Pricing Stock Index Futures with Stochastic Interest Rates

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INTRODUCTION

One of the major financial innovations of the last decade was the introduction of trading in stock index futures. Within three years of their introduction in 1981, the dollar value of index futures trading exceeded the dollar value of traditional equity trading, and by October 1987, accounted for 150–200% of trading on the NYSE. This spectacular growth prompted many researchers to take a close look at the pricing relationship between the futures contracts and the underlying index. Because of the predominance of the S&P 500 futures contracts among the different index futures, most of the research has been directed at this contract.

It is generally agreed that the linkage in prices between the S&P 500 index and the S&P 500 futures is maintained by arbitrage. If interest rates and dividend rates are nonstochastic, then, in the absence of taxes and other market imperfections, the no-arbitrage futures price at time t is given by:

$$F(t,T) = S_t \exp[(r - \delta)(T - t)] \tag{1}$$

where: $F(t,T)$ is the futures price; S_t is the price of the index at time t; r is the risk-free (nonstochastic) interest rate between t and T, we shall refer to $T - t$ as τ; δ is the dividend yield on the index; and T is the maturity of the futures contract.

Equation (1) is the standard cost-of-carry model, and reflects the equivalence between the forward and futures prices when interest rate is nonstochastic. Since the market interest rates historically exceed the dividend rate on common stocks, one might except that the futures price given in eq. (1) will be at a premium compared to the price of the index.

Surprisingly, observations from earlier contracts show that when futures prices are mispriced with respect to eq. (1), such mispricing is usually negative, i.e., the futures sell at a discount from the model price of eq. (1). This mispricing is described as a "temporary phenomenon" by Figlewski (1984) who attributes this tem-

This research was partly funded by a grant from the Coordinating Council of Business Studies at Rutgers University. We gratefully acknowledge the superb research assistance of Steve Alessandrini, and the comments of two anonymous referees. This paper was presented at the 1990 meeting of the Northern Finance Association at Banf, and we have greatly benefited from the comments of Peter Carr who was the discussant for the paper. All remaining errors are our own.

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The Journal of Futures Markets, Vol. 11, No. 4, 441–452 (1991)

CCC 0270-7314/91/040441-12$04.00

porary disequilibrium to the lack of knowledge about the workings of these markets, especially the convention of "marking to market." Figlewski's explanation is supported by Billingsley and Chance (1988).

Modest and Sundaresan (1983) argue that arbitrageurs who sell the index short will rarely obtain the full proceeds of the short-sale; and that the actual futures price has to be at a sufficient discount before arbitrageurs will initiate trades. As pointed out by Figlewski (1984), this may explain why a discount is sustained, but it does not explain why a discount came about in the first place. If the actual futures price is higher than the theoretical price then an arbitrageur would buy the index and sell the futures until equilibrium is achieved. In this case, the problems of short-sales would not apply.

Cornell and French (1983a,b) attribute the discount largely to a tax effect that Constantinides (1983) calls the "timing option." As for the stock index, investors have a valuable tax option because they can reduce their taxes by realizing capital losses and deferring capital gains. However, this tax option is not available on the futures. First, all gains and losses on the futures trades, whether realized or not, are taxable at year end. Second, futures have a fixed maturity. Finally, because index futures are cash settlement instruments, taxes cannot be deferred by taking delivery. Because of these distinctions in the tax law, Cornell and French argue that index futures should not be priced relative to the underlying basket of stocks, but relative to what is called "the restricted security." However, the empirical results of Cornell (1985) do not support this tax hypothesis. Cornell (1985) concludes that "it appears that as the market has matured, actual futures prices and predictions of the perfect-markets-model prices have converged." He goes on to explain this development by suggesting that tax-exempt institutions may now be the marginal investors and that the tax option may be ignored when pricing stock-index futures contracts.

Bhatt and Cakici (1990) not only confirm the gradual disappearance of the earlier discount, but also point out the emergence of a premium. Using daily closing values of the S&P 500 index and two nearest maturity futures contracts for 1982–1987, Bhatt and Cakici show that: (1) the percentage of mispricing is small, but positive; (2) the percentage error is larger in absolute value, the longer the maturity; (3) futures prices become more efficient relative to the model price of eq. (1) as the futures market has matured; (4) more contracts sell at a premium than at a discount; and (5) the mispricing is positively and significantly related to time-to-maturity and dividend yield, rather than being random. The evidence that "mispricing" increases on average with maturity is consistent with the findings of MacKinlay and Ramaswamy (1988). It should be emphasized again that, in all these studies, the mispricing is measured in relation to the model price of eq. (1) which assumes a nonstochastic interest rate.

MacKinlay and Ramaswamy (1988) also document that the variability of futures price changes exceeds the variability of changes in the spot price of the S&P 500 index, even after controlling for the nonsynchronous prices in the index quotes. (This finding is in contrast to Cornell (1985) who finds variability largely equal, except for the September 1982 contract, where the variability of the futures price changes exceeded that of the index by about 25%). If the interest rate and dividend yield are nonstochastic, then the variability of the two price-change series should be equal.

Although it is well recognized in the literature that eq. (1) is misspecified in the presence of stochastic interest rates, simulations and empirical studies by Rendelman and Carabini (1979); Cornell and Reinganum (1981); and Elton, Gruber, and Rentzler (1984) show that the effect of stochastic interest rates is economically insignificant for foreign currency and T-bill futures. However, French (1983) shows that the effect is significant for copper and silver futures. Although no direct empirical tests are available for the S&P 500 futures contracts, Cornell (1985) suggests (see his footnote 1) that the effect is not likely to be significant for S&P 500 futures as well, due to the low correlation between the index price changes and the changes in the short-term interest rates. Bailey (1989) compares the pricing models with stochastic and nonstochastic interest rates for the Japanese stock index futures, and concludes that the more complex model provides no improvement over the simpler model of eq. (1). Bailey attributes this finding to the low variance of Japanese interest rates.

The results reported in this article show, however, that the stochastic interest rate model may give significantly different and substantially better "prices" compared to the nonstochastic interest rate model, even when the correlation coefficient (ρ) is zero. This happens because there are two other important factors in the stochastic rate model. One factor has to do with the difference between the current spot interest rate and its long-term asymptotic mean. The other factor is the "adjustment factor" which accounts for the speed with which the current interest rate adjusts toward the long-term mean. It is found that the stochastic interest rate model is particularly suitable when the adjustment factor is large and when the current spot interest rate is very far from the long-term mean. This issue is explored first by assuming that the correlation is zero, using an explicit closed-form solution for the futures price which is given in Ramaswamy and Sundaresan (1985). This allows explicit comparison of the actual prices to a model price when interest rates are stochastic and $\rho = 0$. The effects of the other factors for both zero and nonzero correlation are reported in the simulation results.

Models of stochastic interest rates seem to do significantly better for certain kinds of contingent claims. For example, Ramaswamy and Sundaresan (1985) argue that mispricing of options on S&P 500 futures can be better explained by stochastic interest rate models. Bailey (1987) provides supporting evidence for options on gold futures. This study attempts to apply stochastic interest rate models to the S&P 500 futures data. In light of the comment in MacKinlay and Ramaswamy, who argue for the need to study alternate stochastic processes, and, in light of Bailey's recent work on Japanese index futures, this article reexamines the futures "mispricing" series when the benchmark theoretical price is changed from eq. (1) to a model price that admits stochastic interest rates. Following Bailey, the continuous time dynamics of interest rates are modeled as a "square root" process, as presented in Ramaswamy and Sundaresan (1985). However, contrary to Bailey's findings on Japanese index futures, it is found that the percentage of errors are systematically smaller with the stochastic interest rate model. This difference is generally small, but is quite pronounced in 1986 and 1987 when the spot interest rate is significantly lower than recent historical trends. This confirms the conclusions from the simulation results.

The simulation exercise examines two versions of the stochastic interest rate model. In one version, the standard deviation of interest rate change is proportional to the square root of the spot interest rate. In the other version, the standard deviation is proportional to the spot interest rate itself. Throughout this article, the first version is called the "square root process," and the second version is called the "lognormal process."

DATA AND METHODOLOGY

Data

The futures and stock index data, both obtained from the Chicago Mercantile Exchange, are daily closing values from April 21, 1982 through June 19, 1987. The stock index closing values represent the value of the S&P 500 index at closing, which is 4:00 p.m. EST. The futures closing value is the nearest and next-to-nearest maturity at closing of the Chicago Mercantile Exchange, which is 4:15 p.m. EST. It is not expected that this 15-minute difference in closing time significantly affects the results. It is assumed that the difference in closing times introduces a "white noise," with no systematic bias. The nearest maturity three-month S&P 500 futures has a maturity from 1–90 days, and the next-to-nearest maturity six-month index futures has a maturity from 91–180 days.

The annualized daily dividend yield (i.e., the dividend yield on the basket of stocks for the number of days to maturity remaining on the futures contract) is calculated from the dividend yield on the value-weighted NYSE and AMEX index obtained from the CRSP tape at the University of Chicago. While it would be preferable to use the dividend yield on the S&P 500 index, the latter is not easily available. Moreover, other authors (e.g., MacKinlay and Ramaswamy (1988)) made a similar substitution calculated as follows:

$$\delta = [\Pi(1 + d_i)]^{252/N} - 1 \tag{2}$$

where N is the remaining days to maturity and d_i is the daily dividend yield obtained from the CRSP tape. The daily dividend yield is the actual rather than the expected yield. This presumes perfect foresight and is adopted by a number of previous researchers, including MacKinlay and Ramaswamy.

The one-month T-bill rate is used as a measure of the riskless rate. It would be theoretically correct to use a T-bill whose maturity matches the maturity of the futures contract. However, for the time frames considered, the various short-term rates are likely to be fairly similar, and the discrepancy created by the one-month T-bill should be relatively insignificant.

METHODOLOGY

It is assumed that the S&P 500 index (S) satisfies the following diffusion process:

$$dS = (\alpha - \delta)Sdt + \sigma_1 Sdz_1 \tag{3}$$

where the parameters α and σ_1 represent the drift and volatility of the process, δ is the dividend yield of the index, and dz_1 is a standard Weiner process.

It is assumed also that the instantaneous riskless interest rate (r) follows a mean reverting "square root" process:

$$dr = \kappa(\mu - r)\,dt + \sigma_2\sqrt{r}\,dz_2 \tag{4}$$

where κ, μ, and σ_2 represent the speed of adjustment, long-run mean, and volatility of the stochastic process, and dz_2 is a standard Weiner process. The covariance between dz_1 and dz_2 is ρdt. If the Local Expectations Hypothesis holds, then the following partial differential equation describes the dynamics of futures price:

$$(1/2)\,\sigma_2^2 rF_{rr} + (1/2)\,\sigma_1^2 S^2 F_{ss} + \rho\sigma_1\sigma_2 S\sqrt{r}F_{rs} + \kappa(\mu - r)F_r + (r - \delta)SF_s = F_\tau \tag{5}$$

Equation (5) above does not, in general, have a closed-form solution, and has to be solved numerically. However, when $\rho = 0$, a closed-form solution is available from Ramaswamy and Sundaresan (1985), and is given by:

$$F = Sa(\tau)\exp[b(\tau)r] \tag{6}$$

where:

$$a(\tau) = \left\{\frac{2\gamma\exp[(\gamma+\kappa)\tau/2}{2\gamma+(\gamma+\kappa)(\exp(\gamma\tau)-1)}\right\}^{2\kappa\mu/\sigma_2{}^2}[\exp\{-\delta\tau\}]$$

$$b(\tau) = \frac{2(\exp\{\gamma\tau\}-1)}{2\gamma+(\gamma+\kappa)(\exp\{\gamma\tau\}-1)}$$

and $\gamma = \sqrt{[\kappa^2 - 2\sigma_2^2]} > 0$ (by assumption).

When ρ is nonzero, the line hopscotch method, a finite difference method presented in Gordon (1965), and Gourlay and McKee (1977) can be used to numerically solve the partial difference equation with appropriate boundary conditions. In the case of futures, this boundary condition is given by the convergence of the futures price to the spot price at maturity.

Two different strategies are adopted to implement the empirical verification of the above model. In the first tests, it is assumed that ρ is zero, and the equation set (6) is employed to estimate the parameters κ, μ, and σ_2. It is assumed first that the futures price F given by eq. (6) differs from the observed price by a mean zero error. A nonlinear least squares method is employed then to estimate the three parameters κ, μ, and σ_2 (a similar procedure is adopted by Brown and Dybvig (1986)). If the error term has a normal distribution, then these estimates are also the maximum likelihood estimates. With these estimates in hand, the in-sample mispricing of eq. (1) is compared to that of eq. (6). Tables I and II report these results.

In the second set of tests, extensive simulations of the futures price in the stochastic interest rate model are presented by assuming different plausible values of κ, μ, ρ, σ_1, and σ_2. Although the "square root" model is the predominant model in previous theoretical and empirical research, some researchers (such as Marsh and Rosenfeld (1983)) examine the empirical relevance of other plausible models that belong to the family of "constant elasticity of variance" models. Marsh and

Table I

OVERALL SAMPLE AND YEAR-BY-YEAR MISPRICING IN CONSTANT INTEREST RATE AND STOCHASTIC INTEREST RATE MODELS

		Constant Interest		Stochastic Interest		t-Value for
Sample	No. of Obs.	Mean	SD	Mean	SD	Diff. in Means
1982–87	2504	0.3101	0.74	0.2522	0.7358	2.776
1982	347	−0.1348	1.09	−0.1731	1.08	−0.465
1983	494	0.1430	0.63	0.0933	0.64	1.230
1984	496	0.7360	0.60	0.6967	0.586	1.044
1985	496	0.6100	0.69	0.5421	0.66	1.584
1986	500	0.1400	0.45	0.0571	0.4465	2.924
1987	171	0.0770	0.28	0.0143	0.2785	2.076

Table II
REPLICATION OF TABLE I AFTER ELIMINATING THOSE FUTURES PRICES THAT HAVE LESS THAN 15 CALENDAR DAYS TO MATURITY

Sample	No. of Obs.	Constant Interest Mean	Constant Interest SD	Stochastic Interest Mean	Stochastic Interest SD	t-Value for Diff. in Means
1982–87	2296	0.3400	0.76	0.2769	0.75	2.832
1982	318	−0.1260	1.12	−0.1680	1.12	−0.473
1983	455	0.1500	0.65	0.0900	0.66	1.382
1984	456	0.7900	0.59	0.7500	0.58	1.032
1985	456	0.6600	0.69	0.5900	0.66	1.566
1986	460	0.1600	0.46	0.0700	0.45	2.999
1987	151	0.0890	0.29	0.0180	0.29	2.127

Rosenfeld report that the "lognormal" model gives a better statistical fit than the "square root" model for the interest rate sample studied by them. Because of these findings, an examination of the "square root" model as well as the "lognormal" model are undertaken. The "lognormal" model is given by the following equation:

$$dr = \kappa(\mu - r)dt + \sigma_2 r\, dz_2 \tag{7}$$

In this case, the futures price dynamics is given by the following partial differential equation:

$$(1/2)\,\sigma_2^2 rF_{rr} + (1/2)\sigma_1^2 S^2 F_{ss} + \rho\sigma_1\sigma_2 SrF_{rs} + \kappa(\mu - r)F_r + (r - \delta)SF_s = F_\tau \tag{8}$$

The line hopscotch method is used for the solution of both eqs. (5) and (8). These prices are compared with the constant interest rate prices of eq. (1) in Tables IV to VII.

RESULTS

Parameter Estimation and Testing of the Square Root Model

As explained before, the parameters of eq. (6) are estimated by assuming that actual prices differ from eq. (6) by a mean zero error, and by applying the method of nonlinear least squares.[1] Based on the entire sample of 2504 observations, the final estimates are $\kappa = 0.3682$, $\sigma_2 = 0.0012$, and $\mu = 0.1171$.[2]

These estimates are used then to derive model prices in eq. (6). Mispricing is defined as [actual price − model price] × 100/(model price). Table I reports the mispricing for the entire sample, as well as for years 1982–1987, separately, for both the stochastic interest rate model of eq. (6) and the constant interest rate model of eq. (1). The statistical significance of the differences between the two models is reported by the t-values in the last column. The t-values show that, for the years 1982–1985, the two models are statistically indistinguishable from each other. But

[1]A convergence criterion of 0.000001 and starting values of $\kappa = 0.4$, $\sigma_2 = 0.10$, and $\mu = 0.10$ are employed.

[2]The parameters on year-by-year subsamples are estimated as well. This gives a better statistical fit for the subsamples as compared to the statistical fit of the total sample. But the parameter estimates are substantially different from one subsample to another. This instability in the parameter estimates is consistent with the findings of other researchers in this field, for example, Brown and Dybvig (1986). Because of this instability, the model is not tested on a separate hold-out sample.

in 1986 and 1987, as well as for the entire sample, the square root model does significantly better. This is surprising in light of the fact that the correlation ρ in all of these years is set equal to zero and, therefore, the difference in results cannot be attributed to a difference in ρ. The estimation and the computation of the mispricing series are replicated after dropping all those contracts that have less than 15 days to maturity. Table II reports these mispricing values and shows that the results are virtually the same as those reported in Table I.

Previous researchers (Figlewski (1984), Bhatt and Cakici (1990)) have examined whether the mispricing of the S&P 500 futures relative to the model price of eq. (1) is systematically related to the dividend yield of S&P 500 and to the maturity of the futures contract. The dividend yield offsets the interest rate and may be viewed as a part of the "cost of carry." These aspects are examined here by regressing the mispricing series generated by eq. (6) against time to maturity, dividend yield, and level of interest rate. The regression equation is similar to that reported in Bhatt and Cakici (1990), except that the level of interest rate is included as an additional independent variable:

$$z_t = \beta_0 + \beta_1 \delta + \beta_2 \tau + \beta_3 r$$

where z_t is the absolute value of percentage of mispricing, δ is the dividend yield described above, τ is the time to maturity for the futures contract, and r is the interest rate. Since there is serial correlation in the errors of this regression model, the regressions are corrected for serial correlation. The regression results are reported in part A of Table III. For contracts with τ less than three months, the R^2 is 0.4088; for contracts with τ between three and six months, the R^2 is 0.5995. Since the mispricing is significantly lower in 1986 and 1987, separate regressions can be run for these years.

Table III shows that, for contracts with τ less than three months, β_0, β_1, β_2, and β_3 are all statistically significant at the 5% level. Thus, for contracts maturing within three months, the mispricing is systematically related to dividend yield, maturity, and level of interest rate. For contracts with τ between three and six months, the level of interest rate is not statistically significant. The R^2 values for the two maturity subsamples are comparable to the R^2 values reported in Bhatt and Cakici (1990).

The relationship is much less pronounced for the subsamples for 1986 and 1987, where the R^2 is also comparatively low. This is understandable, because the mispricing in these two years is comparatively much less.

The autocorrelation in the mispricing series for different lags ranging from 1 to 12 days is reported in part B of Table III. It is found that the autocorrelation gradually reduces as the lag increases; but even at a 12-day lag, the autocorrelation is statistically significant. This suggests the persistence of the mispricing.[3]

Simulation Results

In discussing Tables I and II, the superiority of the stochastic interest rate model for the subsamples of 1986 and 1987 can not be attributed to the correlation coefficient ρ, because ρ is set equal to zero for the other years as well. This result suggests that the suitability of the stochastic interest rate model may depend on factors other than the correlation coefficient. Simulations are designed to explore the impact of

[3]This is similar to the autocorrelation in levels of mispricing reported by MacKinlay and Ramaswamy (1988).

Table III

$z_t = \beta_0 + \beta_1\delta + B_2\tau + \beta_3 r$

PART A. WHERE: z_t = ABSOLUTE VALUE IN PERCENTAGE MISPRICING, δ = DIVIDEND YIELD, τ = MATURITY OF THE FUTURES CONTRACT, AND r = LEVEL OF INTEREST RATE (NUMBERS IN PARENTHESIS DENOTE t-VALUES)

	τ = 0 to 3 Mos.			τ = 3 to 6 Mos.		
	All Years (N = 1273)	1986 (N = 248)	1987 (N = 116)	All Years (N = 1231)	1986 (N = 252)	1987 (N = 55)
β_0	−0.2682	0.2734	−0.2764	−0.4239	−0.1287	−0.7832
	(−3.249)	(2.062)	(−1.197)	(−1.599)	(−0.383)	(−1.394)
β_1	5.0585	0.0325	3.0182	16.3657	23.3323	4.7139
	(3.841)	(0.019)	(0.990)	(2.411)	(2.770)	(0.272)
β_2	1.3683	0.4991	0.6777	0.5660	−0.5040	0.8848
	(5.769)	(1.759)	(1.849)	(1.741)	(−1.362)	(1.851)
β_3	3.9754	−1.0532	6.5696	2.7347	−2.5247	9.6762
	(3.396)	(−0.460)	(1.559)	(0.983)	(−0.633)	(1.314)
R^2	0.4088	0.0578	0.1440	0.5995	0.2370	0.1128

PART B. AUTOCORRELATIONS IN z_t

Lag	Maturity: 0 to 3 Mos.	Maturity: 3 to 6 Mos.
1 day	0.67	0.86
2 days	0.61	0.83
3 days	0.58	0.80
4 days	0.53	0.78
5 days	0.49	0.76
6 days	0.44	0.73
7 days	0.42	0.71
8 days	0.38	0.70
9 days	0.41	0.70
10 days	0.34	0.67
11 days	0.30	0.65
12 days	0.28	0.63

the different parameters of the stochastic processes and to examine the sensitivity of the results to the specification of the stochastic process. These simulation results are reported in Tables IV to VII.

Table IV compares the pricing errors for the two stochastic interest rate models to the pricing error of the constant interest rate model for different values of the spot interest rate and two extreme values of κ. The correlation coefficient, ρ, is assumed to be zero, and μ is assumed to be 10%. It is assumed that δ is 5%, σ_1 is 25%, and σ_2 is 9%. Two conclusions can be drawn from the results in Table IV. (1) For every level of spot interest rate, the prices given by the two stochastic models are virtually identical; and (2) the stochastic interest rate models give significantly different prices, compared to the constant rate model, only when the spot interest rate is either very low or very high compared to the long-term mean and when the adjustment factor κ is high.

Table IV
SIMULATION COMPARISON OF ERRORS FOR DIFFERENT VALUES OF κ IN THE SQUARE ROOT AND THE LOGNORMAL STOCHASTIC INTEREST RATE MODELS

	$\kappa = 0.25$		$\kappa = 2.0$	
Spot Rate	**Square Root**	**Lognormal**	**Square Root**	**Lognormal**
7%	0.0231	0.0229	0.1600	0.1600
8%	0.0155	0.0154	0.1067	0.1066
9%	0.0078	0.0077	0.0535	0.0533
10%	0.0002	0.0001	0.0002	0.0001
11%	−0.0074	−0.0076	−0.0531	−0.0532
12%	−0.0150	−0.0152	−0.1063	−0.1064
13%	−0.0226	−0.0229	−0.1594	−0.1596

Error = Futures price given by stochastic model *minus* futures price by contract interest rate model.
Assumptions: $\mu = 10\%$, $\delta = 5\%$, $\sigma_1 = 0.25$, $\sigma_2 = 0.09$, $\rho = 0$, and $\tau = 3$ mos.

The first result suggests that the pricing of stock index futures is not very sensitive to the specification of the stochastic process. Therefore, although the stochastic model may give substantially different results in some circumstances, the exact specification of the model is a less important factor.

The second conclusion provides an explanation for the earlier evidence regarding the superior performance of the stochastic interest rate model in 1986 and 1987. The evidence in Table IV suggests that an unusually low or unusually high current interest rate makes the stochastic model a better candidate for futures pricing. In fact, the spot interest rates in 1986 and 1987 are significantly lower compared to the recent historical trend. This confirms the simulation results in Table IV.

The evidence from Table IV is also very important because previous researchers (e.g., Cornell (1985)) have identified ρ as the decisive factor in distinguishing between the stochastic and constant interest rate models. These researchers have advanced the view that if ρ is close to zero, then the two models should give fairly identical results. But, Table IV assumes that ρ equals zero, and therefore one can conclude that the more important factors are the level of the current interest rate in comparison to the long-term mean and the adjustment factor κ. To further examine the interaction between κ and ρ, a simulation of the futures prices for different combinations of κ and ρ are undertaken. These results are reported in Tables V, VI, and VII. The futures prices are computed by assuming a spot index price of 100.00. It is seen from Table V that when κ is zero, the prices given by the square root stochastic model are virtually identical to the prices of the constant interest rate model even for different values of ρ. The economic interpretation of $\kappa = 0$ is that there is no mean-reversion in interest rates; however, the prices given by the stochastic model start diverging from the constant interest rate model as the value of κ increases. When κ is 2.0, a larger difference, as compared to $\kappa = 0.25$, is obtained. However, κ is not the only determining factor. Even when $\kappa = 2.0$, the stochastic model is virtually identical to the constant interest rate model when the spot rate is equal to the long-term mean of 10%. For any given value of κ and ρ, the stochastic model's difference from the constant interest rate model increases as the current spot rate moves away from the long-term asymptotic mean interest rate. On the other hand, for any given spot rate and κ, the prices do not seem to be sensitive to ρ, although higher values of ρ give slightly higher futures prices.

Table V

SIMULATED FUTURES PRICES IN THE SQUARE ROOT MODEL FOR DIFFERENT VALUES OF SPOT RATE, κ AND ρ: MATURITY 3 MONTHS

Spot rate	7%	8%	9%	10%	11%	12%	13%
Constant interest rate model:	100.5013	100.7528	101.0050	101.2578	101.5113	101.7654	102.0201
Stochastic interest rate model:							
$\kappa = 0.0$							
$\rho = 0.0$	100.5014	100.7530	101.0052	101.2581	101.5115	101.7657	102.0204
$\rho = -0.5$	100.4921	100.7430	100.9946	101.2468	101.4997	101.7533	102.0075
$\rho = 0.5$	100.5108	100.7630	101.0159	101.2693	101.5234	101.7781	102.0333
$\kappa = 0.25$							
$\rho = 0.0$	100.5245	100.7684	101.0129	101.2580	101.5038	101.7501	101.9970
$\rho = -0.5$	100.5153	100.7586	101.0025	101.2470	101.4922	101.7380	101.9844
$\rho = 0.5$	100.5337	100.7782	101.0234	101.2691	101.5154	101.7622	102.0096
$\kappa = 2.0$							
$\rho = 0.0$	100.6621	100.8603	101.0590	101.2580	101.4574	101.6572	101.8575
$\rho = -0.5$	100.6539	100.8516	101.0498	101.2484	101.4474	101.6468	101.8467
$\rho = 0.5$	100.6703	100.8690	101.0681	101.2676	101.4674	101.6677	101.8683

Assumptions: $\mu = 10\%$, $\delta = 5\%$, $\sigma_1 = 0.25$, and $\sigma_2 = 0.09$.

Table VI

SIMULATED FUTURES PRICES IN THE SQUARE ROOT MODEL FOR DIFFERENT VALUES OF SPOT RATE, κ AND ρ: MATURITY 6 MONTHS

Spot rate	7%	8%	9%	10%	11%	12%	13%
Constant interest rate model:	101.0050	101.5113	102.0201	102.5315	103.0455	103.5620	104.0811
Stochastic interest rate model:							
$\kappa = 0.25$							
$\rho = 0.0$	101.0970	101.5734	102.0521	102.5330	103.0162	103.5016	103.9894
$\rho = -0.5$	101.0608	101.5347	102.0109	102.4895	102.9704	103.4537	103.9393
$\rho = 0.5$	101.1334	101.6124	102.0934	102.5767	103.0601	103.5498	104.0396
$\kappa = 2.0$							
$\rho = 0.0$	101.5654	101.8861	102.2087	102.5323	102.8570	103.1826	103.5094
$\rho = -0.5$	101.5352	101.8554	102.1766	102.4989	102.8223	103.1467	103.4723
$\rho = 0.5$	101.5939	101.9169	102.2409	102.5658	102.8918	103.2187	103.5467

Assumptions: $\mu = 10\%$, $\delta = 5\%$, $\sigma_1 = 0.25$, and $\sigma_2 = 0.09$.

The implication of the combined impact of κ and the level of spot interest rate is that the mean-reversion in the interest rate process is more important than the correlation of the interest rate process and the spot index process. As mentioned previously, this implication is different from the view advanced by other researchers.

Table VII
SIMULATED FUTURES PRICES IN THE LOGNORMAL MODEL FOR DIFFERENT VALUES OF SPOT RATE, κ AND ρ: MATURITY 3 MONTHS

Spot rate	7%	8%	9%	10%	11%	12%	13%
Constant interest rate model:	100.5013	100.7528	101.0050	101.2578	101.5113	101.7654	102.0201
Stochastic interest rate model:							
$\kappa = 0.25$							
$\rho = 0.0$	100.5243	100.8602	101.0588	101.2579	101.4573	101.6571	101.8573
$\rho = -0.5$	100.5219	100.8577	101.0561	101.2548	101.4540	101.6535	101.8534
$\rho = 0.5$	100.5268	100.8627	101.0616	101.2609	101.4606	101.6606	101.8611
$\kappa = 2.0$							
$\rho = 0.0$	100.6621	100.7683	101.0128	101.2579	101.5036	101.7499	101.9968
$\rho = -0.5$	100.6598	100.7655	101.0096	101.2544	101.4997	101.7457	101.9922
$\rho = 0.5$	100.6643	100.7710	101.0159	101.2614	101.5074	101.7541	102.0013

Assumptions: $\mu = 10\%$, $\delta = 5\%$, $\sigma_1 = 0.25$, and $\sigma_2 = 0.09$.

Table VI replicates the results of Table V for a maturity of six months for $\kappa = 0.25$ and 2.0. The results are identical to those of Table V. Table VII replicates these results for the lognormal model and shows, once again, that the results are virtually identical.

SUMMARY

This article compares the constant interest rate model of futures pricing to some alternative stochastic models of interest rates. This comparison is carried out for the "square root" stochastic process on a data set comprising the daily closing values of S&P 500 futures from April 21, 1982 to June 19, 1987. The stochastic process performs better for the total sample as well as for 1986 and 1987 when the analysis is carried out on a year-by-year basis, although there is no significant difference between the stochastic and the constant interest rate models for the years 1982–1985. Simulations are used as a means of further comparison. The results indicate that the stochastic model gives significantly better results when the spot interest rate is far away from the long-term mean, or when the parameter accounting for the speed of adjustment toward this long-term mean is very high. Taken together, this suggests that the superiority of the stochastic model depends on the impact of the mean-reversion factor. The correlation between the interest rate process and the spot index process does not appear to have any significant impact. The results also indicate that the superiority of the stochastic model is not sensitive to the exact specification of the model: virtually identical results are obtained for the square root model and the lognormal model.

Perhaps stochastic interest rate models have not been applied to study the S&P 500 futures before due to the earlier notion that the stochastic interest rate model might not give very different results as compared to the constant interest rate model where ρ is typically close to zero. This article demonstrates that ρ is not the most important parameter in this regard. Rather, the level of the spot interest rate and the adjustment factor, κ, are the most important factors in deciding about the applicability of the stochastic interest rate models. It is hoped that this demonstration will encourage practitioners and academicians to take a closer look at the applicability of the stochastic interest rate model for the pricing of S&P 500 and other index futures.

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