Valuation of Residential Mortgage-Backed Securities with Proportional Hazard Model: *Cumulant Expansion Approach to Pricing RMBS*

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The residential mortgage-backed security (RMBS) market is currently a very large segment in fixed income markets. Under this circumstance, the valuation problem of RMBS is important for practitioners. This article develops a pricing formula for not only RMBS, but also interest only (IO) and principal only (PO) with the proportional hazard model.

Usually, RMBS are a pass-through security with monthly payments. Since mortgage contracts composed of RMBS allow the borrowers to prepay the principal at any time prior to maturity without any penalty, the cash flows of RMBS include their prepayment. That is, there is uncertainty in the RMBS cash flows due to prepayment. Almost all practitioners recognize that prepayment risk is the most important issue for RMBS valuation and risk management. On the other hand, in past literature, many researchers focused upon modeling and evaluating prepayment risk. In general, the approaches for modeling prepayment behavior can be classified into two categories: the structural approach and the intensity-based approach.

In the structural approach, it is assumed that the borrowers behave rationally and exercise their optimal prepayment strategy. Since this strategy can be seen as the early exercise problem of the American option, it is also well-known as the option-based approach. However, due to the existence of various reasons for prepayment, the situation of RMBS is more complicated than that of the American option. The structural approach was pioneered by Dunn and McConnell [1981a, 1981b]. Since their studies, much literature dealing with the structural approach has been published. For example; see McConnell and Singh [1994], Stanton [1995], Kariya and Kobayashi [2000], Nakamura [2001], and Nakagawa and Shouda [2004]. Recently, Longstaff [2005] and Pliska [2005, 2006] have conducted research on sequential refinancing and have considered multi-stage decision models.

Instead of much energetic research on the structural approach, it seems that practitioners in the RMBS market hesitate to employ practical applications of this approach. Several reasons exist: Evaluating RMBS using the structural approach is computationally demanding and extremely time-consuming. Apart from the computational difficulties, the outputs of the structural approach do not completely match with time-series prepayment data and market prices of RMBS. In addition, the assumption of the optimal prepayment strategy is often violated in market practice. This is because in actuality there are many irrational borrowers who do not fit the assumption of the structural approach.

In the intensity-based approach, it is assumed that timing of prepayment is a random time governed by some hazard rate processes.

Thus, a mortgage prepayment event is regarded as a default in the credit risk modeling (e.g., the monograph on the intensity-based credit risk modeling by Bielecki and Rutkowski [2002]). However, in contrast to credit risk modeling, research on the theoretical and mathematical foundations of prepayment modeling in the intensity-based framework is limited. The notable references include Pliska [2005, 2006] and Sugimura [2004].

Conversely, there are a large number of empirical studies based on the intensity-based approach. In the empirical intensity-based approach, a certain prepayment hazard rate function is statistically estimated from historical prepayment data. In particular, the proportional hazard model is frequently applied to describe prepayment behavior both academically and in practice, and its estimation methods have been established. For example, see Schwartz and Torous [1989], Aonuma and Kijima [1998], Ichijo and Moridaira [2001], and Sugimura [2002] for studies on empirical prepayment analysis by the proportional hazard model. Since the proportional hazard model is considered to be a typical prepayment model in the RMBS market, we assume that prepayment behavior follows the proportional hazard model.

It is well-known that evaluating RMBS using numerical calculations such as Monte Carlo and the finite-difference method is highly demanding. Particularly, in multi-dimensional stochastic cases, there is nothing for practical RMBS pricing methods other than Monte Carlo. On the other hand, literature on analytical RMBS valuation is very limited. Gorovoy and Linetsky [2007] derives an analytical pricing formula for RMBS by using the eigenfunction expansion method. Yamazaki [2005] proposes a closed-form pricing formula for RMBS under a simple Gaussian assumption. However, Gorovoy and Linetsky [2007] and Yamazaki [2005] intentionally choose analytically tractable prepayment models, which might possibly be unsuitable to market practice, in order to derive analytical formula.

In this article, assuming the proportional hazard model to describe prepayment behavior, we derive a pricing formula for RMBS by using the cumulant expansion method. Because the proportional hazard model already has empirical evidence as a prepayment model, the assumption of the proportional hazard model is more appropriate than that of Yamazaki [2005] or Gorovoy and Linetsky [2007]. The pricing formula gives very accurate approximate prices of RMBS quickly like a closed-form pricing formula. Moreover, the formula is applicable to various types of the proportional hazard models; i.e., it is able to deal with multi-dimensional stochastic environments and jumps. Through numerical examples, we also show that the formula is very useful from a practical point of view.

The rest of this article is organized as follows: First of all, we provide a cash flow modeling of a mortgage contract with prepayment and develop risk-neutral valuation models for RMBS, IO, and PO. Second, we show that the RMBS valuation problem is equivalent to a survival probability calculation of a mortgage contract under a forward neutral measure. Third, we derive a general pricing formula for IO and PO, as well as RMBS by the cumulant expansion, which is the main contribution to research of this article. Next, we show that the formula can be applied to the proportional hazard model not only with Gaussian processes, but also with Lévy processes. Additionally, we provide numerical examples based on Japanese RMBS market data. Finally, we present concluding remarks.

MODELS

This section develops risk-neutral valuation models for RMBS, IO, and PO. Our interest is prepayment risk, which is the risk of uncertain RMBS cash flow due to borrowers' prepayment, while for simplicity we ignore other risks of RMBS such as default risk, earthquake risk, and commingling risk.

Mortgage Contracts without Prepayment

First, we consider a mortgage contract with fixed-rate c and maturity T without prepayment. The borrower takes out a loan of $M(0)$ dollars at time 0, then he pays back periodically a constant amount denoted by A, where the payment times of A are $t_i = i/m$, $i = 0, 1, \ldots, mT$. It is obvious that A is given by

$$A = M(t_i) - M(t_{i+1}) + \frac{c}{m} M(t_i) = P(t_{i+1}) + I(t_{i+1}) \quad (1)$$

where $M(t_i)$, $P(t_i) := M(t_{i-1}) - M(t_i)$, and $I(t_i) := \frac{c}{m} M(t_{i-1})$ are remaining mortgage principal, payment amount of mortgage principal, and coupon amount at time t_i respectively. From Eq. (1), we obtain

$$M(0) = A\frac{1-(1+c/m)^{-mT}}{c/m}$$

Inversely, A can be rewritten as

$$A = M(0)\frac{c/m(1+c/m)^{mT}}{(1+c/m)^{mT}-1}$$

Thus, the constant payment amount A is determined by the initial principal $M(0)$, maturity T, coupon rate c, and payment interval m. Moreover, remaining mortgage principal at time t_i is given by

$$M(t_i) = M(0)\frac{(1+c/m)^{mT}-(1+c/m)^{mt_i}}{(1+c/m)^{mT}-1}$$

See, for instance, Fabozzi [2001] or Yamazaki [2005] for derivations of the above cash flow models.

MORTGAGE CONTRACTS WITH PREPAYMENT IN THE INTENSITY-BASED FRAMEWORK

We assume frictionless markets that are arbitrage-free. Let τ denote a prepayment time of a mortgage contract on a probability space $(\Omega, \mathcal{G}, \mathbb{Q})$, where $\mathbb{Q}$ is an equivalent martingale measure. We denote the associated filtration of τ by $\mathbb{H} := (\mathcal{H}_t)_{t\geq 0}$, where $\mathcal{H}_t = \sigma(\mathbf{1}_{\{\tau>s\}}: s \leq t)$. Let $\mathbb{G} := (\mathcal{G}_t)_{t\geq 0}$ be an arbitrary filtration on $(\Omega, \mathcal{G}, \mathbb{Q})$. Furthermore, we assume an auxiliary filtration $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ such that $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$; i.e., $\mathcal{G}_t = \mathcal{H}_t \vee \mathcal{F}_t$ for any $t \in [0, T]$. In order to model prepayment, we introduce a positive prepayment intensity (hazard rate) process $(h_t)_{t\geq 0}$ adapted to the filtration $\mathbb{F}$. We model the random time of prepayment τ as

$$\tau = \inf\left\{t \geq 0 : \int_0^t h_s ds \geq e\right\}$$

where $e \sim \text{Exp}(1)$. Here we define that $H_t := \mathbf{1}_{\{\tau \leq t\}}$ is the prepayment indicator, $F_t := \mathbb{Q}(\tau \leq t \mid \mathcal{F}_t)$ is the conditional prepayment probability of τ, and $\Gamma_t := -\ln(1 - F_t) = \int_0^t h_s ds$ is the hazard process of τ under $\mathbb{Q}$. Note that the compensated process $H_t - \Gamma_{t\wedge\tau} = H_t - \int_0^{t\wedge\tau} h_s ds$ is a $\mathbb{G}$-martingale.

Next, we consider a mortgage contract with prepayment. Let M_{t_i}, P_{t_i}, and I_{t_i} be remaining mortgage principal, payment amount of mortgage principal, and coupon amount with prepayment at time t_i, respectively. Then,

$$M_{t_i} = M(t_i)\mathbf{1}_{\{\tau>t_i\}},\ I_{t_i} = I(t_i)\mathbf{1}_{\{\tau>t_{i-1}\}},\ P_{t_i} = P(t_i)\mathbf{1}_{\{\tau>t_{i-1}\}} \quad (2)$$

Moreover, a prepayment amount[1] at time t_i denoted by PR_{t_i} is given by

$$PR_{t_i} = (M_{t_{i-1}} - P_{t_i})\mathbf{1}_{\{t_{i-1}<\tau\leq t_i\}} = M(t_i)(\mathbf{1}_{\{\tau>t_{i-1}\}} - \mathbf{1}_{\{\tau>t_i\}}) \quad (3)$$

Let CF_{t_i} be a cash flow of the mortgage contract with prepayment at time t_i. From Eqs. (2) and (3), and the definition of $P(t_i)$, we obtain

$$CF_{t_i} = P_{t_i} + I_{t_i} + PR_{t_i} = \left(1+\frac{c}{m}\right)M(t_{i-1})\mathbf{1}_{\{\tau>t_{i-1}\}} - M(t_i)\mathbf{1}_{\{\tau>t_i\}}$$

RMBS Valuation Models

We consider RMBS to be composed of a homogeneous mortgage pool. In this case, RMBS pricing can be identified with valuation of an arbitrary mortgage contract in the pool. If RMBS consists of a heterogeneous pool, it is sufficient to divide it into homogeneous pools and to evaluate each homogeneous pool.

Assumption 1 (*Short Rate Process*) *The short rate process $(r_t)_{t\geq 0}$ is adapted to the filtration $\mathbb{F}$ and a unique strong solution of the stochastic differential equation:*

$$dr_t = \mu(t, r_t)dt + \mathbf{b}^\top(t, r_t)d\mathbf{W}_t$$

where $(\mathbf{W}_t)_{t\geq 0}$ is an $\mathcal{F}_t$-adapted d-dimensional Brownian motion under $\mathbb{Q}$, and $\mu : \mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}$ and $\mathbf{b} : \mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}^d$ are deterministic functions. In addition, the discount bond price process with maturity U denoted by $(B_t(U))_{t\geq 0}$ is a unique strong solution of the stochastic differential equation:

$$\frac{dB_t(U)}{B_t(U)} = r_t dt + \mathbf{g}^\top(t, U, r_t)d\mathbf{W}_t$$

where $\mathbf{g}: \mathbf{R}_+ \times \mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}^d$ *is a deterministic function.*

Assumption 1 implies that the equivalent martingale measure $\mathbb{Q}$ is a spot neutral measure. Hence, below we call it a *spot neutral measure* to distinguish it from other measures such as *forward neutral measures*.

Let V_{t*} denote a present value[2] of RMBS at time t^*. Using the standard intensity-based framework, we obtain

$$
\begin{aligned}
V_{t*} &= \mathbb{E}\left[\sum_{i=i*}^{mT} e^{-\int_{t*}^{t_i} r_s ds} CF_{t_i} \mid \mathcal{G}_{t*}\right] \\
&= \sum_{i=i*}^{mT}\left\{\left(1+\frac{c}{m}\right)M(t_{i-1})\mathbb{E}\left[e^{-\int_{t*}^{t_i} r_s ds}\mathbf{1}_{\{\tau>t_{i-1}\}} \mid \mathcal{G}_{t*}\right] - M(t_i)\right. \\
&\qquad \left.\times\ \mathbb{E}\left[e^{-\int_{t*}^{t_i} r_s ds}\mathbf{1}_{\{\tau>t_i\}} \mid \mathcal{G}_{t*}\right]\right\} \\
&= \sum_{i=i*}^{mT}\left\{\left(1+\frac{c}{m}\right)M(t_{i-1})\mathbb{E}\left[e^{-\int_{t*}^{t_i} r_s ds} e^{-\int_{t*}^{t_{i-1}} h_s ds} \mid \mathcal{F}_{t*}\right] - M(t_i)\right. \\
&\qquad \left.\times\ \mathbb{E}\left[e^{-\int_{t*}^{t_i}(r_s+h_s)ds} \mid \mathcal{F}_{t*}\right]\right\}
\end{aligned}
$$

where $\mathbb{E}[\cdot]$ is an expectation operator under a spot neutral measure $\mathbb{Q}$ and $i^* := \inf\{i : t_i > t^*\}$. Taking a forward neutral measure $\mathbb{Q}_{t_i}$ for each cash flow CF_{t_i}, where the numéraire is a discount bond with maturity t_i, RMBS price can be described as

$$
\begin{aligned}
V_{t*} = \sum_{i=i*}^{mT}\left\{\left(1+\frac{c}{m}\right)M(t_{i-1})B_{t*}(t_i)\mathbb{E}^{t_i}\left[e^{-\int_{t*}^{t_{i-1}} h_s ds} \mid \mathcal{F}_{t*}\right]\right. \\
\left. -M(t_i)B_{t*}(t_i)\mathbb{E}^{t_i}\left[e^{-\int_{t*}^{t_i} h_s ds} \mid \mathcal{F}_{t*}\right]\right\}
\end{aligned}
$$

where $\mathbb{E}^{t_i}[\cdot]$ is an expectation operator under a forward neutral measure $\mathbb{Q}_{t_i}$. Similarly, PO and IO prices can be expressed as

$$
\begin{aligned}
\mathrm{PO}_{t*} &= \mathbb{E}\left[\sum_{i=i*}^{mT} e^{-\int_{t*}^{t_i} r_s ds}(P_{t_i} + PR_{t_i}) \mid \mathcal{G}_{t*}\right] \\
&= \sum_{i=i*}^{mT}\left\{M(t_{i-1})B_{t*}(t_i)\mathbb{E}^{t_i}\left[e^{-\int_{t*}^{t_{i-1}} h_s ds} \mid \mathcal{F}_{t*}\right]\right. \\
&\qquad \left. -M(t_i)B_{t*}(t_i)\mathbb{E}^{t_i}\left[e^{-\int_{t*}^{t_i} h_s ds} \mid \mathcal{F}_{t*}\right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{IO}_{t*} &= \mathbb{E}\left[\sum_{i=i*}^{mT} e^{-\int_{t*}^{t_i} r_s ds} I_{t_i} \mid \mathcal{G}_{t*}\right] \\
&= \frac{c}{m}\sum_{i=i*}^{mT} M(t_{i-1})B_{t*}(t_i)\mathbb{E}^{t_i}\left[e^{-\int_{t*}^{t_{i-1}} h_s ds} \mid \mathcal{F}_{t*}\right]
\end{aligned}
$$

According to the above valuation models, we have to only evaluate

$$
\mathbb{E}^{t_i}\left[e^{-\int_{t*}^{t_{i-1}} h_s ds} \mid \mathcal{F}_{t*}\right] \text{ and } \mathbb{E}^{t_i}\left[e^{-\int_{t*}^{t_i} h_s ds} \mid \mathcal{F}_{t*}\right]
$$

for RMBS, PO, and IO pricing. Further generalizing this argument, RMBS, PO, and IO valuations can be reduced to the problem of calculating the following equation:

$$
\mathbb{E}^{U}\left[\exp\left\{-\int_{t*}^{t} h_s ds\right\} \mid \mathcal{F}_{t*}\right] \tag{4}
$$

where $\mathbb{E}^U[\cdot]$ is an expectation operator under forward neutral measure $\mathbb{Q}_U$, and where the numéraire is a discount bond with maturity $U(\geq t)$. Note that Eq. (4) can be seen as a survival probability of a mortgage contract under $\mathbb{Q}_U$.

Assumption 2 (*Prepayment Model*) *The prepayment intensity h_t is described as the proportional hazard model, that is,*

$$
h_t := h_0(t)\exp\left\{\mathbf{w}^{\mathsf{T}}\mathbf{X}_t\right\} \tag{5}
$$

where $h_0 : \mathbf{R}_+ \to \mathbf{R}_+$ called the base-line hazard function is a non-negative deterministic function with respect to time t, $(\mathbf{X}_t)_{t\geq 0}$ called covariate vector is an $\mathcal{F}_t$-adapted m-dimensional stochastic process under a spot neutral measure $\mathbb{Q}$, and $\mathbf{w}$ is a coefficient vector on $\mathbf{R}^m$.

Assumptions 1 and 2 are more general settings than in Yamazaki [2005] or in Gorovoy and Linetsky [2007], in both of which analytically tractable prepayment models and simple interest rate processes are set in order to derive closed-form pricing formulas. In particular, Assumption 2 is meaningful because there is much literature documenting research on empirical analysis of prepayment behavior by the proportional hazard model. For example see Schwartz and Torous [1989], Aonuma and Kijima [1998], Ichijo and Moridaira [2001], and Sugimura [2002].

PRICING FORMULA

This section provides a general formula for not only RMBS, but also IO and PO, which is the main finding of our article. In the previous section, it is shown that the RMBS pricing problem can be reduced to evaluating the survival probability of a mortgage contract under a forward neutral measure. Therefore, we shall concentrate on evaluating Eq. (4) for RMBS valuation. Since the pricing formula is given by an infinite series that is known as the cumulant expansion, it gives an approximate value of RMBS prices.

Theorem 1: *Under Assumptions 1 and 2, Eq. (4) is given by*

$$\mathbb{E}^{U}\left[\exp\left\{-\int_{t^*}^{t} h_s ds\right\} \mid \mathcal{F}_{t^*}\right] = \exp\left\{\sum_{n=1}^{\infty}(-1)^n \frac{1}{n!}\kappa_n\right\} \quad (6)$$

where

$$\begin{aligned}
\kappa_1 &= \chi_1,\\
\kappa_2 &= \chi_2 - \chi_1^2,\\
\kappa_3 &= \chi_3 - 3\chi_1\chi_2 + 2\chi_1^3,\\
\kappa_4 &= \chi_4 - 4\chi_1\chi_3 - 3\chi_2^2 + 12\chi_1^2\chi_2 - 6\chi_1^4,\\
\kappa_5 &= \chi_5 - 5\chi_1\chi_4 - 10\chi_2\chi_3 + 20\chi_1^2\chi_3 + 30\chi_1\chi_2^2\\
&\quad - 60\chi_1^3\chi_2 + 24\chi_1^5,\\
&\quad \ldots\ldots\ldots,
\end{aligned}$$

and

$$\begin{aligned}
\chi_n = n!\int_{t^*}^{t}\int_{t^*}^{s_n}\cdots\int_{t^*}^{s_2}\prod_{k=1}^{n} h_0(s_k)\\
\times \mathbb{E}^{U}\left[\exp\left\{\sum_{k=1}^{n}\mathbf{w}^{\mathrm{T}}\mathbf{X}_{s_k}\right\} \mid \mathcal{F}_{t^*}\right] ds_1 ds_2 \ldots ds_n \quad (7)
\end{aligned}$$

Note that κ_n and χ_n are, respectively, the nth cumulant and moment of $\bar{\Gamma}_t := \Gamma_t - \Gamma_{t^*} = \int_{t^*}^{t} h_s ds$. Thus, it can be said that the pricing formula is an approximation around normal distribution with respect to $\bar{\Gamma}_t$. When $\bar{\Gamma}_t$ happens to be a Gaussian process, the formula coincides with the closed-form pricing formula in Yamazaki [2005], which gives exact prices of RMBS.

Before the proof of Theorem 1, the below lemma is provided.

Lemma 2 *Suppose that $f: \mathbf{R}_+ \to \mathbf{R}$ is an integrable function and*

$$F(x) := \int_{\alpha}^{x} f(u)du$$

where α is an arbitrary non-negative constant. Then, for all $n \in \mathbf{N}$,

$$\begin{aligned}
(F(x))^n = n!\int_{\alpha}^{x}\int_{\alpha}^{u_n}\cdots\int_{\alpha}^{u_2} f(u_n)\\
\times f(u_{n-1})\cdots f(u_1)du_1du_2\cdots du_n \quad (8)
\end{aligned}$$

Proof of Lemma 2 In the case of $n = 2$, Eq. (8) is well-known and omitted. Next, assume that Eq. (8) is valid in the case of $n \leq k$. Then,

$$\begin{aligned}
\frac{d}{dx}\left[\frac{1}{(k+1)!}(F(x))^{k+1}\right] &= \frac{1}{k!}(F(x))^k f(x)\\
&= f(x)\int_{\alpha}^{x}\int_{\alpha}^{u_k}\cdots\int_{\alpha}^{u_2} f(u_k)f(u_{k-1})\cdots\\
&\quad f(u_1)du_1du_2\cdots du_k
\end{aligned}$$

The second equality of the above equation is due to the assumption. Therefore,

$$\begin{aligned}
\frac{1}{(k+1)!}(F(y))^{k+1} &= \int_{\alpha}^{y} f(x)\int_{\alpha}^{x}\int_{\alpha}^{u_k}\cdots\int_{\alpha}^{u_2} f(u_k)f(u_{k-1})\cdots\\
&\quad f(u_1)du_1du_2\cdots du_k dx\\
&= \int_{\alpha}^{y}\int_{\alpha}^{u_{k+1}}\int_{\alpha}^{u_k}\cdots\int_{\alpha}^{u_2} f(u_{k+1})f(u_k)\\
&\quad f(u_{k-1})\cdots f(u_1)du_1du_2\cdots du_k du_{k+1}
\end{aligned}$$

Here we rewrite $u_{k+1} := x$. By mathematical induction, the proof of lemma 2 is completed. ❑

Proof of Theorem 1 First, we shall derive Eq. (6). Let $\psi_{\bar{\Gamma}_t}(\theta)$ denote the moment generating function of $\bar{\Gamma}_t$. Then, by cumulant expansion,

$$\log\psi_{\bar{\Gamma}_t}(\theta) = \kappa_1\theta + \frac{1}{2}\kappa_2\theta^2 + \frac{1}{6}\kappa_3\theta^3 + \cdots + \frac{1}{n!}\kappa_n\theta^n + \cdots$$

where κ_n is the nth cumulant of $\tilde{\Gamma}_t$. When $\theta = -1$, Eq. (6) is obtained.

Next, we consider nth moment of $\tilde{\Gamma}_t$:

$$\chi_n := \mathbb{E}^U\left[\tilde{\Gamma}_t^n \mid \mathcal{F}_{t*}\right] = \mathbb{E}^U\left[\left(\int_{t*}^t h_s ds\right)^n \mid \mathcal{F}_{t*}\right] \quad (9)$$

Applying Lemma 2 to Eq. (9),

$$\begin{aligned}\mathbb{E}^U\left[\left(\int_{t*}^t h_s ds\right)^n \mid \mathcal{F}_{t*}\right] &= \mathbb{E}^U\Big[n!\int_{t*}^t \int_{t*}^{s_n} \cdots \int_{t*}^{s_2} h_{s_n} h_{s_{n-1}} \cdots \\ &\qquad h_{s_1} ds_1 ds_2 \cdots ds_n \mid \mathcal{F}_{t*}\Big] \\ &= n!\int_{t*}^t \int_{t*}^{s_n} \cdots \int_{t*}^{s_2} \\ &\quad \times \mathbb{E}^U\left[\prod_{k=1}^n h_{s_k} \mid \mathcal{F}_{t*}\right] ds_1 ds_2 \cdots ds_n\end{aligned}$$

By Assumption 2,

$$\begin{aligned}\mathbb{E}^U\left[\prod_{k=1}^n h_{s_k} \mid \mathcal{F}_{t*}\right] &= \mathbb{E}^U\left[\prod_{k=1}^n h_0(s_k)\exp\{\mathbf{w}^\top \mathbf{X}_{s_k}\} \mid \mathcal{F}_{t*}\right] \\ &= \prod_{k=1}^n h_0(s_k)\mathbb{E}^U\left[\exp\left\{\sum_{k=1}^n \mathbf{w}^\top \mathbf{X}_{s_k}\right\} \mid \mathcal{F}_{t*}\right]\end{aligned}$$

Therefore, Eq. (7) is obtained. The proof of Theorem 1 is completed. ❑

According to Theorem 1, the RMBS pricing problem is eventually reduced to evaluating the following equation:

$$\mathbb{E}^U\left[\exp\left\{\sum_{k=1}^n \mathbf{w}^\top \mathbf{X}_{s_k}\right\} \mid \mathcal{F}_{t*}\right] \quad (10)$$

Note that, when covariate vector $\mathbf{X}_t$ of the proportional hazard model is decomposed to some independent vectors, it is sufficient to evaluate Eq. (10) of each independent vector in $\mathbf{X}_t$. If a closed-form expression of Eq. (10) is obtained, RMBS, IO, and PO can be evaluated simultaneously by an appropriate numerical method of multiple integrals in Eq. (7). Although numerical multiple integrals are very time-consuming in general, our numerical examples will show that a single or double integral, which can be calculated very quickly, is enough to obtain accurate RMBS prices and their sensitivity.

Corollary 3 *Suppose that the covariate vector* $\mathbf{X}_t = (X_t^1, X_t^2, \ldots, X_t^m)$ *of the proportional hazard model (5) is an m-dimensional Gaussian process under a forward neutral measure* $\mathbb{Q}_U$. *Then,*

$$\mathbb{E}^U\left[\exp\left\{\sum_{k=1}^n \mathbf{w}^\top \mathbf{X}_{s_k}\right\} \mid \mathcal{F}_{t*}\right] = \exp\left\{\mu + \frac{\upsilon}{2}\right\}$$

where

$$\mu := \sum_{k=1}^n \sum_{j=1}^m w_j \mathbb{E}^U[X_{s_k}^j \mid \mathcal{F}_{t*}],$$

$$\upsilon := \sum_{k_1,k_2}^n \sum_{j_1,j_2}^m w_{j_1} w_{j_2} \mathrm{Cov}^U[X_{s_{k_1}}^{j_1}, X_{s_{k_2}}^{j_2} \mid \mathcal{F}_{t*}]$$

and w_j (j = 1,2, …, m) *is the jth component of coefficient vector* $\mathbf{w}$.

STOCHASTIC PROPORTIONAL HAZARD MODELS

In this section, we demonstrate that Theorem 1 is applicable to various types of the proportional hazard models. Presenting four examples, we shall show explicit expressions of Eq. (10). These examples deal with the stochastic proportional hazard model not only with Gaussian processes, but also with Lévy processes. In the following, we set current time $t^* = 0$ to evaluate RMBS without loss of generality.

Example 1 (Multi-Dimensional OU Process) Let $(\mathbf{X}_t)_{t\geq 0}$ be an m-dimensional OU process under a spot neutral measure $\mathbb{Q}$, that is, the covariate vector $\mathbf{X}_t = (X_t^1, X_t^2, \ldots, X_t^m)$ of the proportional hazard model (5) is given by:

$$dX_t^j = (\phi_j(t) - a_j X_t^j)dt + \mathbf{b}_j^\top d\mathbf{W}_t, \quad j = 1,2,\ldots,m \quad (11)$$

where $X_t^1 := r_t$, $(\mathbf{W}_t)_{t\geq 0}$ is a d-dimensional standard Brownian motion under $\mathbb{Q}$, $\phi_j(\cdot)$ is a deterministic function with respect to time t, $\mathbf{b}_j$ is $\mathbf{R}^d$-constant vector, and

a_j is constant. Note that the spot rate process $(r_t)_{t\geq 0}$ is well-known as the Hull-White model (Hull and White [1990]).

Under a forward neutral measure $\mathbb{Q}_U$, SDE(11) is transformed into

$$dX_t^j = (\xi_j^U(t) - a_j X_t^j)dt + \mathbf{b}_j^\top d\mathbf{W}_t^U, \quad j = 1,2,\ldots,m \quad (12)$$

where

$$\xi_j^U(t) := \phi_j(t) - \frac{1-e^{-a_j(U-t)}}{a_j}\mathbf{b}_1^\top \mathbf{b}_j$$

and $(\mathbf{W}_t^U)_{t\geq 0}$ is a d-dimensional standard Brownian motion under $\mathbb{Q}_U$. Solving Eq. (12), we obtain the following Gaussian processes:

$$\begin{aligned} X_t^j &= X_0^j e^{-a_j t} + \int_0^t \xi_j^U(s)e^{-a_j(t-s)}ds \\ &+ \mathbf{b}_j^\top \int_0^t e^{-a_j(t-s)} d\mathbf{W}_s^U, \quad j = 1,2,\ldots,m \end{aligned} \quad (13)$$

Since it is obvious that the covariate vector $\mathbf{X}_t$ is an m-dimensional Gaussian process under $\mathbb{Q}^U$, and

$$\mathbb{E}^U[X_t^j] = X_0^j e^{-a_j t} + \int_0^t \xi_j^U(s)e^{-a_j(t-s)}ds$$

and as $t_1 \geq t_2$,

$$\begin{aligned} \mathrm{Cov}^U[X_{t_1}^{j_1}, X_{t_2}^{j_2}] &= \mathbf{b}_{j_1}^\top \mathbf{b}_{j_2} \mathbb{E}^U\left[\left(\int_0^{t_1} e^{-a_{j_1}(t_1-s)} d\mathbf{W}_s^U\right)^\top \right. \\ &\quad \left. \times \left(\int_0^{t_2} e^{-a_{j_2}(t_2-s)} d\mathbf{W}_s^U\right)\right] \\ &= \mathbf{b}_{j_1}^\top \mathbf{b}_{j_2} \int_0^{\min\{t_1,t_2\}} e^{-a_{j_1}(t_1-s)-a_{j_2}(t_2-s)}ds \\ &= \frac{\mathbf{b}_{j_1}^\top \mathbf{b}_{j_2}}{a_{j_1}+a_{j_2}}(e^{-a_{j_1}(t_1-t_2)} - e^{-(a_{j_1}t_1+a_{j_2}t_2)}) \end{aligned}$$

the closed-form solution of Eq. (10) can be obtained from Corollary 3.

In the following, we consider a component X_t in the covariate vector $\mathbf{X}_t$ of the proportional hazard model, and we assume that X_t is independent of all other components of $\mathbf{X}_t$ and interest rate r_t. Under this assumption, we have only to calculate

$$\mathbb{E}^U\left[\exp\left\{w\sum_{k=1}^n X_{s_k}\right\}\right] \quad (14)$$

as regards $(X_t)_{t\geq 0}$, where w is a coefficient of X_t. Note that $(X_t)_{t\geq 0}$ is invariant under any forward neutral measures due to the independence assumption.

Example 2 (Function of a Gaussian process) Suppose that $X_t := f(Y_t)$, where $f(\cdot)$ is a deterministic function and Y_t is a Gaussian process under a spot neutral measure $\mathbb{Q}$. Obviously, Eq. (14) is given by

$$\begin{aligned} \mathbb{E}^U\left[\exp\left\{w\sum_{k=1}^n X_{s_k}\right\}\right] &= \int_{\mathbf{R}^n} \exp\left\{w\sum_{k=1}^n f(y_k)\right\} \\ &\quad \times p_{s_1,\ldots,s_n}(y_1,\ldots,y_n)dy_1\ldots dy_n \end{aligned} \quad (15)$$

where $p_{s_1, s_2, \ldots, s_n}(y_1, y_2, \ldots, y_n)$ is the density of the n-dimensional random variable $(Y_{s_1}, Y_{s_2}, \ldots, Y_{s_n})$ with a normal distribution. For quick numerical calculation of the improper integral on the right hand side of Eq. (15), we can use the Gauss-Hermite quadrature rule.

Example 3 (Compound Poisson Process) Let $(X_t)_{t\geq 0}$ be a compound Poisson process under a spot neutral measure $\mathbb{Q}$, that is,

$$X_t := \sum_{j=1}^{N_t} Y_j$$

where N_t is a Poisson process with intensity λ, and Y_j, $j = 1, 2, \ldots$, are i.i.d. random variables which are independent of N_t and have the same characteristic function $\varphi_Y(\theta)$. To derive the closed-form solution of Eq. (14), we first see

$$\begin{aligned} \sum_{k=1}^n X_{s_k} &= n(X_{s_1} - X_{s_0}) + \cdots + (n-k+1) \\ &\quad \times (X_{s_k} - X_{s_{k-1}}) + \cdots + (X_{s_n} - X_{s_{n-1}}) \end{aligned}$$

where we set $X_{s_0} = 0$ for convention. Then, Eq. (14) is written as

$$\mathbb{E}^U\left[\exp\left\{w\sum_{k=1}^{n} X_{s_k}\right\}\right]$$
$$= \mathbb{E}^U\left[\exp\left\{w\sum_{k=1}^{n}(n-k+1)(X_{s_k} - X_{s_{k-1}})\right\}\right]$$
$$= \prod_{k=1}^{n}\mathbb{E}^U\left[e^{w(n-k+1)(X_{s_k}-X_{s_{k-1}})}\right]$$
$$= \prod_{k=1}^{n}\mathbb{E}^U\left[e^{w(n-k+1)X_{s_k-s_{k-1}}}\right] \quad (16)$$

The last equality is shown by the stationary increments property of the compound Poisson process X_t. By the Lévy-Khinchine formula,

$$\mathbb{E}^U[e^{\iota\theta X_{s_k-s_{k-1}}}] = \exp\left\{(s_k - s_{k-1})\lambda(\varphi_Y(\theta)-1)\right\} \quad (17)$$

where $\iota := \sqrt{-1}$. Substituting Eq. (17) for Eq. (16) with $\theta = -\iota w(n-k+1)$, we can obtain

$$\mathbb{E}^U\left[\exp\left\{w\sum_{k=1}^{n} X_{s_k}\right\}\right] = \prod_{k=1}^{n}\exp\left\{(s_k - s_{k-1})\lambda \right.$$
$$\left. \times\ (\varphi_Y(-\iota w(n-k+1))-1)\right\} \quad (18)$$

Example 4 (Infinite Activity Lévy Process) Let $(X_t)_{t\geq 0}$ be a pure jump infinite activity Lévy process with a Lévy measure ν under a spot neutral measure $\mathbb{Q}$. By the Lévy-Itô decomposition, X_t can be represented as a sum of a compound Poisson process and an almost sure limit of compensated compound Poisson processes:

$$X_t = \sum_{s\leq t}\Delta X_s 1_{|\Delta X_s|\geq 1} + \lim_{\varepsilon\downarrow 0} N_t^{\varepsilon}$$

where

$$N_t^{\varepsilon} = \sum_{s\leq t}\Delta X_s 1_{\varepsilon\leq|\Delta X_s|<1} - t\int_{\varepsilon\leq|x|<1} x\nu(dx)$$

Therefore, X_t can be approximated by

$$X_t^{\varepsilon} = \sum_{s\leq t}\Delta X_s 1_{|\Delta X_s|\geq 1} + N_t^{\varepsilon}$$

and the residual term is given by

$$R_t^{\varepsilon} = -N_t^{\varepsilon} + \lim_{\delta\downarrow 0} N_t^{\delta}$$

which is a pure jump Lévy process with Lévy measure $1_{|x|\leq\varepsilon}\nu(dx)$ satisfying $\mathbb{E}^U[R_t^{\varepsilon}] = 0$. In the finite variation case, the process $(X_t^{\varepsilon})_{t\geq 0}$ can be written as

$$X_t^{\varepsilon} = \sum_{s\leq t}\Delta X_s 1_{\varepsilon\leq|\Delta X_s|} + \mathbb{E}^U\left[\sum_{s\leq t}\Delta X_s 1_{|\Delta X_s|<\varepsilon}\right]$$

Since the approximation process $(X_t^{\varepsilon})_{t\geq 0}$ is composed of a compound Poisson process and a deterministic term, we can use Eq. (18) in Example 3 for deriving the closed-form solution of Eq. (14) with respect to X_t^{ε}.

Moreover, for the sake of more accurate approximation, using

$$t\sigma^2(\varepsilon) := \mathrm{Var}^U[R_t^{\varepsilon}] = t\int_{|x|<\varepsilon} x^2\nu(x)dx$$

X_t can be approximated by

$$\hat{X}_t^{\varepsilon} := X_t^{\varepsilon} + \sigma(\varepsilon)B_t$$

where B_t is an independent Brownian motion. See pp. 184–192 in Cont and Tankov [2004] for details. Since the infinite activity Lévy process can be approximately decomposed into a compound Poisson process and a Brownian motion, which are mutually independent, we can also obtain the closed-form solution of Eq. (14) with respect to $\hat{X}_t^{\varepsilon}$ by applying Corollary 3 and Eq. (18) in Example 3 to the Brownian motion and the compound Poisson process respectively.

Remark 4. *In Example 4, because an infinite activity Lévy process is approximated to obtain an explicit expression of Eq. (14), there is the approximation error in RMBS price by the pricing formula. On the other hand, even if it is obtained by Monte Carlo, the same approximation scheme is usually implemented (see Cont and Tankov [2004]). Therefore, it can be said that the same error exists in RMBS price by Monte Carlo.*

NUMERICAL EXAMPLES

This section provides two numerical examples: the accuracy test and the sensitivity calculations. First, we specify an interest rate model and the proportional hazard model for the numerical examples. The parameters of these models are set based on the Japanese RMBS market. Second, by comparing RMBS, IO, and PO prices by the pricing formula with the exact prices, we examine the accuracy of the first and second order cumulant expansion prices. The result implies that the prices obtained by the second order cumulant expansion are sufficient in practice. Third, as an application of the pricing formula, we calculate effective duration and convexity of RMBS, IO, and PO. It is shown that the formula is very useful to compute these values accurately and quickly.

Model Specification

First, we specify spot rate dynamics $(r_t)_{t\geq 0}$ using the Hull-White model (Hull and White [1990]):

$$dr_t = (\phi(t) - ar_t)dt + \sigma dW_t$$

where a, $\sigma > 0$ are constant, and $\phi(t)$ is a deterministic function with respect to time t. Next, we suppose that initial yield curves are given by the augmented Nelson-Siegel model (Björk and Christensen [1997]):

$$f(0,t) := z_1 + z_2 \exp(-at) + z_3 t \exp(-at) + z_4 \exp(-2at)$$

where $f(0, t)$ denotes an instantaneous t-forward rate at time 0, and z_k, $k = 1, 2, 3, 4$, are parameters. It is well-known that the augmented Nelson-Siegel model is consistent with the Hull-White model (see Björk and Christensen [1997]). By trivial calculations, we obtain

$$\phi(t) = az_1 + z_3 \exp(-at) - az_4 \exp(-2at) + \sigma^2 \frac{1-\exp(-2at)}{2a}$$

and

$$r_0 = f(0,0) = z_1 + z_2 + z_4$$

The parameters of the Hull-White model are set $a = 0.2$ and $\sigma = 0.008$, which are taken from the numerical example in Sugimura [2004]. In the accuracy test, the augmented Nelson-Siegel model is calibrated to the JGB yield curve as of May 9, 2008; i.e., $z_1 = 0.0308$, $z_2 = -0.0874$, $z_3 = 0.0049$, and $z_4 = 0.0725$. On the other hand, we suppose flat yield curves with parallel shift in the sensitivity calculation; i.e., $z_1 = r_0$ and $z_2 = z_3 = z_4 = 0$. These parameters are listed in Exhibits 1 and 2.

Next, we specify the proportional hazard model as follows:

$$h_t = h_0(t)\exp\{wX_t\} := \lambda(1-\exp(-\gamma t))\exp\{w(R - r_t)\}$$

where $h_0(t) := \lambda(1 - \exp(-\gamma t))$, $X_t := R - r_t$, and λ, γ, R are constant.

In the Japanese RMBS market, a standard prepayment model called PSJ is presented as a benchmark of prepayment rate by the Japan Securities Dealers Association. PSJ model is a simple deterministic function with respect to time t. It rises in a linear fashion for 60 months to a certain level, at which point it remains constant. The base-line hazard function is calibrated to PSJ model; i.e., $\lambda = 0.0614$ and $\gamma = 0.3607$. Thus, we have set up the proportional hazard model if the interest rate does not move, then it behaves like PSJ model (see Exhibit 5).

When w is positive, if the interest rate goes down, then the prepayment

EXHIBIT 1

RMBS Setting and Model Parameters in the Accuracy Test

RMBS	Face Value 100	Coupon 0.0216	Amortization Monthly	Maturity 35
Proportional Hazard Model	λ 0.0614	γ 0.3607	w 3 ~ 50	R 0.0159
Hull-White Model	r_0 0.0159	a 0.2000	σ 0.0080	
Augmented Nelson-Siegel Model	z_1 0.0308	z_2 −0.0874	z_3 0.0049	z_4 0.0725

Notes: The RMBS setting is the same as the 13th RMBS issued by JHFA. See Exhibits 5 and 6 for the parameters of the proportional hazard model. The augmented Nelson-Siegel model is calibrated to JGB yield curve as of May 9, 2008. The parameters of the Hull-White model are taken from Sugimura [2004].

Exhibit 2

RMBS Setting and Model Parameters in the Sensitivity Calculations

RMBS	Face Value 100	Coupon 0.100	Amortization Monthly	Maturity 35
Proportional Hazard Model	λ 0.0614	γ 0.3607	w 20	R 0.05
Hull-White Model Parameters	r_0 0.000 ~ 0.100	a 0.2000	σ 0.0080	
Augmented Nelson-Siegel Model	z_1 r_0	z_2 0.0000	z_3 0.0000	z_4 0.0000

Notes: The setting in Exhibit 2 is a higher interest rate environment than in Exhibit 1 to generate negative convexity effect clearly. Note that the augmented Nelson-Siegel parameters imply flat yield curves.

rate goes up and vice versa. Therefore, the parameter w can be interpreted as prepayment sensitivity with respect to interest rate shift. In the accuracy test, we set $R = r_0$ and $w = 3, 5, 10, 20, 30, 50$ to examine the impact of the prepayment sensitivity. Note that the proportional hazard model approaches a linear function with respect to r_t when w is small enough. In this case, the proportional hazard model can be approximated to a Gaussian function, which gives us a closed-form pricing formula for RMBS (see Yamazaki [2005]). This fact implies that the larger w is, the larger approximation errors are. For RMBS issued by the Japan Housing Finance Agency (JHFA), market consensus of prepayment sensitivity to interest rate is presented by Bloomberg. Exhibit 6 plots prepayment sensitivity curves in the case of the accuracy test against prepayment consensus of May 9, 2008 on the 13th RMBS issued by JHFA. On the other hand, we set $R = 0.05$ and $w = 20$ in the sensitivity calculations. The parameters of the proportional hazard model are also listed in Exhibits 1 and 2.

Accuracy Test

By the pricing formula, we compute RMBS, IO, and PO prices, which are set to the same condition as the 13th RMBS issued by JHFA; i.e., face value = 100, $m = 1/12$, $c = 0.0216$, $T = 35$. We regard the prices by Monte Carlo with 10^7 sample paths as the exact prices (EP). In order to verify the accuracy of the pricing formula, we compare the first and second order cumulant expansion prices (CEP1 and CEP2, respectively) with EP. Moreover, we realize very quick computation of the second order RMBS prices (hereafter, CEP2*) by a simple acceleration scheme introduced by Shibasaki and Nakamura [2001]. That is, we do not compute the monthly cash flow value for each month, but we compute a specific monthly cash flow value per year and interpolate into the interim values by using the spline function. By virtue of the acceleration scheme, the number of the cash flow valuations can be reduced from $35 \times 12 = 420$ to 35, and the computation speed can be dramatically improved. Exhibits 7–9 plot with the spline interpolation each cash flow value of RMBS, IO, and PO, respectively.

Exhibits 3 and 4 show RMBS, IO and PO prices, errors := EP – CEP, and error ratios := $100 \times$ errors/EP. In the case of CEP1, although the errors are small when w is small, they might not be ignored when w is large. In contrast, in the case of CEP2 and CEP2*, even when $w = 50$, a stress scenario of prepayment behavior, the error of RMBS price is only 0.007 and the errors of IO and PO prices are less than 0.05. According to market consensus of prepayment sensitivity of May 9, 2008, w is nearly 10 in the Japanese RMBS market, and the errors of CEP2 and CEP2* with $w = 10$ are 0.002 or below. As a result of the accuracy test, it can be said that CEP2 and CEP2* are substantially accurate in practice. In particular, CEP2* is very practical in terms of computation time.

Sensitivity Calculations

As an application of the pricing formula, we compute effective duration and convexity of RMBS, IO and PO by CEP2*. Effective duration (hereafter, ED) and convexity (hereafter, EC) of RMBS are defined as follows: Let $V(\Delta y)$ be a RMBS price with Δy -yield curve shift at time 0. Then,

$$ED := \frac{V(-\Delta y) - V(\Delta y)}{2\Delta y V(0)}$$

and

$$EC := \frac{V(-\Delta y) - 2V(0) + V(\Delta y)}{(\Delta y)^2 V(0)}$$

EXHIBIT 3
RMBS, IO and PO Prices in the Case of $w = 3, 5, 10$

$w = 3$	RMBS			IO			PO		
	Price	Error	Error ratio	Price	Error	Error ratio	Price	Error	Error ratio
EP	100.855			21.997			78.858		
CEP1	100.854	0.001	0.001	21.994	0.003	0.014	78.860	–0.002	–0.003
CEP2	100.853	0.001	0.001	21.995	0.002	0.008	78.858	0.000	0.000
CEP2*	100.853	0.001	0.001	21.995	0.002	0.007	78.858	0.000	0.000
$w = 5$	RMBS			IO			PO		
	Price	Error	Error ratio	Price	Error	Error ratio	Price	Error	Error ratio
EP	100.783			22.068			78.715		
CEP1	100.854	–0.071	–0.071	21.994	0.074	0.335	78.860	–0.145	–0.185
CEP2	100.781	0.001	0.001	22.066	0.002	0.008	78.715	0.000	0.000
CEP2*	100.781	0.001	0.001	22.066	0.002	0.007	78.715	0.000	0.000
$w = 10$	RMBS			IO			PO		
	Price	Error	Error ratio	Price	Error	Error ratio	Price	Error	Error ratio
EP	100.609			22.229			78.380		
CEP1	100.854	–0.245	–0.245	21.994	0.235	1.057	78.860	–0.480	–0.612
CEP2	100.608	0.001	0.001	22.227	0.002	0.008	78.381	0.000	0.000
CEP2*	100.608	0.001	0.001	22.227	0.002	0.007	78.381	0.000	0.000

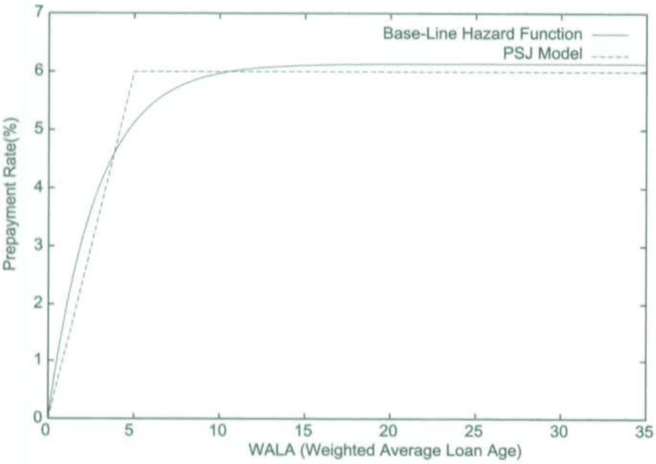
Notes: We regard the prices computed by Monte Carlo with 10^7 sample paths as the exact prices denoted by EP. CEP1 and CEP2 denote the prices by the first and second order cumulant expansion respectively. CEP2 is the prices by the second order cumulant expansion with the spline interpolation. Here we define error := EP – CEP and error ratio := 100 × error/EP.*

EXHIBIT 4
RMBS, IO and PO Prices in the Case of $w = 20, 30, 50$

$w = 20$	RMBS			IO			PO		
	Price	Error	Error ratio	Price	Error	Error ratio	Price	Error	Error ratio
EP	100.292			22.478			77.813		
CEP1	100.854	–0.562	-0.561	21.994	0.485	2.156	78.860	–1.047	–1.345
CEP2	100.290	0.002	0.002	22.478	0.001	0.003	77.812	0.001	0.001
CEP2*	100.290	0.002	0.002	22.478	0.000	0.002	77.812	0.001	0.001
$w = 30$	RMBS			IO			PO		
	Price	Error	Error ratio	Price	Error	Error ratio	Price	Error	Error ratio
EP	100.013			22.631			77.383		
CEP1	100.044	–0.030	–0.030	22.502	0.129	0.568	77.542	–0.159	–0.205
CEP2	100.011	0.003	0.003	22.634	–0.004	–0.016	77.376	0.006	0.008
CEP2*	100.011	0.002	0.002	22.635	–0.004	–0.018	77.376	0.006	0.008
$w = 50$	RMBS			IO			PO		
	Price	Error	Error ratio	Price	Error	Error ratio	Price	Error	Error ratio
EP	99.561			22.655			76.906		
CEP1	100.854	–1.293	–1.299	21.994	0.661	2.919	78.860	–1.955	–2.541
CEP2	99.553	0.007	0.007	22.697	–0.042	–0.187	76.856	0.050	0.065
CEP2*	99.554	0.007	0.007	22.698	–0.043	–0.190	76.856	0.050	0.065

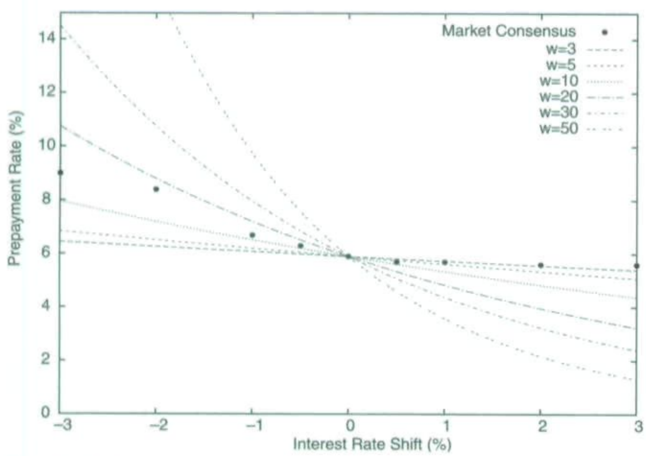
Notes: Exhibit 4 shows RMBS, IO and PO prices, errors, and error ratios in the case of $w = 20, 30, 50$.

EXHIBIT 5
PSJ Model and Base-Line Hazard Function



Notes: The base-line hazard function is calibrated to the PSJ model. See Prepayment Standard Japan Model Guide Book published by Japan Securities Dealers Association [2006] for the definition of PSJ model.

EXHIBIT 6
Prepayment Sensitivity to Interest Rate Shift

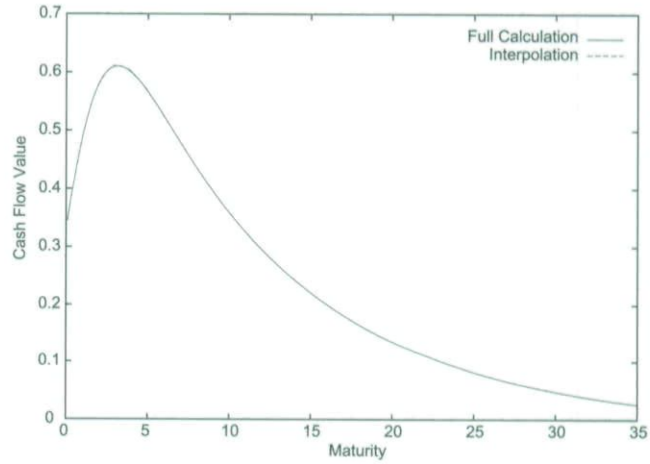


Notes: The dots denote the prepayment rates of market consensus on the 13th RMBS issued by JHFA as of May 9, 2008, whose data are downloaded from Bloomberg. The lines are the exponential curve $h_0 e^{w\Delta r}$.

Here, we set $\Delta y = 0.001$ (10 bp). ED and EC of IO and PO are defined in the same manner.

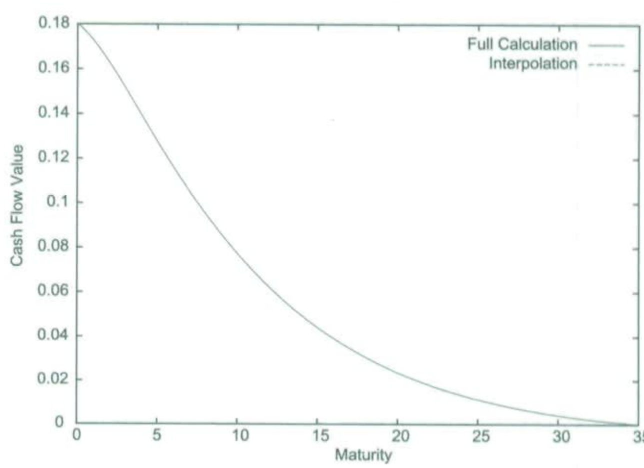
In the sensitivity calculations, we need higher interest rate environments than these of the current Japanese interest rate market. We set $c = 0.100$, $r_0 = 0.00 \sim 0.100$ (see Exhibit 2 for other parameters). This is because negative convexity effects can not be observed clearly in the Japanese RMBS market due to the low interest rate environment. In order to demonstrate the usefulness of the pricing formula, we compare ED and EC by CEP2* with these by Monte Carlo with 10^4 sample paths, which is merely tolerable for practical calculation time without any special techniques. In Monte Carlo, we adopt two

EXHIBIT 7
RMBS Cash Flow Values with Interpolation



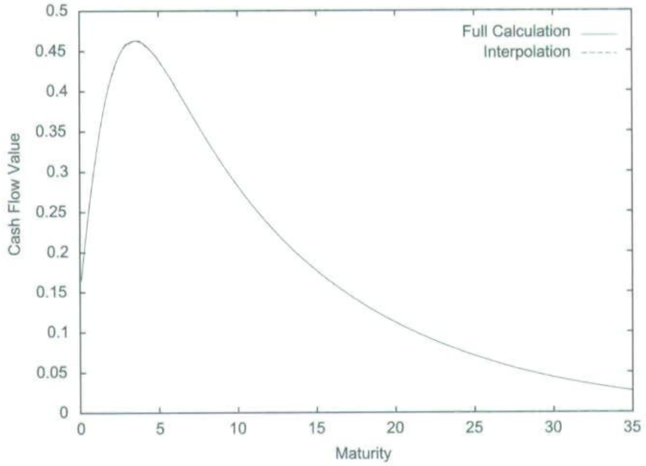
Notes: The solid line denotes RMBS cash flow values without interpolation, in which we compute the monthly cash flow value for each month; i.e., $12 \times 35 = 420$ cash flows, by the second order cumulant expansion. The dashed line denotes RMBS cash flow values with interpolation, in which we compute a specific monthly cash flow value per year; i.e., only 35 cash flows, by the same order expansion, and interpolate into the interim cash flow values by the spline function.

EXHIBIT 8
IO Cash Flow Values with Interpolation



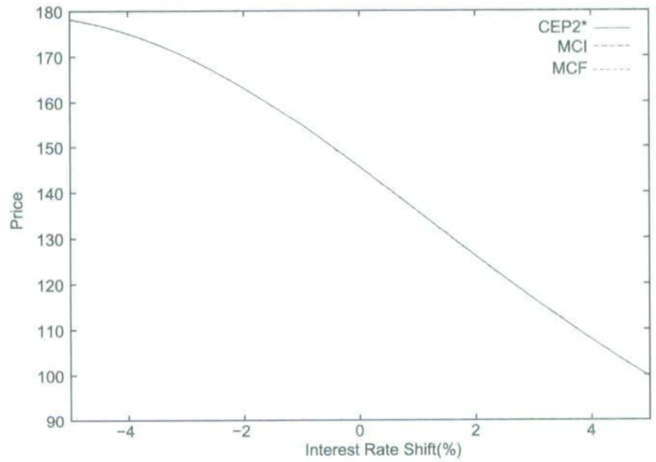
Notes: The solid line denotes IO cash flow values without interpolation, and the dashed line denotes IO cash flow values with interpolation. These calculation methods are the same as in Exhibit 7.

Exhibit 9
PO Cash Flow Values with Interpolation



Notes: The solid line denotes PO cash flow values without interpolation, and the dashed line denotes PO cash flow values with interpolation. These calculation methods are the same as in Exhibit 7.

Exhibit 10
RMBS Price

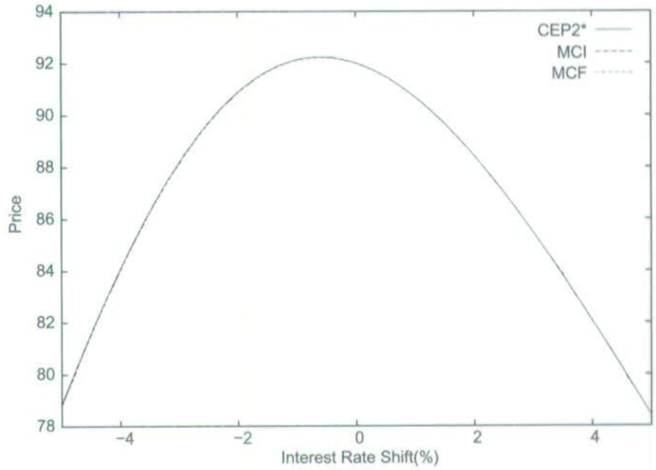


Notes: The solid line denotes RMBS prices by the second order cumulant expansion with interpolation denoted by CEP2. The dashed line denotes RMBS prices by Monte Carlo with independent 10^4 sample paths denoted by MCI. The dotted line denotes RMBS prices by Monte Carlo with fixed 10^4 sample paths denoted by MCF.*

types of sensitivity computation methods; that is, the Monte Carlo difference method with independent sample paths (hereafter, MCI) and fixed sample paths (hereafter, MCF).

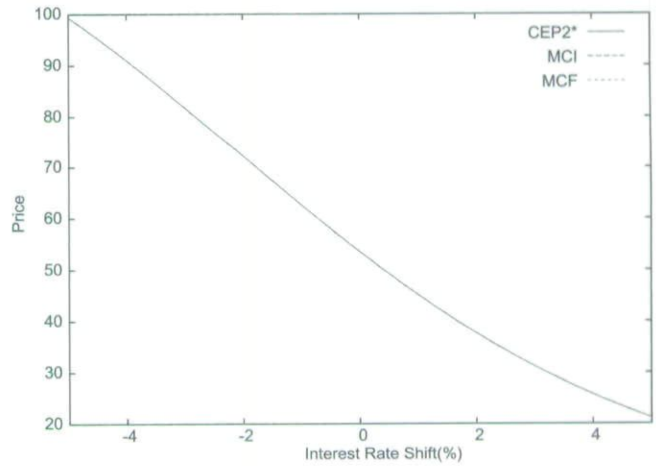
Exhibits 10–12 plot RMBS, IO, and PO prices, respectively, against interest rate shift. Negative convexity effects can be seen on the left-hand side in Exhibits 10

Exhibit 11
IO Price



Notes: The solid line denotes IO prices by CEP2. The dash line denotes IO prices by MCI. The dotted line denotes IO prices by MCF.*

Exhibit 12
PO Price

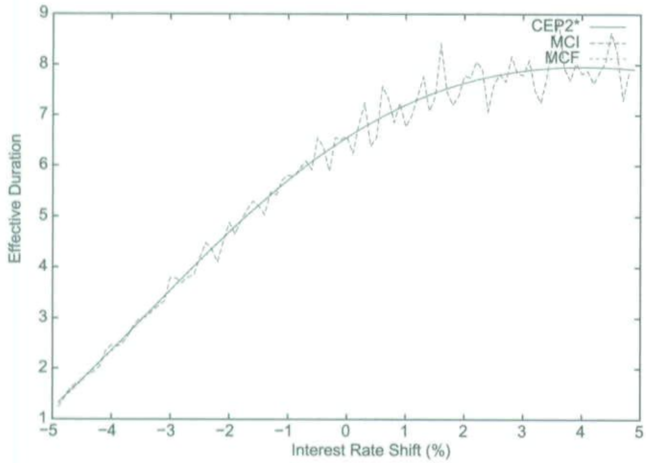


Notes: The solid line denotes PO prices by CEP2. The dashed line denotes PO prices by MCI. The dotted line denotes PO prices by MCF.*

and 12. In Exhibit 11, they are observed everywhere. All prices in Exhibits 10–12 seem to be very stable and smooth. However, the prices by CEP2^* are more accurate than these by MCI and MCF. The Monte Carlo prices with 10^4 paths generate 0.023% absolute error ratio against EP in average. In contrast, the absolute error ratios of CEP2^* are less than 0.003%.

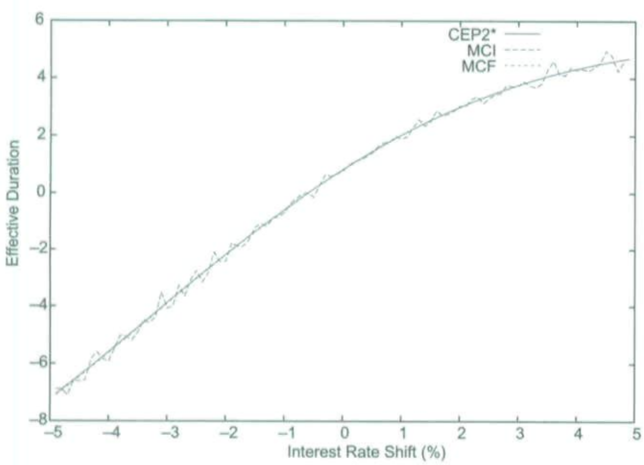
Exhibits 13–15 plot effective duration of RMBS, IO, and PO, respectively; and Exhibits 16–20 plot their effective convexity. Note that all effective duration curves

EXHIBIT 13
Effective Duration of RMBS



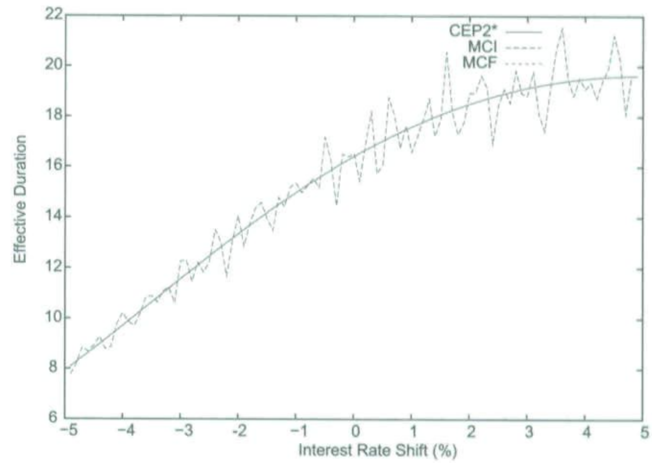
Notes: The effective duration is defined as $ED := \frac{V(-\Delta y) - V(\Delta y)}{2\Delta y V(0)}$. *The solid line denotes the effective duration of RMBS by CEP2*. The dashed line denotes the effective duration of RMBS by MCI. The dotted line denotes the effective duration of RMBS by MCF.*

EXHIBIT 14
Effective Duration of IO



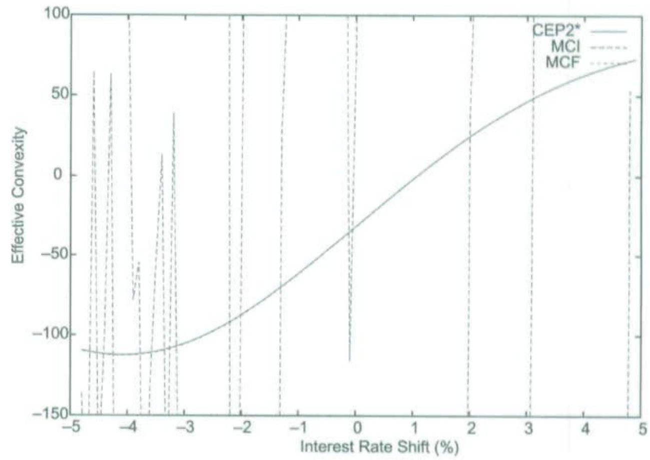
Notes: The solid line denotes the effective duration of IO by CEP2. The dashed line denotes the effective duration of IO by MCI. The dotted line denotes the effective duration of IO by MCF. The definition of IO effective duration is the same as that of RMBS.*

EXHIBIT 15
Effective Duration of PO



Notes: The solid line denotes the effective duration of PO by CEP2. The dashed line denotes the effective duration of PO by MCI. The dotted line denotes the effective duration of PO by MCF. The definition of PO effective duration is the same as that of RMBS.*

EXHIBIT 16
Effective Convexity of RMBS



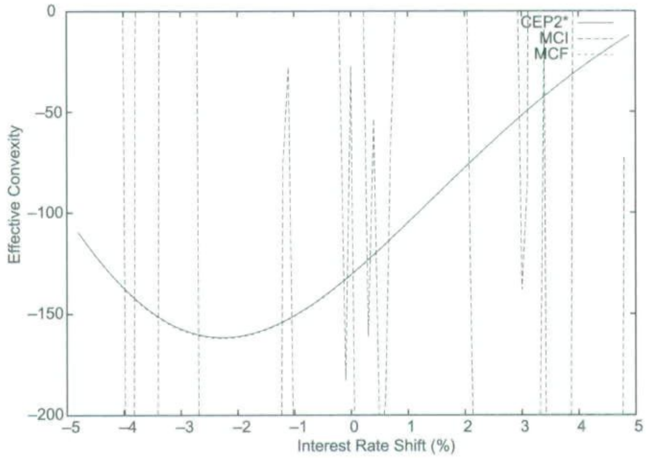
Notes: The effective convexity is defined as $EC := \frac{V(-\Delta y) - 2V(0) + V(\Delta y)}{(\Delta y)^2 V(0)}$. *The solid line denotes the effective convexity of RMBS by CEP2*. The dashed line denotes the effective convexity of RMBS by MCI. The dotted line denotes the effective convexity of RMBS by MCF.*

of MCI are jagged and the effective convexity is highly unstable due to simulation error by Monte Carlo. Obviously, MCI is useless for sensitivity computations in practice. On the other hand, Exhibits 13–18 show that CEP2* and MCF are able to draw stable and smooth ED and EC curves. However, in terms of accuracy, the sensitivity calculations by CEP2* are more appropriate than those by MCF. As a result of the numerical example, it can be said that CEP2* is a very powerful tool for RMBS valuation.

CONCLUDING REMARKS

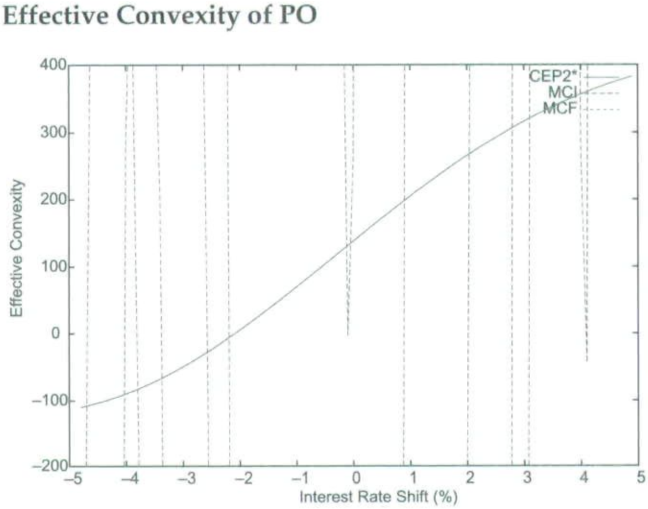
This article presents a general pricing formula for RMBS with the proportional hazard model. Since the

EXHIBIT 17
Effective Convexity of IO



Notes: The solid line denotes the effective convexity of IO by CEP2. The dashed line denotes the effective convexity of IO by MCI. The dotted line denotes the effective convexity of IO by MCF. The definition of IO effective convexity is the same as that of RMBS.*

EXHIBIT 18
Effective Convexity of PO



Notes: The solid line denotes the effective convexity of PO by CEP2. The dashed line denotes the effective convexity of PO by MCI. The dotted line denotes the effective convexity of PO by MCF. The definition of PO effective convexity is the same as that of RMBS.*

assumption that prepayment behavior follows the proportional hazard model has empirical evidence, the formula seems to be more appropriate for practitioners than that in Yamazaki [2005] or Gorovoy and Linetsky [2007]. In addition, it is shown that the formula is applicable to the proportional hazard models not only with Gaussian processes, but also with Lévy processes. Numerical examples demonstrate that the accuracy of RMBS prices by the formula is remarkably better than sufficient, even when the second order approximation is adopted. Moreover, the formula is useful to calculate the effective durations and convexities very quickly.

Finally, our next research topic will be to establish more general pricing formulas for RMBS with other typical prepayment models such as the generalized additive model (see Jegadeesh and Ju [2000]) and the Poisson regression model (see Schwartz and Torous [1993]).

ENDNOTES

The views expressed in the article are those of the authors and do not necessarily reflect those of Mizuho-DL Financial Technology Co., Ltd.

[1]We assume that a prepayment cash flow occurs at time t_i, where $t_{i-1} < \tau \leq t_i$. This is a reasonable assumption when RMBS is considered, because RMBS cash flows including prepayment occur only at time $t_i (i = 0, 1, \ldots, mT)$. Although τ should be called *prepayment decision time* in terms of this assumption, we shall call it *prepayment time* for convenience.

[2]Strictly speaking, V_{t*} denotes a RMBS price per one unit against the *remaining principal*. However, in order to avoid redundant notations, we do not explicitly mention the *remaining principal*.

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