A Note on the Role of Futures Indivisibility: Reconciling the Theoretical Literature

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INTRODUCTION

In a seminal treatment, Cox, Ingersoll, and Ross, hereafter CIR (1981), derive a number of testable implications regarding forward and futures pricing under conditions of uncertainty. CIR maintain that, in the presence of futures resettlement, stochastic interest rates are a sufficient condition to render price divergences between forward and futures contracts. In the absence of uncertainty regarding the time path of stochastic interest rates, forward and futures prices will be the same; a conclusion reached by Black (1976) as well. CIR note that Margrabe (1976), Jarrow and Oldfield (1981), Richard and Sundaresan (1981), and French (1982) also analyze the influence of stochastic rates of interest on the pricing of derivative instruments, and that their (CIR's) article represents a consolidation of a number of results arising from these independent studies.

Recently, Levy (1989) extends the CIR analysis to demonstrate that all future interest rates need not be known for futures and forward prices to be equal. It is sufficient to know with certainty only the next day's interest rate. Acknowledging that this is not possible, Levy maintains, nevertheless, that the variance associated with forecasting the next day's interest rate should be small enough to render forward-futures price differentials to the realm of empirical inconsequence.

On the other hand, Morgan (1981) asserts that forward and futures price differences "must" occur; and, the current futures price is dependent on the future time path of stochastic interest and futures rates. Either uncertain interest or futures movements alone are sufficient to generate spreads between forward and futures prices; but, the equilibrium price of a futures contract will be jointly influenced by both conditions.

The CIR, Levy, and Morgan analyses deal primarily with the pricing effects of uncertain futures payouts. Numerous market imperfections are not considered. In this regard, Kane (1980) discusses the role of differential taxes and contract performance guarantees as potential contributors to forward-futures price divergence.[1]

[1]Additionally, segmented markets (Branch (1978)), a default risk premium in the futures contract (Lang and Rasche (1978)), asymmetric execution costs (Allen and Thurston (1988) Cappoza and Cornell (1979)), and differential liquidity and default risk premia (Kamara (1988)) have been cited in empirical studies as contributors to significant Treasury-bill forward-futures price differentials.

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Nevertheless, the institutional issues brought forth by Kane and others do not alter the lack of a theoretical consensus as to whether stochastic interest rates, stochastic futures prices, or both stochastic interest rates and futures prices should affect futures rates. This article contends that distinct conclusions regarding this issue can be reconciled by focusing on a single underlying market imperfection, the indivisibility of futures contracts.

To highlight the implications associated with futures indivisibility, Levy's analysis needs to be reconsidered. While his extension of the CIR framework might be considered correct, both analyses operate under the assumption that futures contracts are divisible. In other words, traders can position fractions of futures contracts. This is an entirely reasonable treatment in view of the perfect capital markets premise employed in these articles. A review of the implications associated with that approach follows.

THE EFFECT OF FUTURES DIVISIBILITY

Using Levy's notation,

Let F_t = the price, at time t, of a futures contract that matures at time n ($n > t$)
G_t = the price, at time t, of a forward contract that matures at time n ($n > t$)
B_t = the price, at time t, of a pure discount bond that pays \$1 at time n
S_t = the price, at time t, of the assets on which the contract is written
d_t = the futures position taken at time t, as a percentage of 1 contract

Levy's three-period model demonstrates conditions under which forward and futures cash flows will be identical. Suppose an investor assumes a long futures position in a contract that has three days remaining until delivery. To obtain the same cash flow that could be achieved using a forward contract, the investor purchases d_0 futures contracts in period 0 and closes this position in period 1. Simultaneously in period 1, d_1 futures contracts are purchased. In period 2, the d_1 position is closed out, and d_2 futures contracts are bought. The d_2 position matures in the next period and is sold into the spot market. The process of simultaneously closing out and opening new positions after period 0 replicates the resettlement feature distinguishing futures contracts. At the end of three periods, the entire cash flow resulting from this strategy is

$$d_0(-F_0 + F_1)/B_1 + d_1(-F_1 + F_2)/B_2 + d_2(-F_2 + S_3) \tag{1}$$
$$= -F_0 d_0/B_1 + F_1[d_0/B_1 - d_1/B_2] + F_2[d_1/B_2 - d_2] + d_2 S_3 \tag{2}$$

In ultimately establishing the independence of futures prices from the time path of interest rates, Levy takes the intermediate step of setting $d_0 = B_1$ and $d_1 = B_2$. It follows from this treatment that futures contracts are divisible, since B_1 and B_2 represent bonds discounted from a face value of \$1. Continuing Levy's analysis, if the next day's interest rate can be known with certainty, B_1 is known at time period zero and B_2 is known at time period one. Accordingly, positions in d_0 and d_1 can be taken at periods zero and one, respectively, that correspond precisely to B_1 and B_2. At the termination of the three-period time horizon, d_2 must equal 1 since B_2 has reached its face value. Substituting B_1 for d_0 and B_2 for d_1, eq. (2) becomes

$$-F_0[B_1/B_1] + F_1[B_1/B_1 - B_2/B_2] + F_2[B_2/B_2 - 1] + 1[S_3] \tag{3}$$
$$= -F_0 + S_3 \tag{4}$$

Equation (4) represents the same cash flow that could be achieved using a forward contract. Therefore, comparable forward and futures prices should be the same. Attributable to Levy (1989), the implication of eqs. (3) and (4) is that knowledge of the next day's bond price would render irrelevant the time path of future interest rates insofar as they affect futures pricing. This result follows from arbitrage and Law of One Price relationships, but it clearly hinges on the condition that futures contracts are divisible.

THE EFFECT OF AND IMPLICATIONS ASSOCIATED WITH FUTURES INDIVISIBILITY

In practice, futures contracts are not divisible.[2] In assuming that markets are efficient, Morgan implicitly treats futures contracts as indivisible. Otherwise, the treatment underlying the CIR, Levy, and Morgan analyses are essentially equivalent. When the divisibility of futures contracts is amended to correspond to actual trading conditions, $d_0 = d_1 = d_2 = 1$, and eq. (3) becomes

$$-F_0/B_1 + F_1/B_1 - F_1/B_2 + F_2/B_2 - F_2 + S_3 \tag{5}$$

Unlike eqs. (3) and (4), the three-period futures cash flow now includes terms F_1, F_2, B_1, and B_2. This would remain the case even if the next day's interest rate could be known with certainty. Thus, when futures contracts are not divisible, the futures cash flow is no longer independent of either the time path of future interest rates or futures prices. Therefore, the price of the futures contract cannot be independent of future stochastic movements in interest rates or futures prices.

This is precisely the conclusion reached by Morgan, whose results follow from an assumption of futures indivisibility. Morgan's modeling step should not be interpreted casually, for the actual practice of contracting indivisibility has not arisen by happenstance. Indeed, it lies at the heart of the theoretical rationale underlying the emergence of futures markets. Indivisibility assists in product standarization, thus aiding in the establishment of market volume, which, in turn, helps lower transacting costs. Telser and Higinbotham (1977) and Silber (1985) view such cost reductions as the primary economic impetus underlying the appearance of futures markets. Therefore, models that do not incorporate the indivisibility feature exclude a condition essential to the competitive viability of futures contracts. When the implications associated with futures indivisibility are considered, it is readily apparent why there are such differing inferences associated with fundamental theoretical treatment of resettlement. Because futures contracts are not divisible and remain so for important theoretical reasons, Morgan's framework remains the most relevant primary basis for empirical testing. The CIR and Levy approaches correspond to that of Morgan when they are amended to reflect futures indivisibility.

[2]Because forward contracts represent a direct transaction between two parties, they can be tailored to meet both participants' needs and must therefore be considered divisible in practice. As a corollary to direct contracting, there is also a direct mutual assessment of credit risk. Hence, forward contracting eliminates the need for the daily cash resettlement feature associated with futures contracts. Black (1976) states that it is resettlement which represents the fundamental distinction between forwards and futures. From an abstract perspective, forward contracts could be modeled as if they were indivisible and resettled daily. This would result in the replication of cash flows associated with indivisible futures contracts. Such treatment, however, would be misspecified on economic grounds and does not ordinarily occur in practice.

That is, they exhibit the influence of stochastic interest and futures rates on futures prices in the manner that Morgan has posited. The pricing role of stochastic interest and futures rates is not compromised by consideration of factors isolated by Kane and others. These factors can be viewed as additional determinants of fundamental, equilibrium forward-futures price relationships.

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