data. Whereas stock prices rise during 1986, soybean prices trend downward that year. The GARCH(2, 2) model of the November 1986 soybean contract price at time t is

$$x_t = \underset{(0.258)}{-0.143} - \underset{(0.055)}{0.108x_{t-2}} + \underset{(0.055)}{0.065x_{t-3}} + \delta_t \tag{8a}$$

$$h_t = \underset{(2.883)}{21.793} + \underset{(0.343)}{0.143\delta_{t-1}^2} - \underset{(0.166)}{0.541\delta_{t-2}^2} + \underset{(0.343)}{0.365h_{t-1}} + \underset{(0.182)}{0.507h_{t-2}} \tag{8b}$$

On the other hand, during 1987 stock prices are lower on the year, while soybean prices rise. The price of the November 1987 soybean contract at time t is

$$x_t = \underset{(0.349)}{0.197} - \underset{(0.055)}{0.093x_{t-1}} - \underset{(0.055)}{0.086x_{t-5}} + \delta_t \tag{9a}$$

$$h_t = \underset{(6.867)}{39.964} - \underset{(0.050)}{0.738\delta_{t-1}^2} - \underset{(0.049)}{0.853\delta_{t-2}^2} - \underset{(0.080)}{0.703h_{t-1}} - \underset{(0.079)}{0.515h_{t-2}} \tag{9b}$$

The fact that both soybean models have GARCH(2, 2) processes while both S&P 500 series have GARCH(1, 1) processes may indicate a difference in time series properties of commodity and financial futures.

The two nearby futures contract price series produce models more similar to each other. The price of the nearby S&P 500 contract at time t is

$$x_t = \underset{(0.240)}{0.075} + \underset{(0.044)}{0.451x_{t-1}} - \underset{(0.045)}{0.220x_{t-2}} - \underset{(0.045)}{0.094x_{t-4}} + \underset{(0.044)}{0.095x_{t-5}} + \delta_t \tag{10a}$$

$$h_t = \underset{(18.136)}{28.011} + \underset{(0.181)}{0.701\delta_{t-1}^2} + \underset{(0.159)}{0.773h_{t-1}} \tag{10b}$$

Stock prices steadily rise during the period covered by the S&P 500 index futures contract data, despite some "corrections." The nearby soybean contract price data have a spiky pattern in which numerous periods of trending prices are evident. As a whole, the GARCH(1, 1) model for the data is

$$x_t = \underset{(0.415)}{0.831} - \underset{(0.048)}{0.154x_{t-2}} + \underset{(0.053)}{0.168x_{t-3}} + \underset{(0.054)}{0.191x_{t-4}} - \underset{(0.054)}{0.141x_{t-5}} + \delta_t \tag{11a}$$

$$h_t = \underset{(13.266)}{45.168} + \underset{(0.027)}{0.849\delta_{t-1}^2} + \underset{(0.013)}{1.107h_{t-1}} \tag{11b}$$

Correlation Dimension Results

Correlation dimension estimates made from the residuals, δ, of eqs. 6–11 are presented in Table I. Observations which can be made from the results in the table include:

1. Both the S&P index and soybeans have "low" CDs in absolute terms and in comparison to CDs reported in earlier cash stock market studies.
2. Both nearby series have lower CDs than the contract series.
3. All series pass both the "scrambled residuals" and random number tests: their CDs are lower than those of the scrambled and random data.

The general implication of these results is that both the S&P index and soybeans appear to have nonlinearities in their underlying generating processes. More detailed observations are made below before presenting the W statistic results from tests of the nonlinear hypothesis.

Table I presents results for the first ten embedding dimensions to enable the reader to see how the CDs stabilize. The CD estimates are rounded to one decimal point from the regression output for each embedding dimension. Despite this imprecision, it appears that the estimates converge after the fourth or fifth embedding dimension in the case of each data series. Convergence does not occur in most of the scrambled series, nor in the random series.

It appears that the CDs in Table I are lower for soybeans than for the S&P index and that both futures markets evaluated have CDs lower than those Scheinkman and LeBaron (1989) report for cash stock market returns, although no statistic is

Table I
CORRELATION DIMENSION ESTIMATES FOR S&P 500 AND SOYBEAN FUTURES PRICES

	Embedding Dimension									
	1	2	3	4	5	6	7	8	9	10
Contract Series										
S&P 12/86	1.0	1.5	2.1	2.2	2.4	2.5	2.3	2.7	2.4	2.3
S&P 12/86[a]	1.1	1.9	2.8	3.2	4.1	4.2	4.5	5.3	6.1	5.8
S&P 12/87	1.0	1.6	2.1	2.1	2.3	2.0	2.0	2.3	2.7	2.8
S&P 12/87[a]	1.0	2.0	2.9	3.6	4.3	4.4	4.4	5.1	4.8	4.9
Soybean 11/86	0.9	1.4	1.7	1.9	1.9	2.0	1.6	1.7	1.4	1.4
Soybean 11/86[a]	0.9	1.8	2.8	3.0	3.4	3.7	4.5	4.8	5.3	5.8
Soybean 11/87	0.9	1.3	1.6	1.9	2.0	2.2	1.9	2.1	2.2	2.1
Soybean 11/87[a]	0.9	2.0	2.7	4.0	4.4	4.8	5.1	5.5	5.6	6.2
Nearby Series										
S&P nearby	0.9	1.3	1.4	1.6	1.6	1.7	1.5	1.6	1.5	1.4
S&P nearby[a]	1.0	2.1	3.2	3.8	4.1	5.2	6.0	7.2	7.9	8.3
Soybean nearby	0.9	1.0	1.2	1.2	1.3	1.3	1.3	1.3	1.3	1.3
Soybean nearby[a]	0.9	2.2	3.4	3.9	4.4	5.8	6.4	7.1	8.0	9.2
Random Series[b]	0.9	2.1	3.6	4.5	5.1	6.3	7.8	8.9	9.6	10.3

[a]Results from a "scrambled" series created for comparison.
[b]Average results from 100 series of random numbers generated from various distributions for comparison.

available to aid in determining whether one CD estimate is significantly different than another. The difference between stock and soybean futures results is more apparent for the contract price series than for the nearby series. The general CDs for the November 1986 and 1987 soybean contracts, respectively, seem to be 1.7–2.0 and 1.9–2.2. The CDs for both S&P contracts are 2.3–2.7. The nearby soybean series has a CD of 1.3, while the S&P nearby data have a CD of 1.5–1.7. These values may not be significantly different from each other, but it is expected that they are significantly lower than the CDs of 6–7 which Scheinkman and LeBaron (1989) report.

The level and stability of the CD estimates in Table I give rise to the question of sample size effects. Ramsey and Yuan (1987) show that CDs may be underestimated from small data sets. In this case, estimates are lower and more stable for larger samples. For example, the nearby soybean series is the largest with 5823 observations and it has the lowest, most stable CD estimates across embedding dimensions. This indicates that no conclusions can be reached concerning the relative differences in CDs between the two markets noted above. Nevertheless, the fact that even the smallest data sets (the S&P contract series) generated CDs that are "low," compared to the CDs for the scrambled and random series, supports the hypothesis that nonlinearity exists in all six series. To resolve the question of significance, W statistics are calculated.

Table II presents W statistics for each of the six series, their respective scrambled counterparts, and the random series. For each price series the statistic exceeds the 1.96 value needed (at the 95% confidence level) to reject the null hypothesis of a linear process. For the soybean nearby series, in particular, the results are strong. This indicates that nonlinearities are present in the data. However, although the W statistic can distinguish between a linear and nonlinear generating process, it can-

Table II
***W* STATISTICS FOR S&P 500 AND SOYBEAN FUTURES PRICE SERIES**

		Embedding Dimension		
	Observations (n)	4	5	6
Contract Series				
S&P 12/86	250	4.21	5.81	5.95
S&P 12/86[a]	250	1.27	1.55	1.80
S&P 12/87	250	6.99	7.76	8.12
S&P 12/87[a]	250	3.03	3.62	4.15
Soybean 11/86	335	5.86	6.07	7.23
Soybean 11/86[a]	335	1.17	1.31	1.43
Soybean 11/87	337	4.66	5.40	6.92
Soybean 11/87[a]	337	1.05	1.11	1.31
Nearby Series				
S&P nearby	1420	6.61	7.22	8.10
S&P nearby[a]	1420	0.75	0.99	1.31
Soybean nearby	5823	12.32	13.00	14.26
Soybean nearby[a]	5823	1.27	1.37	1.52
Random Series[b]	250	0.92	1.06	1.15

[a]Results from a "scrambled" series created for comparison.
[b]Average results from 100 series of random numbers.

not detect whether that process is stochastic or deterministic. Therefore, additional evidence, such as LE estimates, is needed to make this distinction.

Lyapunov Exponent Results

Lyapunov exponent estimates for soybean and S&P 500 prices are presented, respectively, in Tables III and IV. In the absence of statistics for use in determining the significance level of these estimates, only qualitative observations can be made, using the random number results presented in Table V, including:

1. Both the S&P index and soybeans have positive, stable LEs across embedding dimensions, implying the presence of chaos.
2. LEs for the S&P nearby series appear to be of similar magnitude to those for S&P contract series and are higher than the random number LEs in many cases, while the LEs for the soybean nearby series are consistently lower than both the soybean contract series' LEs and the highest of the random number LEs.
3. For all series, LE values decline as evolution lengths increase, as expected.

The general conclusion is that these results support those of the correlation dimension analysis: both the S&P index and soybeans appear to have chaotic nonlinearities in their underlying generating processes. More detailed observations are made below.

Tables III and IV present estimates of the largest Lyapunov exponent calculated for different embedding dimensions using different evolution lengths. Dimensions 3, 5, and 9 are selected to illustrate results across the range that has stable CDs. For

Table III
LYAPUNOV EXPONENTS FOR SOYBEAN FUTURES PRICE SERIES

		Embedding Dimension		
Price Series	**Evolution**[a]	**3**	**5**	**9**
November 1986 Contract	5	0.0430	0.0613	0.0484
	10	0.0401	0.0525	0.0401
	21	0.0308	0.0337	0.0180
	42	0.0112[b]	0.0136	0.0172
	63	0.0109[b]	0.0129[b]	0.0067[b]
November 1987 Contract	5	0.0382[b]	0.0517	0.0410
	10	0.0285	0.0431	0.0402
	21	0.0240	0.0366	0.0180
	42	0.0233	0.0137	0.0126
	63	0.0263	0.0128[b]	0.0122
Nearby Contract	5	0.0002[b]	0.0051[b]	0.0067[b]
	10	0.0001[b]	0.0050[b]	0.0057[b]
	21	0.0000[b]	0.0031[b]	0.0045[b]
	42	−0.0011[b]	0.0039[b]	0.0037[b]
	63	−0.0012[b]	0.0042[b]	0.0035[b]

[a]Number of observations over which the two pieces of the trajectory are followed.
[b]Values fall between the average and highest estimates for the relevant entry in Table V.
Note: The minimum and maximum scale lengths used here are 1 and 45 cents per bushel, respectively.

Table IV
LYAPUNOV EXPONENTS FOR S&P 500 FUTURES PRICE SERIES

Price Series	Evolution[a]	Embedding Dimension 3	5	9
December 1986 Contract	5	0.0334	0.0316[b]	0.0339
	10	0.0316	0.0261[b]	0.0317
	21	0.0245	0.0232	0.0168
	42	0.0149	0.0195	0.0097[b]
	63	0.0081[b]	0.0058[b]	0.0183
December 1987 Contract	5	0.0960	0.0978	0.0695
	10	0.0623	0.0930	0.0550
	21	0.0708	0.0647	0.0216
	42	0.0012[b]	−0.0018[b]	0.0100[b]
	63	−0.0027[b]	0.0009[b]	0.0065[b]
Nearby Contract	5	0.0216[b]	0.0465	0.0423
	10	0.0211[b]	0.0336	0.0280
	21	0.0199[b]	0.0120[b]	0.0202
	42	0.0066[b]	0.0077[b]	0.0149
	63	0.0071[b]	0.0129[b]	0.0138

[a]Number of observations over which the two pieces of the trajectory are followed.
[b]Values fall between the average and highest estimates for the relevant entry in Table V.
Note: The minimum and maximum scale lengths used here are 1 and 25 index points, respectively.

Table V
LYAPUNOV EXPONENTS FOR 100 SERIES OF RANDOM NUMBERS

Price Series	Evolution[a]	Embedding Dimension 3	5	9
Average Estimates	5	−0.0123	0.0032	−0.0037
	10	−0.0191	0.0006	−0.0112
	21	−0.0265	−0.0069	−0.0131
	42	−0.0281	−0.0080	−0.0143
	63	−0.0254	−0.0101	−0.0162
Highest Estimates	5	0.0238	0.0456	0.0070
	10	0.0223	0.0286	0.0141
	21	0.0185	0.0160	0.0117
	42	0.0140	0.0108	0.0105
	63	0.0152	0.0182	0.0102

[a]Number of observations over which the two pieces of the trajectory are followed.

each of those dimensions, LEs are calculated for evolution lengths equaling one week (five observations), two weeks (ten observations), one month (21 observations), two months (42 observations), and three months (63 observations).

This cross section of LEs is estimated in a "bootstrapping" effort to add robustness to any conclusions reached concerning the presence of chaos and to provide for a preliminary test of the hypothesis that futures prices are detectably deterministic over a long enough time period to make (short-term) price forecasting models possible. As noted by Frank and Stengos (1989, p. 555), "A chaotic system will be quite predictable over very short time horizons. If however the initial conditions are only known with finite precision, then over long intervals the ability to predict the time path will be lost." This means that if chaos can be detected at an evolution interval, the deterministic system generating prices has not been overwhelmed by stochastic noise and the development of forecasting models may be possible. If the largest LE is never positive in experiments across evolution lengths, the system is stochastic. However, if LEs are positive for short evolutions and turn negative at some longer length, the implication is that at that time interval stochastic noise becomes dominant and deterministic forecasting models are not empirically viable, even though the underlying system may be completely deterministic in nature. In summary, experimenting with evolution lengths provides a way in which analysts can assess the potential for developing forecasting models.

Table V provides results for the random numbers for the same cross section of LE experiments. These results illustrate the small sample distribution of the LE statistic and provide a basis for assessing the effects of noise on the estimates. The "Average Estimates" reported in Table V are the mean estimates of the LE from the 100 series of random numbers. The fact that all but two of the values reported are negative indicates that Wolf et al.'s (1985) procedure for calculating LEs does a good job of detecting stochastic processes in general. However, in any particular case there is some chance of mislabeling a stochastic process as being deterministic due to the effects of noise in the small sample (of 250 observations). This is evident in Table V as the "Highest Estimates" are positive for each experiment. In other words, at least one of the 100 random series produces a positive LE for each combination of evolution length and embedding dimension. Therefore, actual data which produce LE estimates falling between the average and highest value reported for the relevant experiment in Table V may or may not have a deterministic generating process.

In Table III all but two LEs are positive, strongly supporting the conclusion that soybean futures prices have a chaotic nonlinear generating process. Also, that process may be predictable over periods as long as the three-month evolutions evaluated here. However, a closer look leads to some interesting issues for future research. For example, the LEs for the two contract price series are very similar across embedding dimensions and evolution lengths, but they differ from LEs for the nearby series. LEs for nearby prices are approximately zero at dimension 3 and, although they stabilize at higher dimensions, no actual estimate exceeds the relevant "Highest Estimate" from Table V. One hypothesis is that the two types of series will have different generating systems because contract series have less noise (possibly indicated by a lower coefficient of variation) and fewer trends, compared to more volatile nearby series which contain multiple trends.

One explanation for this hypothesis comes from expectations concerning the two types of data series. Futures contract price series reflect increasing amounts of information about the relatively few factors known to influence prices expected at one point in time (the contract maturity date). Over the life of a particular contract, trade volume tends to rise as information changes become easier to interpret, making the market more liquid and, hence, more efficient in its reaction to information flows. On the other hand, nearby futures contract prices reflect a liquid market's interpretation of information concerning many supply and demand factors for different points in time. Trading volume in whatever contract is "nearby" at the moment tends to remain high and be closely correlated with spot prices of the commodity [Blank et al. (1991)].

The results in Table III indicate that compared to contract series, nearby soybean futures prices have a more complex generating system requiring analysis at a higher embedding dimension before the effects of chaos (if any) can be captured. The differences in LEs between dimensions may also mean that more complex models will be required for forecasting, making successful specification of those models less likely.

Table IV presents LE estimates for S&P 500 futures prices which, in general, are surprisingly similar to those for soybeans, but which have two subtle differences raising additional issues for future research. First, the LEs for the December 1986 contract and the nearby series are similar to the soybean contracts' LEs. One hypothesis for this phenomenon is that the S&P nearby series is like many contract series in that the data have a single trend around which prices demonstrate relatively low levels of noise (at least at higher embedding dimensions). If multiple trends are present, the S&P nearby series might produce a more complex set of results, as in the case of soybeans.

Second, the LE results for the December 1987 S&P contract are unique compared to the other three contract series studied, drawing attention to the effects of the sharp market correction in October 1987. The LE values for all embedding dimensions are noticeably higher than those of any other series at evolution lengths of 5–21 days, yet for evolutions of 42 and 63 days the LEs are low and occasionally negative. It is hypothesized that the strong uptrend in S&P prices prior to the "crash" had so little noise in it that forecasting was quite possible for periods of up to one month (21 trading days), but longer forecasts necessitate more complex models to account for the effects of "Black Monday," which are interpreted as noise by simple models. This illustrates the frustration which will no doubt continue to face modelers: even if a system such as that underlying stock prices is deterministic, forecasting models cannot be expected to detect every trend or turning point. As long as initial conditions are not perfectly identified, analysts will not know whether a model is correctly specified until it fails. In other words, no matter how complex a deterministic model is, additional complexity may be required to avoid forecasting errors so large as to question the modeling effort's value.

SUMMARY AND CONCLUDING COMMENTS

This study provides results of nonlinear dynamical analysis of two commodity futures markets. It illustrates what type of methodological and empirical procedures must be used to evaluate these markets. It also raises and attempts to address issues concerning identification and measurement of nonlinear generating systems in economic data, providing a guide for research in the future. Although chaos analyses in economics are still very rare, the results presented here and elsewhere give reason for expecting use of these methods to expand, but with difficulty.

Results from analyses of a financial, the S&P 500 index, and an agricultural product, soybeans, are presented as examples. All empirical results in this study are consistent with those of markets with underlying generating systems characterized by deterministic chaos. Comparing results for the stock index and soybean futures reveals surprising similarity, yet the CDs for the index are noticeably different than those reported in earlier stock market studies which use cash price data. Both futures markets are shown to have a "low" correlation dimension of about two, while CDs are slightly lower for soybeans and for nearby series compared to contract series. The statistical significance of this difference in CDs between product types is unclear, but statistical tests do indicate the presence of nonlinearities in both markets. Estimates of Lyapunov exponents indicate that these nonlinearities are (at least partially) deterministic, rather than stochastic in nature. However, the absence of any tests for the statistical significance of estimated LEs makes the results of this study necessary, but not sufficient conditions to prove the existence of deterministic chaos.

Although providing just an introduction to the study of nonlinear dynamics in future markets, the results of this analysis may be useful for commodity market analysts in industry, academia, and government. The fact that futures prices appear to have a nonlinear generating process of a type not recognized previously raises the possibility that short-term forecasting models may be improved by incorporating these new factors. This, in turn, has significant implications for resource allocation and marketing strategies for firms trading in these product markets. Ultimately, the discovery of a nonlinear, nonrandom process in commodity futures markets could raise the level of debate in the economic literature concerning "random walk" hypotheses and definitions of pricing efficiency. However, from a practical viewpoint, chaos analysis procedures provide another test for (deterministic) nonlinearity, but do not easily lend themselves to direct applications in forecasting model construction.

Bibliography

Abraham, N., Gollub, J., and Swinney, H. (1984): "Testing Nonlinear Dynamics," *Physica,* 11D:252–264.

Aradhyula, S., and Holt, M. (1988): "GARCH Time-Series Models: An Application to Retail Livestock Prices," *Western Journal of Agricultural Economics,* 13:365–374.

Ashley, R., and Patterson, D. (1989): "Linear Versus Nonlinear Macroeconomies: A Statistical Test," *International Economic Review,* 30:685–704.

Baillie, R., and DeGennaro, R. (1990): "Stock Returns and Volatility," *Journal of Financial and Quantitative Analysis,* 25:203–214.

Barnett, W., and Chen, P. (1988): "The Aggregation-Theoretic Monetary Aggregates Are Chaotic and Have Strange Attractors: An Econometric Application of Mathematical Chaos," in *Dynamical Econometric Modeling,* Barnett, Berndt, and White (eds.). Cambridge: Cambridge University Press, pp. 199–246.

Barnett, W., and Hinich, M., (1991): "Has Chaos Been Discovered with Economic Data?" in *Evolutionary Dynamics and Nonlinear Economics*, Chen and Day (eds.), Oxford University Press.

Baumol, W., and Benhabib, J. (1989): "Chaos: Significance, Mechanism, and Economic Applications," *The Journal of Economic Perspectives,* 3:77–105.

Blank, S., Carter, C., and Schmiesing, B. (1991): *Futures and Options Markets: Trading Commodities and Financials.* Englewood Cliffs, NJ: Prentice Hall, Chapter 3.

Bollerslev, T. (1986): "Generalized Autoregressive Conditional Heteroscedasticity," *Journal of Econometrics* 31:307–327.

Booth, G., Kaen, F., and Koveos, P. (1982a): "R/S Analysis of Foreign Exchange Rates under Two International Monetary Regimes," *Journal of Monetary Economics,* 10:407–415.

Booth, G., Kaen, F., and Koveos, P. (1982b): "Persistent Dependence in Gold Prices," *Journal of Financial Research,* 85–93.

Brock. W. A. (1986): "Distinguishing Random and Deterministic Systems: Abridged Version," *Journal of Economic Theory,* 40: 168–195.

Brock, W. A. (1988a): "Introduction to Chaos and Other Aspects of Nonlinearity," in *Differential Equations, Stability, and Chaos in Dynamic Economics,* Brock, W., and Malliaris, A., New York: North Holland.

Brock, W. A. (1988b): "Nonlinearity and Complex Dynamics in Economics and Finance," in *The Economy as an Evolving Complex System,* Addison-Wesley Publishing Company: New York.

Brock, W., and Dechert, W. (1991): "Theorems on Distinguishing Deterministic from Random Systems," in *Dynamic Econometric Modeling,* Barnett, Berndt, and White (eds.). Cambridge: Cambridge University Press.

Brock, W., and Sayers, C. (1988): "Is the Business Cycle Characterized by Deterministic Chaos?" *Journal of Monetary Economics,* 22:71–90.

Candela, G., and Gardini, A. (1986): "Estimation of a Non-Linear Discrete-Time Macro Model," *Journal of Economic Dynamics & Control,* 10:249–255.

Cheung, Y., and Ng, L. (in press): "The Dynamics of S&P 500 Index and S&P 500 Futures Intraday Price Volatilities," *The Review of Futures Markets.*

Cornew, R., Town, D., and Crowson, L. (1984): "Stable Distributions, Futures Prices, and the Measurement of Trading Performance," *The Journal of Futures Markets,* 4:531–557.

Day, R. H. (1983): "The Emergence of Chaos from Classical Economic Growth," *The Quarterly Journal of Economics,* 98:201–213.

Deneckere, R., and Pelikan, S. (1986): "Competitive Chaos," *Journal of Economic Theory,* 40:13–25.

Feigenbaum, M. J. (1983): "Universal Behavior in Nonlinear Systems," *Physica,* 7D:16–39.

Frank, M., and Stengos, T. (1989): "Measuring the Strangeness of Gold and Silver Rates of Return," *Review of Economic Studies,* 56:553–567.

Garcia, P., Hudson, M., and Waller, M. (1988): "The Pricing Efficiency of Agricultural Futures Markets: An Analysis of Previous Research Results," *Southern Journal of Agricultural Economics* 20:119–130.

Goodwin, R. (1990): *Chaotic Economic Dynamics.* Oxford: Oxford University Press.

Grassberger, P., and Procaccia, I. (1983): "Measuring the Strangeness of Strange Attractors," *Physica,* 9D:189–208.

Greene, M., and Fielitz, B. (1977): "Long Term Dependence in Common Stock Returns," *Journal of Financial Economics,* 4:339–349.

Helms, B., and Martell, T. (1985): "An Examination of the Distribution of Futures Price Changes," *The Journal of Futures Markets,* 5:259–272.

Helms, B., Kaen, F., and Rosenman, R. (1984): "Memory in Commodity Futures Contracts," *The Journal of Futures Markets,* 4:559–567.

Hinich, M., and Patterson, D. (1985): "Evidence of Nonlinearity in Daily Stock Returns," *Journal of Business and Economic Statistics,* 3:69–77.

Hsieh, D. (1989): "Testing for Nonlinear Dependence in Daily Foreign Exchange Rate Changes," *Journal of Business,* 62:339–368.

Jensen, R.V. (1987): "Classical Chaos," *American Scientist,* 75:168–181.

Just, R., and Rausser, G. (1981): "Commodity Price Forecasting with Large Scale Econometric Models and the Futures Market," *American Journal of Agricultural Economics,* 63:197–208.

Kenyon, D., Kling, K., Jordan, J., Seale, W., and McCabe, N. (1987): "Factors Affecting Agricultural Futures Price Variance," *The Journal of Futures Markets,* 7:73–91.

Lamoureux, C., and Lastrapes, W. (1990): "Persistence in Variance, Structural Change, and the GARCH Model," *Journal of Business and Economic Statistics,* 8:225–234.

Lorenz, H.-W. (1987): "Strange Attractors in a Multisector Business Cycle Model," *Journal of Economic Behavior & Organization,* 8:397–411.

Mandelbrot, B. (1977): *The Fractal Geometry of Nature.* New York: Freeman.

Melese, F., and Transue, W. (1986): "Unscambling Chaos through Thick and Thin," *The Quarterly Journal of Economics,* 101:419–423.

Mirowski, P. (1990): "From Mandelbrot to Chaos in Economic Theory," *Southern Economic Journal,* 57:289–307.

Priestly, M. B. (1988): *Non-Linear and Non-Stationary Time Series Analysis,* San Diego: Academic Press.

Ramsey, J., and Yuan, H. (1987): "The Statistical Properties of Dimension Calculations Using Small Data Sets." New York: C. V. Starr Center for Applied Economics, New York University.

Roll, R. (1984): "Orange Juice and Weather," *American Economic Review,* 74:861–880.

Savit, R. (1988): "When Random is Not Random: An Introduction to Chaos in Market Prices," *The Journal of Futures Markets,* 8:271–289.

Savit, R. (1989): "Nonlinearities and Chaotic Effects in Options Prices," *The Journal of Futures Markets,* 9:507–518.

Scheinkman, J., and LeBaron, B. (1989): "Nonlinear Dynamics and Stock Returns," *Journal of Business* 62:311–337.

Shumway, R. H. (1988): *Applied Statistical Time Series Analysis.* Englewood Cliffs, NJ: Prentice-Hall.

Stevenson, R., and Bear, R. (1970): "Commodity Futures: Trends or Random Walks?" *Journal of Finance,* 25:65–81.

Swinney, H. L. (1983): "Observations of Order and Chaos in Nonlinear Systems," *Physica,* 7D:3–15.

Takens, F. (1985): "On the Numerical Determination of the Dimension of an Attractor," in *Dynamical Systems and Bifurcations,* Braaksma, N., Broer, H., and Takens, F. (eds.). Berlin: Springer-Verlag, pp. 99–106.

Talpaz, H. (1974): "Multi-Frequency Cobweb Model: Decomposition of the Hog Cycle," *American Journal of Agricultural Economics* 56:38–49.

van der Ploeg, F. (1986): "Rational Expectations, Risk and Chaos in Financial Markets," *Economic Journal,* 96:151–161.

Wolf, A. (1986): "Quantifying Chaos with Lyapunov Exponents," in *Nonlinear Science: Theory and Applications,* Holden, A. (ed.) Manchester: Manchester University Press.

Wolf, A., and Swift, J. (1984): "Progress in Computing Lyapunov Exponents from Experimental Data," in *Statistical Physics and Chaos in Fusion Plasmas,* Horton, C., and Reichl, L. (eds.). New York: John Wiley & Sons.

Wolf, A., Swift, J., Swinney, H., and Vastano, J. (1985): "Determining Lyapunov Exponents from a Time Series," *Physica,* 16D:285–317.