Multiperiod Hedging Using Futures: A Risk Minimization Approach in the Presence of Autocorrelation

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The single period model, because of its simplicity and often plausible representation, is used to describe a wide range of economic and financial decision making. Those who advocate such an approach recognize that the world is truly multiperiod and approximately infinite, but find that a single period model is adequate under certain conditions. Either decisions in one period have little or no effect on future periods or the interperiod dependencies are too complex to be captured in the modeling process. These two reasons are the ones most often cited, explicitly or implicitly, as justification for using a single period model.

Futures hedging is no exception. Most models proposed to explain hedging are all single period[1] [see, for example, Ederington (1979); Anderson and Danthine (1980); Howard and D'Antonio (1984, 1986); Morgan et al. (1988); Black (1989); Hilliard and Jordon (1989)]. In addition, most empirical studies use a single period model even if the tests are conducted in a multiperiod setting. Some recent empirical hedging studies are those by Leatham (1988), Black (1989), Gagnon et al. (1989), Gardner (1989), Lein (1989), McCabe and Solberg (1989), and Thomas (1989).

However, hedgers frequently face a multiperiod situation in which a single period approach simply is not adequate. For example, consider an issuer of commercial paper who is paying off old and offering new commercial paper on a weekly basis. If the issuer finds it attractive to hedge interest costs on next week's issue, then all future weekly issues should be hedged, up to and including the final issue needed to fund the assets supported by the issue . This is because short term interest rates tend to follow a random walk and so higher (lower) interest cost next week will lead to higher (lower) interest costs in all the following weeks. Consequently, the hedge position taken today should take into consideration next week's issue as well as all future weekly issues.

Not all series being hedged are autocorrelated. Take, for example, the management of a stock portfolio with negligible cash inflows and outflows and no reinvestment. The series being hedged is the return on the stock portfolio which portrays little or no autocorrelation. Thus, a single period approach is appropriate. Of course, this does not rule out the possibility that the hedger will continuously hold

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[1]See Baesel and Grant (1982) for an example of a multiperiod futures trading model.

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a futures position as would a multiperiod hedger. The difference is that the futures position held is exclusively for hedging this period's spot position as compared to the multiperiod hedger who uses only a portion of the current futures position for hedging this period's spot holding, the remainder being held as a hedge against a later period's spot holdings.[2]

The autocorrelation of the spot series being hedged is a critical element in the multiperiod model that follows.[3] In fact, spot series autocorrelation is used as the measure of interperiod dependency in the price series being hedged. Thus, if autocorrelation is zero, a single period hedging approach is appropriate, whereas if autocorrelation is other than zero, a multiperiod hedging approach is appropriate.

The primary focus of the model is return uncertainty. It is assumed that the quantity being hedged in future periods grows at a known rate g.[4] This simplifying assumption allows for the summation of hedge ratios over different time periods, which produces the total current futures position expressed as a multiple of the current spot position.

The focus on spot return rather than on price may seem inappropriate. However, such a representation captures a wide range of hedging activities. Hedgers usually face price uncertainty. This situation can be translated into the return framework by modeling the percentage change in the price rather than the price itself. On the other hand, in a number of hedging situations, it is quite natural to think in terms of returns. For example, when hedging financial instruments and stock portfolios, returns are the normal focus. Finally, there are hedging situations in which the cost rather than the return is the series of interest. The commercial paper example cited earlier is such a situation. In this case, the interest cost can simply be thought of as representing a negative return to the issuer. Consequently, the focus here is on returns as an effective way to represent a wide range of hedging situations.

FRAMEWORK FOR THE MODEL

Assume a hedger who is facing a t period planning horizon. Hedging decisions are made at the beginning of each of the t periods and each period's spot and futures return uncertainty is resolved at the end of the period. The hedger can choose between the spot commodity and a zero margin futures contract. There is a risk-free asset which pays the same rate i in all periods. The standard deviation of return is the appropriate measure of risk.

Central to the derivation of the model is the specification of the spot return stochastic process, which is assumed to be first order autoregressive:[5]

$$\tilde{r}_{s,t} = \bar{r}_{s,t} + \phi^{t-1}\tilde{u}_1 + \phi^{t-2}\tilde{u}_2 + \ldots + \tilde{u}_t \tag{1}$$

[2]There are other tools besides futures for hedging multiperiod risks. One of the most successful and fastest growing are currency and interest swaps. The choice of whether or not to use the futures, swap, or some other market for multiperiod hedging depends upon the comparative hedging benefits and costs of each market.

[3]See Herbst et al. (1989) for an empirical paper which addresses the issue of autocorrelation and hedging.

[4]See Rolfo (1980) for hedging models in which output is uncertain.

[5]There is a wide range of time series models from which to choose. An AR(1) model or Markov process is selected for its simplicity and because it reasonably represents a wide range of actual return time series, in particular, those most likely to be hedged. For example, interest rates and most commodities can be adequately represented with an AR(1) model. However, the usual representation of an AR(1) model is $\tilde{r}_{s,t} = \mu + \phi\tilde{r}_{s,t-1} + \tilde{u}_t$. Equation (1) is obtained through backward recursion using this standard form. For example, $\tilde{r}_{s,t} = \mu + \phi(\mu + \phi\tilde{r}_{s,t-2} + \tilde{u}_{t-1}) + \tilde{u}_t$; $\tilde{r}_{s,t} = \mu + \phi\mu + \phi^2\tilde{r}_{s,t-2} + \phi\tilde{u}_{t-1} + \tilde{u}_t$. This backward recursion is repeated $t - 1$ times to obtain eq. (1).

where

$\tilde{r}_{s,t}$ = the spot return t periods ahead,
$\bar{r}_{s,t}$ = the expected spot return as of time t,
$= \phi^t r_{s,0} + \phi^{t-1}\mu + \phi^{t-2}\mu + \ldots + \mu,$
$r_{s,0}$ = actual spot return in period 0,
μ = the single period expected spot return (or drift),
ϕ = the autoregressive coefficient between zero and 1, and
$\tilde{u}_t$ = the random disturbance term in period t.

It is further assumed that $\tilde{u}_t$ is i.i.d. and $N(0, \sigma_u)$ for all periods.

To get a better feel for eq. (1), consider the extreme values for ϕ. First assume that $\phi = 0$, which means eq. (1) can be simplified to $\tilde{r}_{s,t} = \mu + \tilde{u}_t$. The actual spot return observed in period t is simply the expected return μ plus a random disturbance. Put another way, the return stochastic process can be characterized as repeated, independent draws from a normal distribution with mean μ and standard deviation σ_u. This is similar to the situation faced by the stock portfolio manager described earlier. Note that the return for period t does not depend upon disturbances from previous periods; and, as a result, there is no autocorrelation in the return series. Now, assume that $\phi = 1$. The series becomes

$$\tilde{r}_{s,t} = r_{s,0} + t\mu + \tilde{u}_1 + \tilde{u}_2 + \ldots. + \tilde{u}_t$$

which is the well-known random walk. The commercial paper hedger described earlier faces a situation similar to this one (more than likely $\mu = 0$ in this case). Autocorrelations are now high since period t's return depends heavily on period t-1's return. Finally, when ϕ lies between 0 and 1, the hedger faces a situation between these two extremes.

The coefficient ϕ also indicates whether the hedger faces a single period or a multiperiod problem. The closer ϕ is to zero, the more realistic it is to use a single period approach to hedging. The closer ϕ is to 1, the more important it becomes for the hedger to operate in a multiperiod context. Consequently, the coefficient ϕ plays a central role in the multiperiod hedging model to be derived.

Three actual hedging situations within the framework of this study's model are presented here to clarify the above. First, consider a flour miller who plans to make 12 monthly wheat purchases over the coming year. In terms of the model, t is equal to 12; μ is equal to the expected periodic change in the price of wheat; ϕ is close to 1, given the autocorrelated nature of wheat prices; and the disturbance term $\tilde{u}$ is the periodic change in the price of wheat. If, for example, the initial price of wheat is \$2.50, $\mu = 0.005$, $r_{s,0} = 0$ (by definition, not by actual result), and $\phi = 1$ then the price of wheat at the end of period 1 if $u_1 = -0.05$ is equal to \$2.50 $(1 + \tilde{r}_{s,1})$ = \$2.50[1 + (0 + 0.005 − 0.05)] = \$2.39.

Second, consider the periodic issuer of commercial paper described earlier. In terms of the model, $\bar{r}_{s,t}$ is equal to the previous period's commercial paper rate ($r_{s,0}$), since μ is likely equal to 0, ϕ is again close to 1, and the disturbance term is the periodic change in the commercial paper rate.

Finally, consider the stock portfolio manager mentioned earlier who is experiencing insignificant inflows and outflows and is not reinvesting funds. In this situation, $\bar{r}_{s,t}$ is equal to the expected periodic return on the stock portfolio (μ), ϕ is close to zero given the lack of autocorrelations in stock returns, and $\tilde{u}$ is equal to the actual return on the portfolio net of μ.

It should be noted that $\tilde{u}_t$ is not in general equal to the change in the spot return. This can be seen by calculating the spot return change, $\Delta\tilde{r}_{s,t}$, using eq. (1). It is

$$\Delta\tilde{r}_{s,t} = \tilde{r}_{s,t} - \tilde{r}_{s,t-1}$$

$$\Delta\tilde{r}_{s,t} = \phi^{t-1}r_{s,0}(\phi - 1) + \phi^{t-1}\mu + \tilde{u}_t + (\phi - 1)\tilde{u}_{t-1} + \phi(\phi - 1)\tilde{u}_{t-2} + \ldots + \phi^{t-2}(\phi - 1)\tilde{u}_1 \tag{2}$$

If $\phi = 1$, (i.e., random walk) then $\Delta\tilde{r}_{s,t} = \mu + \tilde{u}_t$ and the change in the spot return is equal to the disturbance term plus μ. However, if $\phi = 0$, $\Delta\tilde{r}_{s,t} = \tilde{u}_t - \tilde{u}_{t-1}$ and if $0 < \phi < 1$, the change in the spot return depends upon all past disturbance terms. Another way to view $\tilde{u}_t$ is that it is the "surprise" remaining in the spot series after all autoregressive structure and "drift" have been removed.

Turning to the futures market, the percent change in the futures price, $\tilde{r}_{f,t}$ (this is not technically a return since no futures margin is required) is assumed to be i.i.d. and $N(0,\sigma_f)$ for all t. That is, there are no autocorrelations in the time series $\tilde{r}_{f,t}$ and the situation in which the expected futures return is other than zero is being ruled out.[6] The contemporaneous covariance between $\tilde{u}_t$ and $\tilde{r}_{f,t}$ is equal to σ_{uf} for all periods and all noncontemporaneous covariances are assumed equal to zero. Thus for ease of exposition, some of the operational realities of futures markets (i.e., early delivery dates, multiple delivery months, etc.)[7] are ignored, and the existence of a single, long-lived futures contract with a delivery date beyond t is assumed.

SOLVING FOR THE OPTIMAL HEDGE RATIO

Hedging decisions (b_j) are made at the beginning of each of the t periods and both the spot and futures return are realized at the end of each period as follows:

	$\tilde{r}_{s,1}$	$\tilde{r}_{s,2}$	$\tilde{r}_{s,3}$	$\tilde{r}_{s,t-1}$	$\tilde{r}_{s,n}$
	$\tilde{r}_{f,1}$	$\tilde{r}_{f,2}$	$\tilde{r}_{f,3}$	$\ldots$ $\tilde{r}_{f,t-1}$	$\tilde{r}_{f,t}$
0	1	2	3	$\ldots$ $t - 1$	t
b_t	b_{t-1}	b_{t-2}	b_{t-3}	b_1	

Note that the subscript for the hedge ratio b denotes the number of periods remaining and not the periods in which the hedge is taken. To obtain an additive multiperiod return equation, it is assumed that the cash generated from both the spot and futures markets at the end of each period is reinvested at the riskfree rate i and held until the end of period t, and the quantity being hedged grows at a known rate g which is set equal to the riskfree rate i.[8] Based on these assumptions, the total return at the end of period t, $\tilde{r}_t$, is given by

$$\tilde{r}_t = S_1(\tilde{r}_{s,1} + b_t\tilde{r}_{f,1})(1 + i)^{t-1} + S_1(1 + g)(\tilde{r}_{s,2} + b^*_{t-1}\tilde{r}_{f,2})(1 + i)^{t-2} + \ldots + S_1(1 + g)^{t-1}(\tilde{r}_{s,t} + b^*_1\tilde{r}_{f,t}) \tag{3}$$

where

S_1 = quantity being hedged in period 1, and
b_t = hedge ratio when t periods remain.

[6]With the assumption that the expected return on the futures market is zero, a risk-return optimizer becomes a risk minimizer, because the spot position is assumed fixed and the margin requirement for futures is zero. If this assumption is relaxed, the problem must be solved using dynamic programming and the future period hedge ratios themselves become random variables. The solution to this more complex problem is not as straightforward as the one presented in this paper. Recent empirical work by Maddala and Yoo (1990) does not support the existence of hedging costs.

[7]See Castelino (1990) for a discussion of the time to delivery impact on hedge ratios.

The hedge ratio b_t dictates the period 1 futures position. However, only a portion of the first period's futures position is intended to hedge the spot position held during period 1. In addition, the hedger must select the value of the hedge ratio b_t in light of the fact that an optimal hedge position is to be taken in each future period. (* indicates an optimal hedge ratio) and future optimal hedge ratios may be random variables themselves since they may depend upon the random variables $\tilde{r}_s$ and $\tilde{r}_f$. Thus we conclude that

$$b_t^* = f(b^*_{t-1}, ,b^*_{t-2}, \dots, b^*_1, \text{other parameters}) \tag{4}$$

To determine a closed form expression for b^*_t one must first determine the expression for b^*_1 through b^*_{t-1}.

Solving for b^*_1

Since the expected return on the futures contract is assumed to be zero and a zero margin requirement on the futures contract is assumed, there is no "cost" to taking a futures position. The hedger can minimize total portfolio risk while ignoring portfolio return because it will be the same regardless of the futures position taken. The single period variance of portfolio return ($\sigma_1{}^2$) is given by[9]

$$\sigma_1^2 = \sigma_u^2 + b_1^2\sigma_f^2 + 2b_1\sigma_{uf} \tag{5}$$

Taking the derivative of eq. (5) with respect to b_1, setting the derivative equal to zero, and solving for the optimal (i.e., risk minimizing) hedge ratio, one obtains:

$$b_1^* = -\frac{\sigma_{uf}}{\sigma_f^2} \tag{6}$$

which is Ederington's (1979) well-known equation. Note that b_1^* is not a function of either return series, and in fact is not a random variable itself.

Solving for b_2^*

The two period portfolio variance is given by the equation (see the Appendix, Part B for a detailed derivation):

$$\sigma_2^2 = (2 + 2\phi + \phi^2)\sigma_u^2 + (b_2^2 + b^{*2}_1)\sigma_f^2 + 2(b_2(1 + \phi) + b_1^*)\sigma_{uf} \tag{7}$$

Taking the derivative of eq. (7) with respect to b_2, setting it equal to zero, recognizing that b_1^* is not a function of b_2, and solving for b_2, the two period optimal (i.e., risk minimizing) hedge ratio is obtained.

$$b_2^* = -(1 + \phi)\frac{\sigma_{uf}}{\sigma_f^2} \text{ or if } \phi < 1,$$

$$b_2^* = -\frac{(1 - \phi^2)\sigma_{uf}}{(1 - \phi)\sigma_f^2} \tag{8}$$

[8]The assumptions concerning reinvestment at the riskfree rate and certainty with regard to quantity being hedged are essential in obtaining a tractable formulation of the multiperiod hedging problem. The assumption that $i = g$ is relaxed in Part D of the Appendix, where the optimal hedge ratios are derived assuming $i \neq g$.

[9]Recall that it is assumed that $i = g$. As a result i, g, and S_1 have been dropped from eq. (5) and subsequent variance equations because the terms involving i, g, and S_1 cancel out as a multiplicative constant. See the Appendix, Part A for a proof of this statement.

Again, note that b_2^* is not a function of either random variable $\tilde{r}_s$ or $\tilde{r}_f$, and so is not itself a random variable.

Solving for b_t^*

Following the same procedure used above, the t period variance is (see Appendix, Part C for a detailed derivation)

$$\sigma_t^2 = C_t\sigma_u^2 + (b_t^2 + b^{*2}_{t-1} + \ldots + b^{*2}_{1})\sigma_f^2 + 2[b_t(1 - \phi^t)/(1 - \phi) + b^*_{t-1}(1 - \phi^{t-1})/(1 - \phi) + \ldots + b_1^*]\sigma_{uf} \tag{9}$$

where

$$C_t = \sum_{j=1}^{t}\left[\frac{1 - \phi^j}{1 - \phi}\right]^2 \tag{10}$$

and $\phi < 1$. Taking the derivative and solving for b_t: we obtain

$$b_t^* = -\frac{(1 - \phi^t)\sigma_{uf}}{(1 - \phi)\sigma_f^2} \tag{11}$$

This is the optimal (risk minimizing) hedge ratio for a t period horizon. If t is very large, then the optimal hedge ratio approaches

$$b^* = -\frac{\sigma_{uf}}{(1 - \phi)\sigma_f^2} \tag{12}$$

where the t subscript is dropped because the hedge ratio no longer depends on t.

ANALYSIS OF THE OPTIMAL HEDGE RATIO

Equation (11) reveals that, among other things, the optimal hedge ratio depends upon the number of periods to be hedged. The more periods to be hedged, the larger the hedge ratio as long as $\phi > 0$. This increase in the hedge ratio is apparent in Figure 1, which shows the relationship between b_t^* and t assuming that

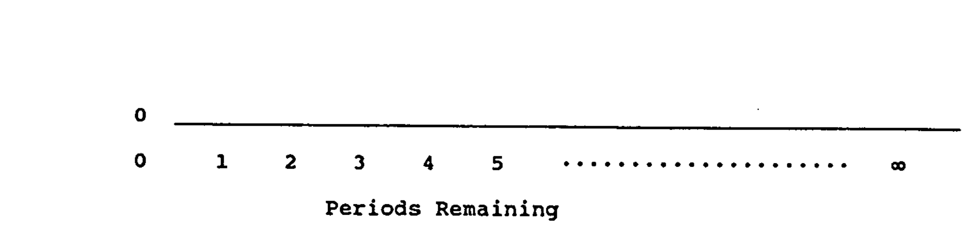

Figure 1
Optimal hedge ratio as a function of the number of periods remaining (assumes $b_1^* = -1.00$, $\phi = 0.5$).

$b_1^* = -1.00$ and $\phi = 0.5$. If $t = 2$, $b_2^* = -1.50$, which represents an increase in the short futures position of 50% when compared to the single period case. For $t = 3$, $b_3^* = -1.75$, and if $t = \infty$, $b^* = -2.00$, which is an increase of 100% in the short futures position.

Looking at the hedge ratio differently, if the hedger wishes to hedge until a fixed time in the future, the hedge ratio grows smaller in absolute terms as the terminal period grows closer. Using the previous example, a hedger with a three period horizon would use a hedge ratio of -1.75; with two periods remaining, a hedge ratio of -1.50; and with one period remaining, a hedge ratio of -1.0.[10] Note this time pattern of hedge ratios is similar to a technique known as "strip"hedging. In strip hedging a substantial short futures position is taken and then a fixed portion of the futures position is closed out each period. Eventually the entire position is closed out and the process is repeated. An important difference between the "strip" hedge used in practice and this theoretical "strip" hedge is that the amount of futures "stripped" away each period in the theoretical case is not constant; in fact, the short futures position may actually increase rather than decrease. (See footnote 10.)

It is implicitly assumed that there is a single futures contract. However, for any one commodity, there are generally a number of delivery months available. If this is the case, then the problem becomes one of choosing an optimal portfolio of futures rather than a single futures contract. This issue is not addressed in this model since a portfolio of futures, due to generally high correlations among different delivery months, is little different from a single futures contract.[11]

Of greater practical concern is the contention that a transaction anticipated in six months should be hedged using a futures contract with a six-month delivery date. If, however, the spot series is represented by the stochastic process shown in Eq. (1), and if the covariance term σ_{uf} is constant in all periods, then a futures position involving a rollover into the closest delivery contract does just as well as taking a large number of futures positions, each with a delivery date closely matched to each spot date. For the purposes of this model, then, it is assumed that eq. (1) holds and, therefore, there is no need to match the spot date with an identical futures delivery date.

It may seem that if the hedger is facing an infinite horizon, an infinite futures position should be taken. However, an examination of eq. (12) reveals that an infinite position should be taken only when $\phi = 1$. The futures position taken at time zero is intended to hedge the random disturbance $\tilde{u}$ in the current spot return as well as this same disturbance as it enters into future spot returns through the autoregressive process described by eq. (1). In each subsequent period, the disturbance term is multiplied by ϕ taken to a higher power, so the influence of the disturbance term diminishes and, consequently, so does the ability to hedge distant spot returns. Only when $\phi = 1$ does the current spot disturbance term enter fully into all future spot returns, leading to an infinite futures position.

[10]The actual short futures position taken is affected also by the growth rate (g) in the spot position. As the spot position grows over time, the short futures position also increases. Thus for this example, the actual futures position does not decrease in relative terms as much as does the hedge ratio; in fact, the short futures position might actually increase as the number of periods decreases. This can be seen by examining the equation for the optimal futures position (F_t^*): $F_t^* = b_t^* S_1(1 + g)^{t-1}$. Using eq. (11) along with the equation above, it can be shown that F^*_{t-1} will exceed F_t^* if $(1 + g)(1 - \phi^{t-1}) > (1 - \phi^t)$. Otherwise, F^*_{t-1} either declines or remains the same when compared to F_t^*.

[11]See Hilliard and Jordon (1989) for a simple solution to the futures portfolio problem.

Table I shows the composition of an infinite period futures position for various values of ϕ, again assuming that $b_1^* = -1$. If $\phi = 0$, the hedger does not face a multiperiod problem, uses a hedge ratio of -1, and the entire futures position is held to hedge this period's spot position. If ϕ increases to 0.25, the hedger facing an infinite horizon establishes a hedge ratio of -1.33, in which case 75% of the futures position is hedging this period's spot position, 19% next period's, 5% period three, and 1% period 4. If ϕ increases to 0.9, the hedge ratio increases dramatically to -10 and fully 66% of the futures position is intended to hedge spot positions five periods and further into the future.

A MULTIPERIOD MEASURE OF HEDGING EFFECTIVENESS

Since the goal for the hedger is to reduce risk as much as possible, a reasonable measure for hedging effectiveness is the fraction of risk, as measured by portfolio variance, eliminated by the hedge. For the single period case, the maximum reduction in variance can be calculated as

$$HE_1 = 1 - \frac{\sigma_1^{*2}}{\sigma_u^2} \tag{13}$$

where HE_1 is the one period measure of hedging effectiveness and σ_1^{*2} is the minimum risk obtainable by taking the optimal futures position b_1^*. Using eq. (5) and (6) it can be shown that

$$HE_1 = \frac{\sigma_{uf}^2}{\sigma_u^2 \sigma_f^2} = \rho_{uf}^2 \tag{14}$$

which is Ederington's familiar hedging effectiveness measure, where ρ_{uf}^2 is the squared correlation between $\tilde{u}$ and $\tilde{r}_{f,1}$. Note that this is the correlation between the spot disturbance term and the futures price change and not between the spot change and futures change as is the typical case when measuring hedging effectiveness.

Parallel to the single period case, the multiperiod measure of hedging effectiveness (HE_t) can be represented as

$$HE_t = 1 - \frac{\sigma_t^{*2}}{C_t \sigma_u^2} \tag{15}$$

where σ_t^{*2} is the minimum risk obtainable when the optimal futures position b_t^* is taken this period, b^*_{t-1} is taken one period from now, and so forth. That is, HE_t is based on the assumption that the optimal multiperiod hedge ratio is taken in each

Table I
COMPOSITION OF AN INFINITE PERIOD FUTURES POSITION FOR VARIOUS VALUES OF ϕ ($B_1^* = -1$).

		% of Futures Hedge Held for Period:				
ϕ	(b^*)	1	2	3	4	5 and Beyond
0.00	−1.00	100	0	0	0	0
0.25	−1.33	75	19	5	1	0
0.50	−2.00	50	25	13	3	9
0.75	−4.00	25	19	14	4	38
0.90	−10.00	10	9	8	7	66

and every future period through time t. Using eq. (9) and (10), it can be shown that (see the Appendix, Part E for a derivation)

$$\sigma_t^{*2} = C_t \sigma_u^2 - C_t \frac{\sigma_{uf}^2}{\sigma_f^2} \tag{16}$$

where

$$C_t = \sum_{j=1}^{t} \left[\frac{1 - \phi^j}{1 - \phi} \right]^2$$

Substituting eq. (16) into eq. (15) and simplifying: we obtain

$$HE_t = \rho_{uf}^2 \tag{17}$$

which is identical to the single period result in eq. (14). Eq. (17) clearly shows that if the correct multiperiod hedge ratio is taken in each and every one of the t periods remaining, the potential for risk reduction depends directly upon the correlation between the spot disturbance term and the futures price change. Therefore, the potential for risk reduction is invariant to the number of periods being hedged.

However, this straightforward result is based upon two very important considerations. First, the hedger must execute the optimal multiperiod hedge in each and every period. If for instance the hedger mistakenly uses a single period hedge ratio when a multiperiod hedge ratio is called for (i.e., when $\phi > 0$ and $t > 1$), then eq. (17) is not the appropriate measure of hedging effectiveness. In fact, in this situation eq. (16) provides an overestimate of the benefit of hedging and the overestimate increases as t and ϕ increase. Second, the correlation coefficient in eq. (17) is that between the spot series disturbance term (i.e., all structure and drift has been eliminated) and the futures price change. If, instead, the correlation coefficient is calculated using spot changes and the futures price change, the square of this incorrect estimate generally underestimates hedging effectiveness.[12]

CONCLUSION

A multiperiod hedge ratio is derived in which the only source of interperiod statistical dependency is the autocorrelations in the spot series being hedged. The quantity being hedged is known with certainty. It is shown that if spot return autocorrelation is near zero, a single period approach to hedging is appropriate. However, as the autocorrelation increases, it becomes more and more important to view the hedging problem in a multiperiod context. The multiperiod hedge ratio is larger in absolute terms than is the single period hedge ratio and increases as the number of periods being hedged increases. As long as the spot return autocorrelation is less than perfect, the infinite period hedge ratio will itself be finite.

Unlike the multiperiod hedge ratio, the multiperiod measure of hedging effectiveness is identical to the single period measure of hedging effectiveness: the square of the correlation between the spot disturbance term and the futures price change. Thus, if a hedger mistakenly uses a single period hedge ratio when a multi-

[12]Using eq. (2), it can be shown that if t is large, the underestimate is given by one minus the value $\sqrt{(1 + \phi)/2}$. That is, if $\phi = 1$, there is no underestimate whereas, if ϕ is less than 1, there is an underestimate. For example, if $\phi = 0.5$, the underestimate is 0.134 of the true value. The maximum underestimate occurs when $\phi = 0$ (i.e., when the spot series is a random walk) and this maximum underestimate is equal to roughly 0.3 of the true value.

period ratio is necessary, the square of the correlation coefficient greatly overestimates the true benefit of hedging. The complete hedging benefit can only be obtained if the correct multiperiod hedge ratio is used in each and every period.

APPENDIX

In this Appendix more detail is provided regarding the derivation of the multiperiod hedging model. Specifically, the focus is on the reason why the constants i, g, and S_1 drop out of the final equation for the optimal hedge ratio (i.e., the proof for the statement contained in footnote 9 in the paper), the derivation of the variance eqs. (7) and (9), the optimal hedge ratio when $i \neq g$, and the derivation of the minimum variance eq. (16).

A. The Irrelevance of i, g and S_1

Footnote 9 states that the risk-free rate (i), the growth in the quantity being hedged (g), and the period one quantity being hedged (S_1) are all irrelevant when determining the optimal hedge ratio and so are excluded from variance eqs. (5), (7), and (9). To show that this is in fact true, rewrite the equation [eq. (3)] for the period t return, $\tilde{r}_t$, as

$$\tilde{r}_t = S_1(1 + i)^{t-1} \sum_{j=1}^{t} (\tilde{r}_{s,j} + b_{t+1-j}\tilde{r}_{f,j}) \left(\frac{1 + g}{1 + i}\right)^{j-1} \tag{A1}$$

where

S_1 = quantity being hedged in period 1,
i = risk-free rate of return,
$\tilde{r}_{s,j}$ = spot return in period j,
b_{t+1-j} = the hedge ratio when $t + 1 - j$ periods remain,
$\tilde{r}_{f,j}$ = change in futures price for period j, and
g = constant growth rate in amount being hedged.

If one assumes $i = g$ as is assumed in the paper, the last term in ( )'s reduces to 1. In addition, if the variance $V(\tilde{r}_t)$ is calculated, one obtains

$$V(\tilde{r}_t) = S_1^2(1 + i)^{2t-2}V(\cdot) \tag{A2}$$

where $V(\cdot)$ is shorthand for the variance of the summation term in eq. (A1). As stated in the paper, the risk (i.e., variance) minimizing hedge ratios must be determined in a recursive fashion. However, for each hedge ratio, it is necessary to take the derivative of eq. (A2) with respect to the hedge ratio and set the resulting derivative equal to zero as follows:

$$\frac{dV(\tilde{r}_t)}{db_k} = S_1^2(1 + i)^{2t-2}\frac{dV(\cdot)}{db_k} = 0 \tag{A3}$$

Clearly this can be simplified to

$$\frac{dV(\cdot)}{db_k} = 0 \tag{A4}$$

Since the constants i, g, and S_1 do not enter into the variance $V(\cdot)$, they will not enter into the equation for b_k^*. End of proof.

B. Derivation of Variance Equation (7)

Based on the result just obtained, the two period return variance σ_2^2 can be represented as

$$\sigma_2^2 = V(\tilde{r}_{s,1} + b_2\tilde{r}_{f,1} + \tilde{r}_{s,2} + b_1^*\tilde{r}_{f,2}) \tag{A5}$$

Keep in mind that the subscripts for the hedge ratios represent the time remaining, not the period in which the hedge is taken. Using the spot return stochastic process described in eq. (1), one obtains:

$$\sigma_2^2 = V(\bar{r}_{s,1} + \tilde{u}_1 + b_2\tilde{r}_{f,1} + \bar{r}_{s,2} + \phi\tilde{u}_1 + \tilde{u}_2 + b_1^*\tilde{r}_{f,2}) \tag{A6}$$

Recall that $\tilde{u}_t$ is assumed i.i.d. and $N(0, \sigma_u)$, that $\tilde{r}_{f,t}$ is also i.i.d. and $N(0, \sigma_f)$, and that the contemporaneous covariance σ_{uf} is a constant for all t while all noncontemporaneous covariances are equal to zero. Keeping these assumptions in mind and carefully collecting terms, one obtains:

$$\sigma_2^2 = (2 + 2\phi + \phi^2)\sigma_u^2 + (b_2^2 + b_1^{*2})\sigma_f^2 + 2(b_2(1 + \phi) + b_1^*)\sigma_{uf} \tag{A7}$$

which is eq. (7).

C. Derivation of Variance Equation (9)

The derivation of variance eq. (9) follows the same steps as those above. The major difference is that instead of hedging for 2 periods, the task is now to hedge for t periods into the future. This leads to complicated expressions for the "coefficients" for the σ_u^2, σ_f^2, and σ_{uf} terms in eq. (9). Each of these "coefficients" is dealt with separately.

The "coefficient" C_t is dependent upon the frequency with which the disturbance terms enter into future spot returns. For example, $\tilde{u}_1$ impacts period 1's return as well as all future returns. However, the impact on future returns is muted by multiplication by ϕ taken to an even greater power the farther into the future is the return. Since $\tilde{u}_1$ is assumed uncorrelated with any other disturbance term, the contribution of $\tilde{u}_1$ to the variance (ignoring covariance with futures for now) is

$$V(\tilde{u}_1 + \phi\tilde{u}_1 + \ldots + \phi^{t-1}\tilde{u}_1) = (1 + \phi + \ldots + \phi^{t-1})^2\sigma_u^2 \tag{A8}$$

A similar argument can be made with respect to $\tilde{u}_2, \tilde{u}_3, \ldots,$ and $\tilde{u}_t$. The difference is that there is now one fewer period, which is impacted by the disturbance term for each period one moves into the future. Putting these all together one obtains

$$C_t = (1 + \phi + \ldots + \phi^{t-1})^2 + (1 + \phi + \ldots + \phi^{t-2})^2 + \ldots + (1 + \phi)^2 + 1 \tag{A9}$$

Recognizing that each of the terms in eq. (A9) represents a geometric progression and assuming $\phi < 1$, eq. (A9) can be simplified to obtain

$$C_t = \sum_{j=1}^{t}\left[\frac{1 - \phi^j}{1 - \phi}\right]^2 \tag{A10}$$

which is eq. (10). If, for example, t is set equal to 2, one obtains:

$$C^2 = \sum_{j=1}^{2}\left[\frac{1 - \phi^j}{1 - \phi}\right]^2 = 1 + \left[\frac{1 - \phi^2}{1 - \phi}\right]^2 = 2 + 2\phi + \phi^2$$

The same value is obtained for the two period case in eq. (7) [or (A7)].

The "coefficient" for σ_f^2 in eq. (9) is simply the sum of the squared hedge ratios, since it is assumed that $\tilde{r}_{f,t}$ is i.i.d. Finally, the "coefficient" for σ_{uf} is a straightforward combination of the other two "coefficients."

D. Relaxing the Assumption That $i = g$.

If the assumption that $i = g$ is relaxed, the last term in eq. (A1) does not reduce to 1. Therefore i and g now enter into the equation for the optimal hedge ratio (S_1 does not, however). To capture the impact of i and g, each of the hedge ratios given by eq. (6), (8), and (11) are rederived.

Solving for b_1^*. An examination of eq. (A1) reveals that the last term in ( )'s drops out when $t = 1$. Therefore eq. (6) is unchanged when $i \neq g$.

Solving for b_2^*. For ease of representation let

$$a = \frac{1 + g}{1 + i} \tag{A11}$$

Using this definition, eq. (A6) can be written as

$$\sigma_2^2 = V[\tilde{r}_{s,1} + \tilde{u}_1 + b_2\tilde{r}_{f,1} + (\tilde{r}_{s,2} + \phi\tilde{u}_1 + \tilde{u}_2 + b_1\tilde{r}_{f,2})a] \tag{A12}$$

This leads to a revised eq. (7).

$$\sigma_2^2 = (1 + a^2 + 2a\phi + a^2\phi^2)\sigma_u^2 + (b_2^2 + a^2b^{*2}_1)\sigma_f^2 + 2(b_2(1 + a\phi) + ab^*{}_1)\sigma_{uf} \tag{A13}$$

Taking the derivative of eq. (A13) with respect to b_2, setting the derivative equal to zero, and solving for b_2, one obtains

$$b_2^* = -(1 + a\phi)\frac{\sigma_{uf}}{\sigma_f^2}, \text{ or}$$

$$b_2^* = -\frac{[1 - (a\phi)^2]\sigma_{uf}}{(1 - a\phi)\sigma_f^2} \tag{A14}$$

Solving for b_t^*. In similar manner, eq. (9) can be derived, which becomes (assume $a\phi < 1$):

$$\begin{aligned}\sigma_t^2 = {} & C_t'\sigma_u^2 + (b_t^2 + a^2b^*{}_{t-1}{}^2 + \ldots + a^{2t-2}b^{*2}_1)\sigma_f^2 \\ & + 2[b_t(1 - (a\phi)^t)/(1 - a\phi) + b^*{}_{t-1}a^2(1 - a\phi)^{t-1})/(1 - a\phi) \\ & + \ldots + b_1^*a^{2t-2}]\sigma_{uf}\end{aligned} \tag{A15}$$

where

$$C_t' = \sum_{j=1}^{t} a^{2(t-j)}\left[\frac{(1 - (a\phi)^j)}{1 - a\phi}\right]^2 \tag{A16}$$

Taking the derivative of eq. (A15) with respect to b_t, setting the derivative equal to zero, and solving for b_t, one obtains:

$$b_t^* = -\frac{(1 - (a\phi)^t)\sigma_{uf}}{(1 - a\phi)\sigma_f^2} \tag{A17}$$

The similarity between eqs. (9), (10), and (11), for which $i = g$, and eqs. (A15), (A16), and (A17), for which $i \neq g$, is striking. The most important difference, most easily seen by comparing hedge ratio eqs. (11) and (A17), is that the single term ϕ has been replaced by the product $a\phi$. Thus, one can think of the term a as modifying the autocorrelation term ϕ. If $g < i$, then $a < 1$ and the product $a\phi < \phi$. Thus $g < i$ means that fewer short futures contracts will be taken as a hedge and vice versa if $g > i$.

E. Derivation of Minimum Variance Equation (16)

Equation (16) can be derived using eqs. (9), (10), and (11) as follows. The "coefficient" for the σ_f^2 term in eq. (9) can be rearranged using eqs. (10) and (11) as follows:

$$b^{*2}_t + b^*_{t-1}{}^2 + \ldots + b^{*2}_1 = \sum_{j=1}^{t}\left[\frac{1-\phi^j}{1-\phi}\right]^2 \frac{\sigma_{uf}{}^2}{\sigma_f^4} = C_t \frac{\sigma_{uf}{}^2}{\sigma_f^4} \qquad (A18)$$

The "coefficient" for the σ_{uf} term can be represented as

$$= 2[b_t^*(1-\phi^t)/(1-\phi) + b^*_{t-1}(1-\phi^{t-1})/(1-\phi) + \ldots + b_1^*]$$

$$= -2\sum_{j=1}^{t}\left[\frac{1-\phi^t}{1-\phi}\right]^2 \frac{\sigma_{uf}}{\sigma_f^2}$$

$$= -2C_t\frac{\sigma_{uf}}{\sigma_f^2} \qquad (A19)$$

Substituting eqs. (A18) and (A19) into eq. (9), one obtains:

$$\sigma_t{}^{*2} = C_t\sigma_u^2 + C_t\frac{\sigma_{uf}{}^2\sigma_f^2}{\sigma_f^4} - 2C_t\frac{\sigma_{uf}}{\sigma_f^2}\sigma_{uf} = C_t\sigma_u^2 - C_t\frac{\sigma_{uf}{}^2}{\sigma_f^2} \qquad (A20)$$

which is eq. (16).

If the $i = g$ assumption is relaxed, it can be shown that HE_t is still equal to ρ_{uf^2}. The proof is virtually identical to the one above and, therefore, is not replicated here.

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