The Relationship Between Forward and Futures Contracts: A Comment

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In a recent article Levy (1989) demonstrates that a sufficient condition for futures prices to equal the corresponding forward prices is that market participants can make perfect forecasts of certain bond prices one day ahead. Even though this is an accurate observation, he is not correct in claiming that this is a significantly weaker condition than interest rates being deterministic, a condition first shown by Black (1976) to imply that futures contracts are priced as if they are forward contracts. This article shows that if investors can, on every day in the future, perfectly forecast the next day's term structure of interest rates, then the preclusion of intertemporal arbitrage opportunities requires that interest rates be deterministic.

The model used here follows Levy in being a simple discrete time, perfect market model where consideration is given only to relative pricing based on replication arguments and a no arbitrage assumption. Let $P(t,s)$ denote the price at time t of a pure discount bond that will pay off \$1 at time s, where time (for convenience of notation) is measured in terms of days from some arbitrary time 0. It is assumed that there are bonds traded of all maturities up to time T which can be taken to be any date beyond the maturity date of all traded futures and forward contracts. Define interest rates to be deterministic if all future bond prices are known with certainty at time 0. The result below demonstrates that Levy's assumption of investors' being able to accurately forecast discount bond prices one day ahead is, in the absence of arbitrage opportunities, equivalent to interest rates being deterministic.

Proposition

Assume that $\forall t \in \{0,1,2,\ldots,T\}$ and $\forall n \in \{t+1,\ldots,T\}$ $P(t+1,n)$ is known at time t, that markets are perfect, and that there exist no arbitrage opportunities.

Then $\forall t \in \{0,1,2,\ldots,T\}$ and $\forall n \in \{t+1,\ldots,T\}$ $P(t,n)$ is known at time 0, and is given by

$$P(t,n) = \frac{P(0,n)}{P(0,t)}. \tag{1}$$

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Proof

To prove the proposition, by induction first notice the trivial fact that for $k = 0$, one has $\forall n \in \{k, k + 1,...,T\}$ that $P(k, n)$ is known at time 0 and given by

$$P(k,n) = \frac{P(0,n)}{P(0,k)}. \tag{2}$$

Next, it will be shown that if (2) holds for some k and if discount bond prices are perfectly forecastable one day ahead, (2) must also hold for $k + 1$ to prevent intertemporal arbitrage.

Thus, *assume* that for some $k \in \{0,1,2,\ldots,T - 1\}$ one has $\forall n \in \{k,k + 1,...,T\}$ that $P(k,n)$ is known at time 0 and given by (2). A bond of any given maturity, $s \in \{k + 1, k + 2,\ldots,T\}$, can be purchased at time k for a price of $P(k,s)$. Also, since its price on the following day, $P(k + 1,s)$, is assumed to be known at time k one can effectively acquire it at that time by investing an amount of $P(k + 1,s)^{*}P(k,k + 1)$ overnight and using the proceeding $P(k + 1,s)$ to buy the bond at time $k + 1$. Since the bond has no intermediate cash flow, and it is assumed that markets are perfect, the no-arbitrage assumption dictates that the time k cash flows for the two strategies be equal, i.e., that

$$P(k,s) = P(k + 1,s)^{*}P(k,k + 1), \tag{3}$$

or rearranging

$$P(k + 1,s) = \frac{P(k,s)}{P(k,k + 1)} \tag{4}$$

Substituting (2) into the numerator and denominator of (4), with $n = s$ and $n = k + 1$, respectively, and simplifying yields:

$$P(k + 1,s) = \frac{P(0,s)}{P(0,k + 1)}. \tag{5}$$

Thus, if the term structure of interest rates at time k is known at time 0 and given by (2), it follows that the term structure of interest rates at time $k + 1$ is known at time 0 and given by (5), which is the exact same relationship. This completes the induction step.

Finally, return to the initial observation that for $k = 0$ the term structure is known at time 0 and given by (2). Thus, by forward induction on k, it follows that this must hold for any $k \in \{0,1,2,\ldots T\}$ and the proof is complete.

Acknowledging that the assumption of perfect one day ahead interest rate forecasts is too strong, Levy nevertheless claims that his result has "practical implications" in that "the variance of the forecast is much smaller than the variance of forecasting all interest rates from the purchase (sale) of the contract to delivery date."

Since he never explicitly models how the resulting forecasting errors affect the investor's cash flow, it is not entirely clear what is meant by this statement. However, it might be interpreted to imply that futures prices arising from idealized contracts that are marked-to-market every instant should be virtually identical to the corresponding forward prices; since one can surely make excellent forecasts of the interest rates to prevail an instant from now. Unfortunately, it appears to be true in general, and is shown explicitly for special cases (see Flesaker (1989)), that the magnitude of the pricing effect of marking futures contracts to the market is monotonically *increasing* in the frequency of the marking-to-market, and thus at its highest when such cash flows are paid every instant.

In conclusion, it is shown that Levy's condition of one day ahead forecastability of interest rates is equivalent to the standard condition of interest rates being deterministic. It is therefore not surprising that he can demonstrate equality between futures and forward prices under such circumstances. Also, it is argued that attempts to make further inferences from his results regarding the effect of marking-to-market when interest rates are stochastic can be misleading.

Bibliography

Black, F. (1976): "The Pricing of Commodity Contracts," *Journal of Financial Economics,* 3.

Flesaker, B. (1989): "Arbitrage Free Pricing of Interest Rate Futures and Forward Contracts," unpublished manuscript, University of California, Berkeley.

Levy, A. (1989): "A Note on the Relationship between Forward and Futures Contracts," *Journal of Futures Markets,* 9 (2):171–173.