Equilibrium Treasury Bond Futures Pricing in the Presence of Implicit Delivery Options

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INTRODUCTION

The concern and contribution of this research is the development of an analytical framework for the determination of equilibrium futures prices in the presence of delivery options. Delivery options arise due to contract designs, which give the short position a great deal of flexibility in regard to when, where, how much, and what may be delivered. Several authors have looked at the various delivery options on an individual basis.[1] In practice, the value of the imbedded delivery options is not observable by the market and their collective value can only be inferred from the reduction in the futures price due to their presence. Furthermore, the value of each delivery option cannot, under most conditions, be determined separately, as the estimation of the equilibrium futures price must reflect their simultaneous influence. This article develops an analytical framework for valuing the T-bond futures price, which explicitly and simultaneously accounts for the joint influence of the end-of-month, quality, accrued interest, and wild card options. This model is the first that takes comprehensive account of all the delivery options. While specific emphasis centers on the T-bond futures contract, the framework may easily be adapted to any futures contract.

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[1]The literature on delivery options is extensive, for example, see Garbade and Silber (1983), Gay and Manaster (1984, 1986), Kane and Marcus (1986a, b), Arak and Goodman (1987), Carr (1987), LaBarge (1988), Brodie and Sundaresan (1988), Hegde (1989), Boyle (1989), and Hemler (1990).

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The empirical results show that the equilibrium model provides useful information. A policy of taking a short futures position when observed market prices are greater than the estimated prices and of taking a long position when observed prices are less than estimated prices appears to be a profitable strategy. This trading strategy is shown to provide profits in excess of the naive strategies of always "sell and hold" and "buy and hold" and outperforms a discretionary trading strategy based on a forward pricing model, which ignores the influence of delivery options.[2] A comparison of observed market prices with calculated prices shows that the results are consistent with the claims of Gay and Manaster (1986) (hereafter GM) that during the 1977–1983 period, delivery options were undervalued and futures prices "too high" relative to equilibrium. The purpose of the empirical tests is to validate the model and the numerical methods used to implement it, but is not intended to be a survey of current market conditions. The results provide evidence that the model is a useful tool and, hence, should be amenable to data taken from any period.

A MODEL FOR EQUILIBRIUM FUTURES PRICES

The delivery rules of the bond contract provide the short position with four types of delivery options: a quality option, which allows the short position to deliver any bond in the deliverable set; and three timing options: an accrued interest option, which allows short to choose any business day in the delivery month to make a delivery; an end-of-month option, which arises because the invoice prices are set (with accrued interest adjustments) on the last day that the futures contract trades for all deliveries occurring during the end of the month (the last seven business days); and a "wild-card" option, which arises because deliveries on any day may be announced as late as 8:00 PM for invoice prices that are fixed as of 2:00 PM. For further discussion of these delivery rules see GM (1986).

Description of Equilibrium

To model the equilibrium T-bond futures prices, the following notation is employed:

- t = the current day.
- T = the last day that the futures contract trades.
- T' = the last possible delivery date (note $t \leq T$ and $T < T'$).
- J = the number of days in the wild card period, which is defined as the number of days between the first delivery announcement day and the last trading day.
- $T - J$ = the first day that delivery may be initiated.
- F_t = the futures price at 2:00 PM on day t.
- C_i = the conversion factor for bond i.[3]
- $\mathbf{C}$ = the vector of conversion factors.

[2]It's believed that the strategies and profits reported are feasible. The empirical results support this belief. However, due to execution risk, bid-ask vibrations in reported prices, and problems associated with nonsynchronous data, absolute proof of the profitability of the simulated strategies in difficult to achieve (for a review of these problems, see Phillips and Smith (1980)). At the very least, however, the methods produce estimates of equilibrium futures prices that are superior to reported settlement prices.

[3]The conversion factor is the price that a $1 face value bond with coupon rate and time to maturity equal to that of the delivered bond would have if it were priced to yield 8% compounded semiannually.

B_{it} = the spot price of bond i on day t at 8:00 PM.

$\mathbf{B}_t$ = the vector of spot prices of deliverable bonds on day t.

I_{it} = C_iF_t the invoice price of bond i on day t, provided t is within the eligible delivery period ($T - J \leq t \leq T'$).[4]

$\mathbf{I}_t$ = $\mathbf{C}F_t$ the vector of invoice prices for day t.

$E_t[\,]$ = The expectation operator conditional on information available at 2:00 PM on day t. Rather than introduce additional notation it should be understood that when writing $E_t[\,|\,\mathbf{B}_t]$, we are conditioning the expectation on the bond prices as of 2:00 PM on day t. Except when used as a conditioning argument in an expectation, $\mathbf{B}_t$ will refer to bond prices at 8:00 PM on day t. For example, $E_t[\mathbf{B}_t\,|\,\mathbf{B}_t]$ indicates the expected bond price at 8:00 PM on day t, conditioned on the prices observed at 2:00 PM on day t.

The T-bond futures price equilibrium must be consistent with two facts: first, positions in the T-bond futures contract are opened at zero cost; and second, the short-term holding period return on deliverable bonds is approximated by the risk-free spot rate.[5] From these two conditions, it follows that the equilibrium futures price is set such that the expected cash payoff to a T-bond futures positions, conditioned on available information, for any day in the future is zero.

Because of the contract design, it is necessary to consider three different families of payoff functions; the payoff on the last possible delivery date (date T'),[6] the payoff during the wild-card period when delivery is allowed but the futures contract continues to trade (days $T - J$ to T), and the period before delivery is allowed (days prior to $T - J$).

Consider the payoff to the short position in a futures contract at day T'.[7] If a short position is maintained past day T, delivery must occur; there is no longer the possibility of taking an offsetting long position. Still, the short position will want to fully exploit the end-of-month option. Define $G_{T'}(\mathbf{B}_{T'}, F_T)$ as the payoff to the short position on day T'.

$$\begin{aligned} G_{T'}(\mathbf{B}_{T'}, F_T) &= \mathrm{Max}\{I_{1T'} - B_{1T'}, I_{2T'} - B_{2T'}, I_{3T'} - B_{3T'}, \ldots, I_{nT'} - B_{nT'}\} \\ &= \mathrm{Max}\{C_1F_T - B_{1T'}, C_2F_T - B_{2T'}, C_3F_T - B_{3T'}, \ldots, C_nF_T - B_{nT'}\} \\ &= \mathrm{Max}\{\mathbf{C}F_T - \mathbf{B}_{T'}\} \qquad (1) \end{aligned}$$

[4] $\mathbf{I}_t = \mathbf{C}F_T$ for $T \leq t \leq T'$ since there is no new futures price between T and T'. Also, the invoice price is the product of the futures price and the conversion factor. This is not strictly correct since the accrued interest of the delivered bond, measured to the delivery date, must be added to this product to produce the correct invoice price. In all the calculations, correct adjustments are made for accrued interest. For now, the discussion omits the accrued interest, since it would needlessly complicate the notation without altering any of the conclusions.

[5] By employing this approximation, implicit acceptance is given to the "local expectations" hypothesis of the term structure (see Ingersoll (1987), pp. 389–391). This is a reasonable assumption as it enables one to simply set the values of the expected payoffs to the futures position to equal zero. Furthermore, the results of the model are quite general in that the expected payoffs may be set to any value of one's choice. Alternatively, in an economy having one risk-neutral agent conducting arbitrage, the expected payoff should equal zero.

[6] In deriving equilibrium prices it is assumed that any deliveries that are made during the end-of-month period are expected to occur at the last date (T'). This assumption is consistent with valuing maximizing behavior by the short position. Furthermore, GM (1986) show that the value of the end-of-month option is greater the longer the time to maturity.

[7] Without any loss of generality payoff functions are considered strictly from the point of view of the short position. The payoff to the long position is simply the negative of short payoff.

Equilibrium requires that the expectations of $G_{T'}(\mathbf{B}_{T'}, F_T)$, conditional on information available at any time t, be zero. For purposes of this model, the set of available information is restricted to two prices, the set of current bond prices ($\mathbf{B}_t$) and the prevailing futures price (F_t). One equilibrium requirement is stated as

$$E_t[G_{T'}(\mathbf{B}_{T'}, F_T) \mid F_t, \mathbf{B}_t] = 0 \tag{2}$$

On day $T - 1$ the process is complicated by a wild-card option. The short position can deliver or mark to market. The delivery of bond i will yield $C_i F_{T-1} - B_{iT-1}$, while marking to market will yield $F_{T-1} - F_T$. Optimal behavior by the short position on day $T - 1$ will be to maximize the expected value of these alternatives. Define the function $G_{T-1}(\mathbf{B}_{T-1}, F_{T-1}, F_T)$ as the payoff function for the optimal policy on day $T - 1$.

$$G_{T-1}(\mathbf{B}_{T-1}, F_{T-1}, F_T) = \text{Max}\{\mathbf{C}F_{T-1} - \mathbf{B}_{T-1}, F_{T-1} - F_T\} \tag{3}$$

Eq. (3) is very similar to eq. (1). The fundamental difference is the addition, at time $T - 1$, of the possibility of marking to market on $T - 1$ to obtain $F_{T-1} - F_T$ for the short position. The function $G_{T-1}(\mathbf{B}_{T-1}, F_{T-1}, F_T)$ is the value of the optimal wild card policy at time $T - 1$.[8]

Based on information available at t, equilibrium requires that the expectation of $G_{T-1}(\mathbf{B}_{T-1}, F_{T-1}, F_T)$ over $\mathbf{B}_{T-1}$, F_{T-1} and F_T equal zero.

$$E_t[G_{T-1}(\mathbf{B}_{T-1}, F_{T-1}, F_T) \mid F_t, \mathbf{B}_t] = 0 \tag{4}$$

Eqs. (3) and (4) can be generalized, as shown in eqs. (5) and (6) below, for any day during the wild-card period (days $T - J$ to $T - 1$).

$$G_{T-k}(\mathbf{B}_{T-k}, F_{T-k}, F_{T-k+1}) = \text{Max}\{\mathbf{C}F_{T-k} - \mathbf{B}_{T-k}, F_{T-k} - F_{T-k+1}\} \quad \text{for } k = 1, 2, \ldots, J \tag{5}$$

$$E_t[G_{T-k}(\mathbf{B}_{T-k}, F_{T-k}, F_{T-k+1}) \mid F_t, \mathbf{B}_t] = 0 \quad \text{for } k = 1, 2, \ldots, J \tag{6}$$

Finally, prior to the wild card period ($t < T - J$), there is no possibility of delivery. The gains or losses to the futures contract are given by the function $G_{T-k}(F_{T-k}, F_{T-k+1})$ reported below.

$$G_{T-k}(F_{T-k}, F_{T-k+1}) = F_{T-k} - F_{T-k+1} \quad \text{for } k = J + 1 \text{ to } T - t \tag{7}$$

The function given in eq. (7) is the payoff to the short position prior to the first allowable delivery date. As before, equilibrium requires that the expected payoff conditional on information available at time t equal zero.

$$E_t[G_{T-k}(F_{T-k}, F_{T-k+1}) \mid F_t, \mathbf{B}_t] = 0 \quad \text{for } k = J + 1 \text{ to } T - t. \tag{8}$$

Summarizing, there are three families of payoff functions for the short position in the T-bond futures contract, which are given as equations (1), (5), and (7). These equations merely define, day by day, the payoffs to the optimal delivery policy by the short position over the life of the futures contract. The futures market on day t is in equilibrium if the expectations given in eqs. (2), (6), and (8) all equal zero.

The model for the equilibrium futures price for any day t is motivated by the $T - t + 1$ equilibrium conditions given in eqs. (2), (6), and (8). Ideally, for a given

[8]The function recognizes that delivery at $T - 1$ can be optimal, even if the investor desires to maintain a short position on day T. If delivery is expected to be more profitable than marking to market, delivery always should be made. If desired, a new short position can be established the next morning and, ignoring transactions fees, at zero cost (see GM, 1986, section 3.5).

value of $\mathbf{B}_t$ and a joint probability distribution for all the F_{T-k}s and $\mathbf{B}_{T-k}$s one would like to solve eqs. (2), (6), and (8) for the value of F_t that satisfies all the equilibrium conditions. However, the analytical and numerical complexity required for those calculations renders that procedure infeasible.[9] An alternative procedure is proposed that has intuitive appeal and, based on empirical tests to be reported below, is a successful compromise between economic reality and empirical tractability.

Approximation for Equilibrium Futures Prices

At time t there are $T - t + 1$ equilibrium conditions given in eqs. (2), (6), and (8). Each condition is of the form $E_t[G_\tau] = 0$, for $\tau = T', T - 1, T - 2, \ldots, t$. Inasmuch as it is not practical to solve all $T - t + 1$ equilibrium conditions simultaneously, the problem is approached with some simplifying approximations.

First, the current bond prices, $\mathbf{B}_t$, are taken as exogenous. Implicitly, this assumes that observed spot prices are in equilibrium, and that futures prices will adjust to accommodate spot bond prices. Although bond and futures prices are believed to be jointly determined, the assumption of exogenously determined spot prices is a standard one for many models of derivative securities and does not appear to limit their usefulness.

Second, all $T - t + 1$ conditions are not solved simultaneously; rather, each equation is considered separately. For each equation of the form $E_t[G_\tau] = 0$, a parameter $f_t(\tau)$ is estimated that represents an anticipated value of F_τ which, if actually realized at time τ, would result in the expectation of G_τ, conditional on $\mathbf{B}_t$, equaling zero. The calculations are recursive. The calculated value of $f_t(\tau + 1)$ is employed to evaluate $f_t(\tau)$. Formally, the values of $f_t(\tau)$ are defined to satisfy eqs. (9), (10), and (11).

$$E_t[G_\tau(\mathbf{B}_\tau, F_\tau) \mid \mathbf{B}_\tau, F_\tau = f_t(\tau)] = 0 \quad \text{for } \tau = T' \tag{9}$$

$$E_t[G_\tau(\mathbf{B}_\tau, F_{\tau-1}, F_\tau) \mid \mathbf{B}_t, F_{\tau-1} = f_t(\tau - 1), F_\tau = f_t(\tau)] = 0 \quad \text{for } \tau = T - J \text{ to } T - 1 \tag{10}$$

$$E_t[G_\tau(F_{\tau-1}, F_\tau) \mid \mathbf{B}_t, F_{\tau-1} = f_t(\tau - 1), F_\tau = f_t(\tau)] = 0 \quad \text{for } t \le \tau < T - J \tag{11}$$

The values of $f_t(\tau)$ are calculated recursively, starting with $\tau = T$ and working backward to $\tau = t$. The set of $f_t(\tau)$ for $\tau = t$ to T is a fictitious time series of futures prices which, if realized, would be consistent with eqs. (9), (10), and (11). Finally, the calculated value of $f_t(t)$ is the estimate of the equilibrium value of F_t.

[9]Stultz (1982) and Johnson (1987) report on techniques that solve problems somewhat similar to the ones posed by eqs. (2), (6), and (8). Their techniques, however, are limited to cases with a single exercise price. For the delivery options there will be a different exercise price for each bond in the deliverable set. In what follows, estimates are obtained of the expectations given in eqs. (2), (6), and (8) using Monte Carlo techniques. These attempts at direct calculation using a numerical integration algorithm by Milton (1972) failed. This failure was probably due to the large number of bonds (four or more) with high mutual correlation employed in the calculations. Bohrer and Schervish (1981) and Schervish (1984) review the problem of the Milton algorithm failing to converge under certain conditions. They introduce auxiliary algorithms for alleviating these problems. These improvements, however, are only applicable to those cases where the region of integration is rectangular (limits of integration constant), and do not alleviate the problems when working with iterated integrals as in this case. This, therefore, remains an unsolved problem.

Characteristics of the Approximation

Consider the calculation that solves for $f_t(T)$. The parameter $f_t(T)$ is the solution to the approximate equilibrium condition eq. (9). The corresponding true equilibrium condition is eq. (2). Eq. (9) differs from eq. (2) in that $f_t(T)$ is substituted for F_T, and the expectation is taken only over $\mathbf{B}_{T'}$, conditional only on $\mathbf{B}_t$. This approach eliminates the need to identify the joint density of the $\mathbf{B}_t$'s and the F_t's. Economically, this is appealing. Given the terms of the contract, the futures price should be fully determined from the distribution of the spot bond prices.

Using eq. (9) instead of eq. (2) will introduce some bias into the procedure. The direction of the bias can be identified by an application of Jensen's inequality. Note that $G_{T'}(\mathbf{B}_{T'}, F_T)$ is positive and convex in F_T; it then follows that for any joint density of $\mathbf{B}_{T'}$ and F_T expression (12) holds.

$$E_t[G_{T'}(\mathbf{B}_{T'}, F_T) \mid \mathbf{B}_t] > E_t[G_{T'}(\mathbf{B}_{T'}, E_t[F_T]) \mid \mathbf{B}_t] \tag{12}$$

Using eqs. (2), (9), and (12), it can be concluded that the value of $f_t(T)$ will be greater than $E_t[F_T]$.

$$f_t(T) > E_t[F_T] \tag{13}$$

Expression (13) illustrates the approximate nature of the model, and the direction of bias introduced.

The second step in the procedure uses the calculated value of $f_t(T)$ as if it were known to be the future realization of F_T. In eq. (3), F_T is replaced with $f_t(T)$. Since it is known that $f_t(T) > E_t(F_T)$ and that F_T enters $G_{T-1}(\mathbf{B}_{T-1}, F_{T-1}, F_T)$ negatively, intuition suggests that relation (14) holds.[10]

$$E_t[G_{T-1}(\mathbf{B}_{T-1}, F_{T-1}, f_t(T))] < E_t[G_{T-1}(\mathbf{B}_{T-1}, F_{T-1}, F_T)] = 0 \tag{14}$$

Recall that the parameter $f_t(T - 1)$ is defined such that eq. (15) is satisfied.

$$E_t[G_{T-1}(\mathbf{B}_{T-1}, f_t(T - 1), f_t(T)) \mid \mathbf{B}_t] = 0 \tag{15}$$

Note that eq. (15) is a special case of eq. (10) for $\tau = T - 1$. Following an argument parallel to that preceding expression (13), expression (16) obtains

$$f_t(T - 1) > E_t(F_{T-1}) \tag{16}$$

There are two reasons why expression (16) holds. First, instead of taking the expectation in (6) over F_T, F_{T-1}, and $\mathbf{B}_{T-1}$, F_T is replaced by $f_t(T)$, which is greater than $E_t(F_T)$. Once this substitution is made, the value of $E_t[G_{T-1}(\mathbf{B}_{T-1}, F_{T-1}, f_t(T))]$ will be less than zero, as indicated in expression (14). However, the value of $f_t(T - 1)$ is calculated to make the value of $E_t[G_{T-1}(\mathbf{B}_{T-1}, f_t(T - 1), f_t(T))]$ equal to zero. Therefore, the calculated value of $f_t(T - 1)$ will be high relative to $E_t(F_{T-1})$.

The second reason that $f_t(T - 1)$ is biased high is from Jensen's inequality. For convex functions, the function of an expectation is less than the expectation of the function. The equilibrium condition in this model is that the expectation of the G_t functions all equal zero. However, these expectations are not actually calculated. Instead, the parameters $f_t(T - k)$, that must be greater than the means of the F_{T-k}, are substituted.

[10]Although plausible, one cannot analytically prove this intuition. The empirical analysis investigates whether this intuition is supported in data. Also, the difficulties associated with analytical solutions are explained.

Following the procedures analogous to those employed to evaluate the parameter $f_t(T - 1)$, the value of $f_t(T - 2)$ is calculated as the solution to eq. (17), another special case of eq. (10).

$$E_t[G_{T-2}(\mathbf{B}_{T-2}, f_t(T - 2), f_t(T - 1))] = 0 \tag{17}$$

Eq. (17) can be generalized to solve for the value of $f_t(T - k)$ as shown below.

$$E_t[G_{T-k}(\mathbf{B}_{T-k}, f_t(T - k), f_t(T - k + 1)) \mid \mathbf{B}_t] = 0 \tag{18}$$

Repeated applications of expression (18) can be used to solve for all the $f_t(T - k)$s for $k = 1, 2, \ldots, T - t$. In general, the bias associated with the parameters $f_t(T - k)$ should become greater as the number of days to maturity, k, increases. This is shown in expression (19) below.

$$f_t(T - k) - E_t[F_{T-k}] > f_t(T - j) - E_t[F_{T-j}] \quad \text{if } k \geq j \tag{19}$$

The final value of this process is $f_t(t)$, and is the approximate equilibrium value of F_t at 2:00 PM on day t. The model can then be tested by comparing the calculated values of $f_t(t)$ with the observed futures prices F_t.

EMPIRICAL METHODS AND TESTS

The data employed in this study are described in GM (1986). Included are the bid and ask spot prices of all bonds in the deliverable set during each delivery period from 1977–1983. These data are provided by the Interactive Data Corporation—Chase Econometrics, and are represented to be "afternoon quotes." The procedures call for prices as of 2:00 PM for the conditioning variables, and as of 8:00 PM to calculate the value of G_t functions. The potential errors introduced by using "afternoon quotes" are unavoidable. However, these errors, should they occur, are neither systematic nor severe (this is discussed in detail in GM (1986), p. 52).

For the same time period, the T-bond futures settlement prices are compiled on a daily basis. It is from these prices that the daily invoice prices are calculated for each day of the delivery month. Also, these are the official prices from which all positions are "marked to market." In the calculations of which bond is cheapest to deliver, use is made of the overnight repurchase rates (repo rates) that would be needed to finance bond delivery under the terms of the futures contracts. The repo rates are supplied by the Board of Governors of the Federal Reserve System.[11]

Finally, the estimation procedure also makes use of the daily rates of return on T-bills. Each day, the yield to maturity for the T-bill that matures closest to the last day of the relevant futures contract delivery month is recorded. These T-bill rates are employed to estimate the expected short-term daily return on all the bonds in the deliverable set. (This is consistent with the "local expectations" hypothesis, see footnote 5.)

Consider the calculations required to solve for $f_t(\tau)$. In general, for a given value of $f_t(\tau + 1)$, the value of $f_t(\tau)$ must be found that satisfies eq. (18). Solving eq. (18) for $f_t(\tau)$ requires knowledge of the joint density of $\mathbf{B}_{T-k}$ conditional on $\mathbf{B}_t$. It is assumed that all prices of all bonds in the deliverable set follow a joint lognormal

[11] The analysis that follows explicitly recognizes the timing of all cash flows. All relevant financing costs are included in the calculations, although in the interest of simplicity and clarity, discussion of these technical issues is suppressed in the text.

density. Specifically, the marginal distribution for bond i at time τ is given by $\mathbf{B}_{i\tau} = B_{it} \exp(z_i(\tau - t))$ where z_i is normally distributed with mean μ_i and standard deviation σ_i. The distribution of the several z_is is a joint normal density given by $\mathbf{z} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where $\mathbf{z}$ is the vector of the individual z_i's, $\boldsymbol{\mu}$ is the vector of means, and $\boldsymbol{\Sigma}$ is the covariance matrix.

The parameters of the joint density, $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$, are estimated from the data as follows:

1. A correlation matrix is estimated using daily rates of return calculated from the spot ask prices of all bonds in the deliverable set, over the entire sample period.
2. The standard deviations of each bond's daily rates of return are calculated from the daily returns (based on ask prices) in the month that the futures contract matures.
3. The mean rate of return is assumed to be the same for all bonds in the deliverable set. The means are assumed to equal the yield to maturity of the treasury bill that matures nearest to the final delivery date in each contract month. The mean vector, $\boldsymbol{\mu}$, is a constant vector that changes daily.

By using these procedures, the mean vector of the joint normal density changes each day based on the observed T-bill prices. However, the means calculated in this way are "pure" in that the data used to make the estimates of means on day t do not employ an information regarding events that occur after day t.[12]

The correlation matrix is computed once using all available data. This has the virtue of increasing the sample size, but the disadvantage that the experiments performed use data in the parameter estimation that are not available to traders in a timely fashion. The results of these experiments, however, are not strongly affected by the procedures for estimating correlations. In fact, an alternative procedure that assumes that all the pairwise correlation coefficients are equal does not appear unreasonable. The average value of all correlations employed in the study is 0.89 with a standard deviation of 0.04.

The covariance matrix is calculated for each delivery month from the sample standard deviations of each deliverable bond within the delivery month. Each element of the covariance matrix is computed as the product of the appropriate correlation coefficient and each appropriate sample standard deviation.

Finally, for the $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ estimated for each month, equilibrium futures prices are calculated for each day in the wild-card period ($T - J$ to T) as follows:

1. On day $T - J$ identify the four bonds that are cheapest-to-deliver based on the spot and futures prices observed on day $T - J$.[13]
2. For these four bonds identify the joint distribution using the parameter estimates listed above.

[12]The methodology for the determination of equilibrium futures prices developed here are amenable to a variety of methods for estimating terminal bond prices. The empirical procedures used in this article may technically allow for arbitrage opportunities. For arbitrage-free methodologies, see techniques presented in Ho and Lee (1986), Ball and Torous (1983), and Kane and Marcus (1986a).

[13]Only the four cheapest-to-deliver bonds are considered, while as many as 9 to 21 bonds are in the deliverable set. By considering only four bonds, the value of the delivery options is understated and the value of the equilibrium futures price is overstated.

3. Generate computer realizations of 10,000 sets of the four bond prices for day T' based on the current spot prices of the bonds and the appropriate joint lognormal distribution.
4. Engage a computer search for $f_t(T)$ that makes the average of the 10,000 realizations of $G_{T'}(\mathbf{B}_{T'}, f_t(T))$ equal to zero.
5. Generate 10,000 realizations for the four bond prices for day $T - 1$. Using the value of $f_t(T)$ calculated in (4) above, search for the value of $f_t(T - 1)$ that makes the average of the 10,000 realizations of $G_{T-1}(\mathbf{B}_{T-1}, f_t(T - 1), f_t(T))$ equal to zero.
6. Repeat step 5 for each day moving back in time until $f_t(t)$ is determined.
7. Repeat, beginning again with step one, for each day in the delivery month.

Empirical Results

Comparison: Estimated Prices with Observed Prices. Table I provides a comparison of the estimated equilibrium futures prices and the observed futures prices. For the section of the table labeled "Results Based on Bid Prices," the equilibrium futures prices, as estimated by the $f_t(t)$, are calculated using the bid prices of the deliverable bonds as inputs into the estimation procedure. Similarly, in the section labeled "Results Based on Ask Prices," the ask prices of the deliverable bond are used as inputs for the estimation of $f_t(t)$. Recall that the current spot prices, $\mathbf{B}_t$, (bid or ask) are the only prices that are employed in the estimation techniques. Furthermore, it is assumed that the parameters of the joint lognormal distribution for bond prices, μ and Σ, are the same whether bid or ask prices are used. Recall from prior discussion that $f_t(t)$ will be an upward biased estimate of the equilibrium futures price at time t. Assuming as is reasonable, that the reported ask prices are an upper limit of the "true equilibrium" spot bond prices, then using the ask prices in the calculations as the relevant bond prices should add an additional upward bias to the equilibrium estimates. Conversely, the use of bid prices may, in part, offset the bias in the procedures. The evidence from Table I, therefore, requires careful interpretation.

For both the bid-based and ask-based estimates, Table I provides the average in each month of the excess of the observed futures price over the estimated futures price. These differences are reported in the columns labeled "Observed FP Less Estimated FP." t-Statistics testing the null hypothesis that the difference between the observed and estimated prices equal zero are reported adjacent to the means. Table I shows that the equilibrium futures price calculated using bond bid prices are usually less than the observed futures price. In 20 of the 23 months, the differences between the observed futures prices and the bid-based equilibrium prices are positive and significant. The average excess of the observed futures price over our bid-based estimate is \$235, an amount that is reliably different from zero. The table also shows that the ask-based estimates are remarkably close to the observed futures prices. In the total sample, the ask-based estimates differ from observed prices by a statistically insignificant \$6. It is ironic that the ask-based prices are so close to the observed prices, inasmuch as the ask prices are expected to be the most biased.

Despite the fact that the ask-based prices conform most closely to the observed prices, it is argued below that the bid-based prices are better estimates of the equilibrium prices. The model is not intended to "fit observed prices"; rather, it is a

Table I
SUMMARY STATISTICS BY CONTRACT MONTH FOR ESTIMATED EQUILIBRIUM T-BOND FUTURES PRICES (FP)[a]
(t-STATISTICS IN PARENTHESES)

Contract Month	Obs.	Results Based on Bid Prices: Observed FP Less Estimated FP		Results Based on Bid Prices: "Forward Price" of Cheapest-to-Deliver Bond Less Estimated FP		Results Based on Ask Prices: Observed FP Less Estimated FP		Results Based on Ask Prices: "Forward Price" of Cheapest-to-Deliver Bond Less Estimated FP		Percent of Time Rank Order Cheapest-to-Deliver Bond. Expected to be Cheapest — Rank Order (Based on Bids): First	Second	Third	Fourth
7712	16	$208	(7.94)[b]	$ 38	(5.44)[b]	$ −56	(−2.26)[c]	$111	(4.43)[b]	58.46	25.19	12.80	3.55
7803	16	103	(5.75)[b]	48	(8.36)[b]	−149	(−8.29)[b]	49	(8.02)[b]	51.69	34.06	11.70	2.55
7806	17	178	(8.94)[b]	24	(4.46)[b]	−83	(−4.06)[b]	24	(4.54)[b]	82.48	10.48	5.07	1.97
7809	15	91	(2.76)[b]	34	(4.64)[b]	−49	(−1.70)	32	(2.68)[b]	77.31	14.36	5.51	2.82
7812	15	165	(8.27)[b]	48	(5.19)[b]	27	(1.25)	135	(9.26)[b]	67.14	22.74	8.10	2.02
7903	17	183	(4.62)[b]	32	(7.20)[b]	14	(0.48)	103	(5.81)[b]	72.77	20.37	6.72	0.14
7906	16	147	(3.20)[b]	68	(5.34)[b]	6	(0.13)	84	(4.23)[b]	63.15	20.75	12.64	3.46
7909	14	9	(0.34)	45	(7.26)[b]	−119	(−4.56)[b]	42	(4.40)[b]	62.76	21.35	10.03	5.85
7912	14	399	(6.32)[b]	157	(6.15)[b]	138	(2.14)[c]	174	(5.33)[b]	47.17	24.21	18.62	10.00
8003	16	524	(4.72)[b]	164	(7.63)[b]	118	(1.07)	164	(7.11)[b]	52.26	22.68	13.06	12.00

8006	16	485	(5.11)[b]	175	(5.81)[b]	67	(0.69)	162	(5.20)[b]	49.96	25.02	13.38	11.64
8009	16	305	(8.47)[b]	212	(7.71)[b]	−110	(−2.78)[c]	437	(5.55)[b]	45.83	24.03	16.94	13.20
8012	16	377	(5.73)[b]	299	(7.23)[b]	110	(1.72)	274	(6.81)[b]	40.10	23.48	21.04	15.38
8103	17	796	(3.89)[b]	672	(4.21)[b]	502	(2.47)[c]	700	(4.27)[b]	37.73	30.90	20.87	10.50
8106	17	82	(1.41)	138	(6.98)[b]	−156	(−2.79)[c]	159	(5.86)[b]	52.58	22.33	16.34	8.75
8109	16	287	(4.58)[b]	165	(8.39)[b]	33	(0.56)	149	(5.74)[b]	41.77	27.92	18.76	11.55
8112	17	263	(3.68)[b]	187	(5.38)[b]	−2	(−0.04)	223	(4.79)[b]	56.13	20.11	13.83	9.93
8203	18	142	(2.94)[b]	146	(7.84)[b]	−99	(−2.05)	155	(6.54)[b]	44.34	25.94	17.22	12.50
8206	17	405	(7.64)[b]	125	(5.97)[b]	173	(3.31)[b]	128	(6.14)[b]	54.25	22.76	14.10	8.89
8209	16	−51	(−0.75)	96	(4.65)[b]	−292	(−4.39)[b]	123	(3.47)[b]	62.48	21.33	10.64	5.55
8212	16	64	(1.34)	89	(7.67)[b]	−112	(−2.53)[c]	210	(1.70)	51.03	31.09	17.41	0.47
8303	18	73	(2.91)[b]	83	(6.97)[b]	−85	(−3.33)[b]	79	(7.00)[b]	56.03	26.63	17.18	0.16
8306	17	161	(7.10)[b]	170	(7.24)[b]	−3	(−0.14)	163	(7.09)[b]	41.41	29.12	22.17	7.31
Row averages		235	(5.80)[b]	140	(4.94)[b]	−6	(−0.17)	169	(5.55)[b]	55.17	23.78	14.09	6.96

[a]Numbers in table are monthly averages. Sample includes observations from all trading days during the delivery month beginning with the first position day through the last trading day for all contracts maturing between December 1977 and June 1983. Total observations: 374.

[b]Significantly different than zero at the 0.01 significance level, two-tailed test.

[c]Significantly different than zero at the 0.05 significance level, two-tailed test.

model of equilibrium. GM (1986) find that during the 1977–1983 period, T-bond futures prices are "too high" relative to equilibrium. It is believed that the upward biased procedures using ask prices track the observed prices closely due only to the bias. Below, evidence is provided that bid-based estimates provide a closer approximation of equilibrium despite the obvious departure of the these bid-based estimates from the observed prices.

In an attempt to quantify the influence of delivery options on the futures price, Table I also reports on the "Forward Price of the Cheapest-to-Deliver Bond Less Estimated FP." The forward prices used in the calculations in this column are determined by taking the price of the cheapest-to-deliver bond on day t, and based on the prevailing T-bill rate, appreciated forward to day T'. The theory of delivery options forecasts that the "forward price" of the cheapest-to-deliver bond calculated in this way should be greater than the equilibrium futures price. The "forward price" makes no consideration of the delivery options. Consequently, the difference between the "forward prices" and the estimated futures prices is a crude estimate of the influence of delivery options. Table I shows that, consistent with delivery option theory, both the bid- and ask-based estimates of equilibrium prices are lower than the cheapest-to-deliver "forward price." This is found to be true despite the upward bias in the equilibrium estimates.

Finally, Table I reports the percent of time the bond that is cheapest-to-delivery on day t is in the Monte Carlo estimations, cheapest-to-deliver on day T'. As reported earlier, the table only considers four bonds at a single time, the bonds that are cheapest-to-deliver on day t. The section "Percent of Time Rank Order Cheapest-to-Deliver Bond Expected to Be Cheapest" shows the bond which is cheapest-to-deliver on day t $(T - J \leq t \leq T)$ can be expected to be cheapest day on T' only about 55% of the time. Nonetheless, the three cheapest-to-deliver bonds on day t account for roughly 93% of the expected deliveries on day T'. This information is reported to provide a rough indication of the magnitude of the problems that may be encountered by restricting the analysis to the four cheapest-to-deliver bonds.[14]

From the difference between the cheapest-to-deliver forward price and the model's equilibrium price one can infer a value for the delivery options. Define the variable REDUCTION as the reduction in the futures price expected from the presence of delivery options expressed as a proportion of the cheapest-to-deliver forward price. This variable is an estimate of the proportional effect delivery options will have on futures prices. Below, the results are supported of the 374 observations in which REDUCTION (based on the bid-based estimates of the equilibrium futures price) is regressed on the spot rate of interest (r), the time to maturity (T), and the generalized variance (V)[15] of the four cheapest-to-deliver bonds (t-statistics in parentheses).

$$\begin{array}{l} \text{REDUCTION} = \underset{(-2.88)}{-0.001} + \underset{(-0.063)}{2.521E-03r} + \underset{(9.42)}{1.291E-04T} + \underset{(30.40)}{4.142\mathrm{E}-08V} \quad R^2 = 0.754 \end{array}$$

Option theory predicts that increases in the time to maturity and increases in the price volatility of the deliverable bonds will increase the value of the delivery op-

[14]See Boyle (1989) for a more complete analysis of the problems that may arise by not using all deliverable bonds in the pricing model.

[15]The generalized variance is the determinant of the covariance matrix. It is a measure of the joint variability of all the deliverable bonds.

ations and thereby cause an increase in REDUCTION. The results shown above support this theoretical prediction.[16] REDUCTION increases both with time to maturity and the generalized variance of the deliverable assets. The coefficients of time to maturity and generalized variance are both positive and highly significant. The spot rate of interest does not enter the regression as a significant variable. Evidently, the cost of carry model's forward price captures the influence of the spot rate of interest as well as the richer model that explicitly accounts for delivery options. Nonetheless, the results provide additional evidence that the equilibrium estimates have been successful in capturing the influence of delivery options on futures prices.

Table I and the regression results together demonstrate that the estimation procedure produces prices that correspond well to futures prices. Moreover, the departures between the forward prices and the estimated equilibrium prices follow the predicted pattern. Yet to be explained, however, is the fact that the ask-based estimates, although they conform more closely to the observed prices, are, in fact, biased high. Next, it is demonstrated that the bid-based estimates provide a usful benchmark for the equilibrium futures prices.

Which Is Closer to Equilibrium: Market or Model Prices? A simulated trading strategy is employed to determine whether the model prices are a close approximation to equilibrium prices. At the start of each delivery period (day $T - J$), whenever the observed futures price differs from the model price by more than a pre-set screen, a position is initiated. Long positions are taken when the model price exceeds the observed price by more than the screen, short positions are taken if the observed price exceeds the model price by more than the screen. No new positions are initiated if the absolute difference is less the screen. Once initiated, a position is held until either the price difference changes sign or the trading of the contract ends on day T. Closing of any position (long or short) depends only on the sign of the difference between the model and market price and not on the screen level imposed. Once a position is closed, new positions are initiated again only if the required screen is exceeded. When the screen is set to zero it is assumed that a position (long or short) is taken every day.

To illustrate, if on day $T - J$ the observed price is 90 and 6/32s and the model price 90 and 2/32, then the observed price exceeds the model price by 4/32s or \$125. If the screen level is less than \$125, a short position is opened at the market price (90 6/32). If the screen is greater than \$125, then no position is taken. Suppose the next day ($T - J + 1$), the market price falls to 90 3/32, but the model price is 90 4/32. Because of $T - J + 1$ the model price exceeds the market price, the short position is closed at the market price (90 3/32) for a gross profit of \$93.75. New long positions are then opened only for screens less than 1/32 (\$31.25). If the screen is greater than \$31.25, the strategy is neutral. This procedure produces profits if the model price is systematically closer to equilibrium than the observed price. To the extent that the observed prices are equilibrium prices, the strategy outlined above should, at best, produce no losses.

Tables IIa, IIb, IIIa, and IIIb provide comprehensive information regarding the results from the trading strategy. Tables IIa and IIIa use, as equilibrium estimates, the bid-based prices. Tables IIb and IIIb employ the ask-based estimates. Tables IIa

[16]Note that the regression results are corrected for autocorrelation, however, the results are essentially unchanged.

Table IIa

AVERAGE MONTHLY PROFITS AND DAYS VARIOUS POSITIONS HELD FOR *STRATEGIC* TRADING VERSUS SELL AND HOLD (SAH), BUY AND HOLD (BAH), AND FORWARD PRICE (FORW) STRATEGIES FOR VARIOUS SCREENS AND TRANSACTION COSTS.[a] EQUILIBRIUM FUTURES PRICES CONDITIONED ON BID SPOT BOND PRICES

(a) *Transaction Costs* = $0:

	(1) Trading up to Last Trading Date								(2) Trading on Last Trading Date							
	STRATEGIC Trading								STRATEGIC Trading							
		Days Position Held			Advantage to STRATEGIC Over[b]					Days Position Held			Advantage to STRATEGIC Over		
Screen	Profits	Short	Long	Neutral	SAH ($−204)	BAH ($204)	FORW		Profits	Short	Long	Neutral	SAH ($1067)	BAH ($−1067)	FORW
$ 0	$147	13.1	2.2	0.0	$351	$ −57	$190		$788[c]	0.65	0.35	0.00	$−279	$1855[d]	$ 0[c]
25	260	13.0	1.8	0.5	464[c]	56	449		901[d]	0.65	0.26	0.09	−166	1968[d]	113[c]
50	268	12.7	1.6	1.0	472[c]	64	385		658	0.61	0.17	0.22	−409	1725[d]	−153
100	257	12.4	0.9	2.0	461[d]	53	119		581	0.48	0.17	0.35	−486	1648[d]	−130
200	220	10.9	0.5	3.9	424[d]	16	134		285	0.35	0.13	0.52	−782	1352[c]	36
300	270	8.8	0.2	6.3	474[d]	66	303		487	0.30	0.04	0.66	−580	1554[d]	276[c]
400	1	6.0	0.2	9.1	205	−203	−108		297	0.17	0.04	0.79	−770	1364[d]	228

(b) *Transaction Costs* = $50:

	STRATEGIC Trading							STRATEGIC Trading						
		Days Position Held			Advantage to STRATEGIC Over[b]				Days Position Held			Advantage to STRATEGIC Over		
Screen	Profits	Short	Long	Neutral	SAH ($−254)	BAH ($154)	FORW	Profits	Short	Long	Neutral	SAH ($1017)	BAH ($−1117)	FORW
$ 0	$−45	13.1	2.2	0.0	$209	$−199	$288	$738	0.65	0.35	0.00	$−279	$1855[d]	$ 0
25	79	13.0	1.8	0.5	333	−75	535	855	0.65	0.26	0.09	−162	1972[d]	117[c]
50	107	12.7	1.6	1.0	361	−47	478	619	0.61	0.17	0.22	−398	1736[d]	−151
100	126	12.4	0.9	2.0	380[c]	−28	198	548	0.48	0.17	0.35	−469	1665[d]	−131
200	127	10.9	0.5	3.9	381[c]	−27	165	261	0.35	0.13	0.52	−756	1378[d]	38
300	207	8.8	0.2	6.3	461[d]	53	320	470	0.30	0.04	0.66	−547	1587[d]	278[c]
400	−36	6.0	0.2	9.1	218	−190	−86	286	0.17	0.04	0.79	−731	1403	226

[a]Sample includes trading conducted during all days in the delivery month beginning with (1) the first position day up to the last trading date, and (2) the last trading date through the last delivery date for all contracts maturing between December 1977 and June 1983. STRATEGIC trading refers to using discretion in taking long or short futures positions, or remaining neutral, depending on the relationship between the observed futures price and the estimated equilibrium futures price with the designated screen size. SAH refers to taking a short futures position on the first position date and holding it. BAH refers to taking a long position on the first position date and holding it. FORW refers to using discretion in taking long or short futures positions, or remaining neutral, depending on the relationship between the observed futures price and the forward price of the cheapest-to-deliver bond. Short (long) positions on the last trading date are assumed to deliver (receive) on the last delivery date the cheapest-to-deliver bond.

[b]The average number of days position held short (long) for the SAH (BAH) strategy was 15.3 days through the last trading date.

[c]Significantly greater than $0 at the 0.10 significance level, one-tailed test.

[d]Significantly greater than $0 at the 0.05 significance level, one-tailed test.

Table IIb
AVERAGE MONTHLY PROFITS AND DAYS VARIOUS POSITIONS FOR *STRATEGIC* TRADING VERSUS SELL AND HOLD (SAH), BUY AND HOLD (BAH), AND FORWARD PRICE (FORW) STRATEGIES FOR VARIOUS SCREENS AND TRANSACTION COSTS.[a] EQUILIBRIUM FUTURES PRICES CONDITIONED ON ASK SPOT BOND PRICES

(a) *Transaction Costs* = $0:

	(1) Trading up to Last Trading Date							(2) Trading on Last Trading Date						
	STRATEGIC Trading							STRATEGIC Trading						
		Days Position Held			Advantage to STRATEGIC Over[b]				Days Position Held			Advantage to STRATEGIC Over		
Screen	Profits	Short	Long	Neutral	SAH ($−204)	BAH ($204)	FORW	Profits	Short	Long	Neutral	SAH ($1067)	BAH ($−1067)	FORW
$ 0	$−84	6.1	9.2	0.0	$120	$−288	$82	$−585	0.26	0.74	0.00	$−1393	$233	$441
25	−19	5.6	8.3	1.4	185	−223	132	−358	0.26	0.70	0.04	−1166	450	440
50	−5	5.0	7.9	2.4	199	−209	144	−342	0.22	0.65	0.13	−1150	466	437
100	38	4.1	7.1	4.1	242	−166	57	−242	0.17	0.57	0.26	−1050	566	501[c]
200	190	3.0	4.8	7.5	394	−14	225	8	0.09	0.22	0.69	−800	816[c]	22
300	307	2.2	2.1	11.0	511	103	−64	8	0.09	0.22	0.69	−800	816[c]	0
400	77	1.8	1.0	12.5	281	−127	−162	−219	0.04	0.13	0.83	−1027	589	30

(b) *Transaction Costs* = $50:

	STRATEGIC Trading							STRATEGIC Trading						
		Days Position Held			Advantage to STRATEGIC Over[b]				Days Position Held			Advantage to STRATEGIC Over		
Screen	Profits	Short	Long	Neutral	SAH ($−254)	BAH ($154)	FORW	Profits	Short	Long	Neutral	SAH ($1017)	BAH ($−1117)	FORW
$ 0	$−397	6.1	9.2	0.0	$−143	$−551	$−5	$−635	0.26	0.74	0.00	$−1393	$223	$441
25	−302	5.6	8.3	1.4	−48	−456	68	−406	0.26	0.70	0.04	−1164	452	440
50	−255	5.0	7.9	2.4	−1	−409	108	−386	0.22	0.65	0.13	−1144	472	439
100	−160	4.1	7.1	4.1	94	−314	48	−279	0.17	0.57	0.26	−1037	579	506[c]
200	53	3.0	4.8	7.5	307	−101	236	−7	0.09	0.22	0.69	−765	851[d]	25
300	220	2.2	2.1	11.0	474	66	−57	−7	0.09	0.22	0.69	−765	851[d]	0
400	19	1.8	1.0	12.5	273	−135	−164	−227	0.04	0.13	0.83	−985	631	33

[a]Samples includes trading conducted during all days in the delivery month beginning with (1) the first position day up to the last trading date, and (2) the last trading date through the last delivery date for all contracts maturing between December 1977 and June 1983. STRATEGIC trading refers to using discretion in taking long or short futures positions, or remaining neutral, depending on the relationship between the observed futures price and the estimated equilibrium futures price with the designated screen size. SAH refers to taking a short futures position on the first position date and holding it. BAH refers to taking a long position on the first position date and holding it. FORW refers to using discretion in taking long or short futures positions, or remaining neutral, depending on the relationship between the observed futures price and the forward price of the cheapest-to-deliver bond. Short (long) positions on the last trading date are assumed to deliver (receive) on the last delivery date the cheapest-to-deliver bond.

[b]The average number of days position held short (long) for the SAH (BAH) strategy was 15.3 days through the last trading date.

[c]Significantly greater than $0 at the 0.10 significance level, one-tailed test.

[d]Significantly greater than $0 at the 0.05 significance level, one-tailed test.

Table IIIa
AVERAGE DAILY PROFIT PER TRADE FOR STRATEGIC TRADING VERSUS SELL AND HOLD (SAH), BUY AND HOLD (BAH), AND FORWARD PRICE (FORW) STRATEGIES FOR VARIOUS SCREENS.[a] EQUILIBRIUM FUTURES PRICES CONDITIONED ON BID SPOT PRICES (NUMBER OF TRADES IN PARENTHESES)

	Average Profit per Trade up to the Last Trading Date						Average Profit per Trade on Last Trading Date					
	Strategy						Strategy					
	STRATEGIC				FORW		STRATEGIC				FORW	
Screen	Short	Long	SAH	BAH	Short	Long	Short	Long	SAH	BAH	Short	Long
$ 0	$ −2	$79	$−13	$ 13	$−12	$ 16	$1422	$−401	$1067	$−1067	$1422	$−401
	(300)	(51)	(351)	(351)	(233)	(118)	(15)	(8)	(23)	(23)	(15)	(8)
$ 25	−2	155	−13	13	−16	−7	1422	−102	1067	−1067	1422	−401
	(298)	(42)	(351)	(351)	(225)	(108)	(15)	(6)	(23)	(23)	(15)	(8)
$ 50	−2	186	−13	13	−24	25	1461	−1330	1067	−1067	1461	−361
	(291)	(36)	(351)	(351)	(217)	(102)	(14)	(4)	(23)	(23)	(14)	(5)
$100	−1	313	−13	13	−6	52	1698	−1330	1067	−1067	1817	−361
	(285)	(20)	(351)	(351)	(195)	(83)	(11)	(4)	(23)	(23)	(10)	(5)
$200	11	203	−13	13	−22	135	1381	−1496	1067	−1067	1381	−1330
	(250)	(12)	(351)	(351)	(145)	(38)	(8)	(3)	(23)	(23)	(8)	(4)
$300	21	400	−13	13	−40	181	1481	836	1067	−1067	1557	−1496
	(202)	(5)	(351)	(351)	(101)	(18)	(7)	(1)	(23)	(23)	(6)	(3)
$400	−12	422	−13	13	−10	263	1497	836	1067	−1067	249	836
	(138)	(4)	(351)	(351)	(64)	(12)	(4)	(1)	(23)	(23)	(3)	(1)

[a]Sample includes trading conducted during all days in the delivery month beginning with (1) the first position day up to the last trading date and (2) the last trading date through the last delivery date for all contracts maturing between December 1977 and June 1983. STRATEGIC trading refers to using discretion in taking long or short futures positions, or remaining neutral, depending on the relationship between the observed futures price and the estimated equilibrium futures price with the designated screen size. SAH refers to taking a short futures position on the first position date and holding it. BAH refers to taking a long position on the first position date and holding it. FORW refers to using discretion in taking long or short futures positions, or remaining neutral, depending on the relationship between the observed futures price and the forward price of the cheapest-to-deliver bond. Short (long) positions on the last trading date are assumed to deliver (receive), on the last delivery date, the cheapest-to-deliver bond.

Table IIIb

AVERAGE DAILY PROFIT PER TRADE FOR STRATEGIC TRADING VERSUS SELL AND HOLD (SAH), BUY AND HOLD (BAH), AND FORWARD PRICE (FORW) STRATEGIES FOR VARIOUS SCREENS.[a] EQUILIBRIUM FUTURES PRICES CONDITIONED ON ASK SPOT PRICES (NUMBER OF TRADES IN PARENTHESES)

	Average Profit per Trade up to the Last Trading Date						Average Profit per Trade on Last Trading Date					
	Strategy						Strategy					
	STRATEGIC				FORW		STRATEGIC				FORW	
Screen	Short	Long	SAH	BAH	Short	Long	Short	Long	SAH	BAH	Short	Long
$ 0	$−23	$ 7	$−13	$ 13	$−73	$ 1	$427	$−943	$808	$−808	$−625	$−1110
	(141)	(210)	(351)	(351)	(58)	(293)	(6)	(17)	(23)	(23)	(4)	(19)
$ 25	−30	17	−13	13	−79	3	427	−674	808	−808	−625	−881
	(128)	(192)	(351)	(351)	(54)	(291)	(6)	(16)	(23)	(23)	(4)	(18)
$ 50	−18	11	−13	13	−70	0	497	−690	808	−808	−625	−907
	(114)	(182)	(351)	(351)	(51)	(287)	(5)	(15)	(23)	(23)	(4)	(17)
$100	−14	14	−13	13	−114	15	−625	−235	808	−808	−625	−973
	(95)	(162)	(351)	(351)	(4)	(271)	(4)	(13)	(23)	(23)	(4)	(15)
$200	−55	75	−13	13	−69	6	332	−94	808	−808	332	−165
	(69)	(109)	(351)	(351)	(31)	(238)	(2)	(5)	(23)	(23)	(2)	(6)
$300	−43	195	−13	13	−145	68	332	−94	808	−808	332	−94
	(50)	(147)	(351)	(351)	(16)	(159)	(2)	(5)	(23)	(23)	(2)	(5)
$400	−58	192	−13	13	−239	86	−1288	−1246	808	−808	1288	−1112
	(42)	(22)	(351)	(351)	(11)	(95)	(1)	(3)	(23)	(23)	(1)	(4)

[a]Sample includes trading conducted during all days in the delivery month beginning with (1) the first position day up to the last trading date and (2) the last trading date through the last delivery date for all contracts maturing between December 1977 and June 1983. STRATEGIC trading refers to using discretion in taking long or short futures positions, or remaining neutral, depending on the relationship between the observed futures price and the estimated equilibrium futures price with the designated screen size. SAH refers to taking a short futures position on the first position date and holding it. BAH refers to taking a long position on the first position date and holding it. FORW refers to using discretion in taking long or short futures positions, or remaining neutral, depending on the relationship between the observed futures price and the forward price of the cheapest-to-deliver bond. Short (long) positions on the last trading date are assumed to deliver (receive), on the last delivery date, the cheapest-to-deliver bond.

and IIb are broken into four sections. Trading profits from the first-day delivery announcements are allowed ($T - J$) to the last day the contract trades (day T) are shown in the left-hand side of each table. The profits earned from time T to delivery at time T' (the end-of-month period) are shown in the right-hand side of the tables. In the upper panel of the tables, profits are tabulated ignoring transactions costs. In the lower panel, a $50 transaction cost is assumed for a round trip transaction in the futures market.

In the upper left-hand portion of Table IIa, it is shown that positive profits are obtained for all levels of screens when the bid-based prices are employed. The table also shows the average number of days in each delivery month that the strategy produces a short, long, or neutral position. Notice that when the screen is zero, the strategy is never neutral. The next three columns indicate the excess profits of the strategy over three alternative strategies. The first two of these columns (labeled "Advantage to STRATEGIC over SAH" and "Advantage to STRATEGIC over BAH") indicate the excess profits of the discretionary strategy over the naive strategies of sell and hold (SAH) and buy and hold (BAH). The third column of these results, labeled FORW, presents the dollar advantage of using the model of futures prices, as opposed to a naive model, in the discretionary trading strategy. This naive pricing model merely estimates the equilibrium futures price as the forward price of the current day's cheapest-to-deliver bond (based on either the bid or ask price) and therefore ignores most of the subtleties of delivery options. The trading strategy, based on the naive model price, is conducted on an identical basis as the formal model and the profits compared. These comparisons allow one to analyze whether the inclusion of delivery options into the equilibrium model is providing useful information and controlling for potential data errors.

Consider the case for a $25 screen and zero transactions costs. The discretionary strategy using the bid-based prices generated an average profit of $260. This profit is $464 more than the SAH strategy produces, $56 more than BAH, and $449 more than FORW. In obtaining these profits, the discretionary strategy generally maintaines a short position, as positions are sold short an average of 13.0 of the 15.3 possible trading dates per month. It is of interest that profits are made from such a "short intense" policy inasmuch as being short 100% of the time produces an average loss of $204.

The most obvious characteristic of the strategic policy is the frequency of short positions. This is consistent with the conjecture that observed prices during the sample period are "too high" relative to equilibrium. With hindsight, it is obvious that being long during delivery months, prior to day T in this period, is, on average, a winning strategy.[17] This is manifested in the observation that the SAH strategy is a loser and BAH a winner. However, the bid-based estimates also show that, during this period, futures prices are higher than can be justified by observed spot bond prices. Consequently, the trading rule using the bid-based estimates recommend a short position many times more frequently than the long. The fact that the highly "short intense" policies prove superior, especially during a period when a long position is, on average, a winner, suggests that the equilibrium model is providing use-

[17] In the simulated trading strategies, no consideration is given for any potential profits from the "wild-card" option nor is allowance made for the short position to make delivery until the last day of delivery month. Consequently, the reported profits for the short position are less than could actually be obtained if optimal exercise of all delivery options is allowed. Symmetrically, the reported profits to long positions are overstated.

ful information.[18] The policy usually directs the use of a short position; however, it is also sensitive enough to detect periods of underpricing.

The right-hand side of Table IIa can be used to test whether the estimates of futures prices on the last day the futures contract trades (T), are close to equilibrium. The results are similar to the days prior to T, in that a short position is indicated many times more frequently than long. Although positive profits are earned by the strategic policy, for the last trading day a naive strategy of always being short turned out, for this sample of 23 days, to be superior to the strategic policy. Nonetheless, the ability to show positive profits (revealed in Table IIa, both with and without transactions costs) suggests that the model has empirical merit.

The conclusion can be drawn from Table IIa that, as estimates of equilibrium, the bid-based prices are highly successful. The experiment shows that futures prices are generally higher than equilibrium during the test period, consequently a "short intense" policy is usually indicated. For part of the sample period (days $T - J$) to $T - 1$) the evolution of prices shows that unanticipated price movements make the short position, on average, a loser. Despite on-average losses to short positions during this period, the "short intense" policy does better (ignoring transactions costs) than a policy of always taking a long position.

For the last trading day, the policy using the bid-based prices is highly short intense. A short position is indicated between a high of 65% to a low of 17% of the time, a long position from 35% down to 4%. Despite the propensity for the policy to be short, it is better to be short even more often. Being short on the last trading day and waiting to the last delivery day and making an optimal delivery produces greater profits than the discretionary strategy. This, however, is not an indictment of the equilibrium prices. As shown in Table IIa, in nearly all the experiments based on the bid-based calculations, positive profits result.

Table IIb repeats the experiments reported using the ask-based prices. Recall that using ask-based prices adds to the bias, whereas using bid prices, to some extent, may offset the bias. Therefore, the ask-based estimates are expected to be higher than the true equilibrium prices. Consequently, the trading rule based on the ask-based prices results in too many long positions. Table IIb reveals this to be true. In most cases, the trading strategy produces losses where, in Table IIa, the profits are positive. Evidently, unlike the bid-based prices, the ask-based policy fails to identify as many cases where prices are "too high." Consequently, the trading rule based on ask prices is not as successful.

Tables IIIa and IIIb repeat the information compiled in Tables IIa and IIb but averages are now on a per day rather than a per month basis. For example, in Table IIIa, the average profit per trade up to the last trading date for a SAH strategy is −$13. That means that being short in each of the 351 days in the sample produces on average, a $13 loss per day. For a zero screen, the bid-based strategy indicates a short position 300 of the 351 days. The average loss to the short position based on the strategic strategy for the 300 days is only $2 per day. This shows that the discretionary policy is able to eliminate a large percentage of the losses that would have occurred for the nondiscretionary SAH strategy.

Similarly, the zero screen policy in Table IIIa shows that, for 51 days, a long position is recommended. These 51 days produce an average profit per day of $79. This compares favorably to the naive BAH average profit of $13. Similarly, the average

[18]It is interesting to note that the FORW strategy is also generally short intensive. However, the discretionary strategy is short even more often by, on average, approximately 3 to 4 days per month, depending on the screen size.

profits per day for both short and long positions based on the model exceed those under the forward price based model. Under FORW, the average short position loses $12 per day and the average long position earns only $16 per day. It is concluded that by using the bid-based prices as an indication of equilibrium, one can generally do better than by using either a naive long or short strategy or by using a model that ignores the influence of delivery options (FORW).

The right-hand side of Table IIIa uniformly shows that for all screen levels the bid-based prices identify better short positions than a naive short strategy. The results for the long position are less conclusive due to a smaller sample size. However, in view of the positive profits for all levels of screens observed in Table IIa, it is reasonable to conclude that the bid-based prices are a useful approximation to equilibrium on day *T*.

Finally, Table IIIb shows the same information as Table IIIa using the ask-based prices. The results are what is expected from employing an equilibrium estimate that is biased high. More long positions are taken than is economically efficient. Most entries in Table IIIb reveal that the average daily profits to a long position and to a short position under the discretionary policy are both less than would be obtained from either a naive long strategy or a naive short strategy (but still greater than those based on a forward price based strategy). The failure of the ask-based prices to improve on the naive strategy is additional evidence that, although the ask-based prices track the observed prices quite closely, they, together with the observed prices, are higher than fair equilibrium prices.

CONCLUSIONS

This article presents and tests a model that provides an equilibrium price for the CBOT T-bond futures contract. The model explicitly incorporates all delivery options in the equilibrium pricing. The inclusion of all the delivery options brings the model closer to reality than previous models, which ignore one or more of these important features. This realism is gained at some cost. Due to the computational and analytical difficulties in considering all the options simultaneously, the estimated equilibrium futures prices are upward biased. Nonetheless, based on extensive empirical experiments, the bias appears to be small, probably smaller than the bid-ask spread in the spot prices of the deliverable bonds.

Several experiments are then performed to examine the usefulness of the model prices. It is found that the observed price corresponds closely to the ask-based prices despite the fact that the ask-based prices are expected to be the most severely biased. It is found that the bid-based prices provide a better estimate of the equilibrium prices than do the observed market prices. It is found also that trading rules based on the presumption that the bid-based estimates are the true equilibrium prices are able to produce positive profits. This demonstrates that, despite the simplifying assumptions required to make the analysis tractable, the methods employed are a meaningful tool for economic analysis and provide useful estimates of equilibrium futures prices.

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