Modeling Swap Spreads in Normal and Stressed Environments

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The authors develop a simple integrated model for the term structure of swap spreads. They begin with a model that explains the dynamics of the riskless treasury curve in terms of two factors. They add to the basic model additional inputs that describe the evolution of the implied hazard rate intensity for interest rate swaps. Based on the model, they derive closed form expressions for observables such as the shape of the term structure of swap spreads and the term structure of volatilities. Their model is economically motivated and is appealing in its simplicity and robustness in being able to explain the dynamics of the swap-spread term structure in both normal and stressed markets. They apply the technique to swap-spread term structures for various international markets.

The interest rate swap market has been one of the most successful derivative markets of the last two decades. In its plain vanilla form, the swap market effectively allows a participant to swap a long-term asset or liability for a shorter one. Every swap has a fixed leg and a floating leg, and typically the floating leg is set to the three-month LIBOR rate. The modeling of swap spreads, defined as the difference between the fixed swap rate and a related government security, has a rich history both in academia and in practice and has evolved with our understanding of the dynamics of the market. The swap market is used by both hedgers and speculators, and it is critical to understand the dynamics that describe the movement of swap spreads. During the early years of the swap market, it was used as a proxy for the broad corporate bond market risk. As the swap market began to be used by hedgers, especially mortgage participants, and as the terms of swap contracts became more standardized, the swap market began its long reign as the market's benchmark for the reference rate on which relative value of various securities was measured. Since the swap contract has exposure to the movement of interest rates, it has increasingly been used as a tool to hedge the interest rate or duration risk by mortgage hedgers. More recently, we have seen a new phase in the development of this market. The floating rate (LIBOR) is a survey rate; the liquidity crisis of 2008 has called the viability of the rate under question for setting the pricing of trillions of dollars of derivatives contracts. The liquidity crisis has resulted in the inversion of the swap spread curve in many countries, i.e., shorter-term swap spreads are higher than longer-term swap spreads since the possibility of defaults in the banking sector has overwhelmed the compensation for risk as the maturity of contracts extends. Further, the dynamics of asset–liability matching in the U.K. and many European countries has driven the swap spreads in these countries to negative values as liability hedgers receive swaps to match their long-term liabilities. In a world where sovereign risk is deemed less risky than banking sector risk, this is clearly a situation that a good model should be able to handle. A similar situation developed in Japan a few years ago where the increased issuance of Japanese government bonds (JGBs) resulted in swap spreads going negative.

Clearly, a robust model of swap spreads has to be able to achieve what seems to be a trio of insurmountable tasks. First, the model has to be parsimonious, with just the right amount of, not too many and not too few, parameters, so that it can be used in real-time. Second, it has to be robust under the various dynamics and structural changes of the character of the market we described above. Finally, it has to be grounded in reality, i.e., the dynamics should be intuitive and the variables should capture the key factors that we know describe the movement of swap spreads. These factors include the level and shape of the sovereign yield curves, the relative supply of risky to riskless debt, the activity of mortgage hedgers (convexity hedging), and the movements in credit spreads (Cortes [2003]; Kobor, Shi, and Zelenko [2005]). Applying the same framework to the swap markets of multiple countries makes this task even more challenging, since we need to also be able to account for the idiosyncracies of their specific markets (for example the ALM flows in the U.K.).

The main purpose of the simple model introduced in this article is to explicitly include the impact of the shape and level of the yield curve on swap spreads. Previous work by Grinblatt [2001] focused on modeling the liquidity driver of the swap spread, and the same approach was later used by Duffie and Singleton [1997, 1999] in a two-factor affine model of the swap spread term structure. This modeling approach was later used in numerous studies including: He [2000], Liu, Longstaff, and Mandell [2002], Solnik and Collin-Dufresne [2000], and Johannes and Sundaresan [2003]. A more elaborate model may attempt to explicitly include the effects of several other economic drivers of swap spread such as the credit and liquidity factors. The model by Liu, Longstaff, and Mandell [2002], which introduced both liquidity and credit risk, used a five-factor affine model which results in a larger and more complicated model.

TERM STRUCTURE MODEL

The spread model depends on an underlying term structure model with two sources of uncertainty. The spot yield $i(T)$ for a given maturity T is given by the price of a one-dollar zero-coupon bond

$$P(T) = e^{-i(T)T} \tag{1}$$

If we denote the discount factor as $m(T)$ the discounted bond price is given by P. For a zero coupon bond to be a martingale we require

$$\mathrm{E}[d(mP)] = 0 \tag{2}$$

where E is the expected value at the present time. We define the instantaneous interest rate $z(t)$ for which the discount factor can be written in terms of z as

$$m(T) = e^{-\int_{t=0}^{T} z(t)dt} \tag{3}$$

which obeys

$$dm = -z(t)mdt \tag{4}$$

We define the instantaneous interest rate to evolve as follows

$$\begin{aligned} dz &= k(x + y - z)dt \\ dx &= -\mu_x(x - \theta_x)dt + \sigma_x dW_x \\ dy &= -\mu_y y dt + \sigma_y dW_y \end{aligned} \tag{5}$$

where both dW_x and dW_y are uncorrelated Wiener processes. Similarly to typical term structure models, the underlying structure of the yield curve is governed by two random variables x and y. The random variable x follows a mean-reverting random walk towards θ_x corresponding to the long-term rate. The random variable y impacts the slope of the yield curve and performs a mean-reverting walk towards zero. The parameters μ_x and μ_y determine the rate in which the variables mean revert, and the larger they are, the faster the variable mean reverts. The instantaneous rate z decays exponentially to the variables $x + y$ with a decay rate given by the parameter k. In this article we take the parameters in the model to be constants independent of time.[1]

The martingale condition (2) is written as

$$m\left[P_t dt + P_x dx + P_y dy + P_z dz + \frac{1}{2}(P_{xx}\sigma_x^2 + P_{yy}\sigma_y^2) - zP\right] = 0 \quad (6)$$

where we have used the notation $P_\alpha \equiv \partial P / \partial \alpha$ to denote a partial derivative. For a price of the form

$$P(x,y,z,t) = \exp[A(t)x + B(t)y + C(t)z + D(t)] \quad (7)$$

the solution is given by solving the following ODEs

$$\begin{aligned} 0 &= A' + kC - \mu_x A \\ 0 &= B' + kC - \mu_y B \\ 1 &= C' - kC \\ 0 &= D' + \mu_x \theta_x A + \frac{1}{2}(A\sigma_x)^2 + \frac{1}{2}(B\sigma_y)^2 \end{aligned} \quad (8)$$

with the boundary condition that the functions A, B, C, and D vanish for $T = 0$.

The zero yield for a horizon T is given by

$$i(T) = -\frac{1}{T}(Ax_0 + By_0 + Cz_0 + D) \quad (9)$$

which has a closed analytical form. Due to the length of the solution we omit the analytical form for $i(T)$ and display it in Appendix A. Thus we have a functional representation for the yield with 9 parameters: x_0, y_0, z_0, θ_x, μ_x, μ_y, k, σ_x, σ_y.

Out of these parameters, we identify 5 parameters μ_x, μ_y, k, σ_x, and σ_y as slow varying and they are assigned a constant value throughout the period of interest. The remaining 4 parameters x_0, y_0, z_0, and θ_x are fast varying in the sense that they are expected to change daily and as such are fitted to data daily. Thus, we have essentially reduced the large number of parameters to 4 by identifying the time scales on which each parameter is expected to change.

HAZARD RATE MODEL FOR SWAP SPREADS

For bonds with credit, liquidity, and other risks beyond the interest rate risks, the price of a one-dollar principal swap is given by

$$P(T) = e^{-T(i(T)+s(T))} \quad (10)$$

where $s(T)$ is the spread in the risky and riskless spot rates for maturity T bonds. As was described in the first section, the spread, to a good approximation, depends on four key parameters: the level of the yield curve, the steepness of the yield curve, market volatility, and the fraction of risky debt in the economy. In the term structure model described in the previous section the variables x and y are the drivers of the shape of the yield curve, and as such they drive both the yield curve level and the steepness. We are assuming that these term structure variables also drive part of the spread.

The swap discount factor can be written in terms of the instantaneous short rate z and an instantaneous hazard rate λ as

$$m(T) = e^{-\int_{t=0}^{T}(z(t)+\lambda(t))dt} \quad (11)$$

and we assume that the hazard rate λ has the following time evolution:

$$d\lambda = -\mu_\lambda(\lambda - \theta_\lambda - \gamma_x x - \gamma_y y)dt + \sigma_\lambda dW_\lambda \quad (12)$$

where x and y are the drivers of the yield curve shape and evolve according to the two-factor model of Equation (5). Their values are taken from fitting to the term structure model. The parameters γ_x and γ_y determine the response to the yield curve variables x and y on the spread evolution. Since the fraction of risky debt affects the swap spread, the model parameters—λ_0, μ_λ, θ_λ, γ_x, γ_y, and σ_λ—all have an implicit dependence on the fraction of risky debt. The parameter μ_λ determines the rate at which the swap spread hazard rate λ mean reverts toward its long-term level θ_λ. As for the term structure model we take the parameters in the model to be constants independent of time. By time independent parameters we do not mean that the values of the parameters are constant throughout the different trading days but that at a given day their value does not depend on the time horizon we consider. If this parsimonious model for the swap spread is well-specified, we should find not only that the parameters capture the shape of the swap spread curve over time and across economies, but also that their magnitudes and signs are consistent with our economic understanding of the variables underlying the dynamics of swap spreads.

By defining for clarity $\gamma_1 = \mu_\lambda \gamma_x$ and $\gamma_2 = \mu_\lambda \gamma_y$, the time evolution for the hazard rate can be written as

$$d\lambda = -\mu_\lambda(\lambda - \theta_\lambda)dt + \gamma_1 x dt + \gamma_2 y dt + \sigma_\lambda dW_\lambda \quad (13)$$

We solve for the swap spread in a similar fashion as was done previously in this article for the interest rate by demanding that the price of a swap is a martingale

$$\mathrm{E}[d(mP)] = 0 \quad (14)$$

Since

$$dm = -[z(t) + \lambda(t)]m dt \quad (15)$$

the martingale condition (14) is written as

$$m\left[P_t dt + P_x dx + P_y dy + P_z dz + P_\lambda d\lambda + \frac{1}{2}(P_{xx}\sigma_x^2 + P_{yy}\sigma_y^2 + P_{\lambda\lambda}\sigma_\lambda^2) - (z+\lambda)P\right] = 0 \quad (16)$$

for which we guess a solution

$$P(x,y,z,t) = \exp[A(t)x + B(t)y + C(t)z + F(t)\lambda + D(t)] \quad (17)$$

The solution is given by solving the following ODEs

$$\begin{aligned}
0 &= A' + kC - \mu_x A + \gamma_1 F \\
0 &= B' + kC - \mu_y B + \gamma_2 F \\
1 &= C' - kC \\
1 &= F' - \mu_\lambda F \\
0 &= D' + \mu_x \theta_x A + \mu_\lambda A \theta_\lambda + \frac{1}{2}(A\sigma_x)^2 \\
&\quad + \frac{1}{2}(B\sigma_y)^2 + \frac{1}{2}(F\sigma_\lambda)^2
\end{aligned} \quad (18)$$

with the boundary condition that the functions A, B, C, D and F, vanish for $T = 0$. The model has a closed form analytical solution for the swap spread that is given in the appendix.

This model for the swap rate has 6 more parameters on top of the 9 for the two plus interest rate model. These parameters are μ_λ, λ_0, θ_λ, γ_1, γ_2, and σ_λ. Out of these 6 parameters we identify three parameters, μ_λ, θ_λ, and σ_λ as slow-varying parameters, and they are assigned a constant value throughout the period of interest. This leaves 3 parameters to be fitted for each day. By transforming one or more of the slow-varying parameters into fast-varying parameters one can achieve a slightly better fit. Nevertheless, the model becomes less robust with regard to the dataset and for a certain dataset one of the parameters can become redundant, and even though the fit is still good the model volatility can dramatically increase through large terms canceling with each other. The fitting procedure is explained in more detail in the following section.

FITTING THE MODEL TO EMPIRICAL DATA

We proceed to examine the model by fitting the resulting spread using historical data on the swap spreads for three different markets: the U.S., U.K. and Japan. The dataset consists of five years of weekly values starting October 17, 2003, up to October 17, 2008. For each market we use the following time horizons: 3 m, 6 m, 1 yr, 2 yr, 5 yr, 10 yr, and 30 yr for both the treasury and the swap rates. We used Bloomberg and PIMCO databases for this data.

The fitting will be conducted in the following manner: first we fit the yield curve to the term structure model described previously. Once we have obtained the term structure parameters we fit the spread model to data. For each of the days we define the goodness of fit, ψ as the least squared value of the difference between the fit and the spread values divided by the sum of the squared spread values.

$$\psi = \frac{\Sigma_i (s_{fit}(T_i) - s(T_i))^2}{\Sigma_i s(T_i)^2} \quad (19)$$

where $s(T_i)$ is the value of the $T = T_i$ year spread from the data and $s_{fit}(T_i)$ is the value of the best fit.

Another statistic we can look at is the annualized spread volatility implied by the model. We define this annualized volatility σ_s as

$$\sigma_s(T) = \sqrt{\left(\sigma_x \frac{\partial s(t)}{\partial x_0}\right)^2 + \left(\sigma_y \frac{\partial s(T)}{\partial y_0}\right)^2 + \left(\sigma_\lambda \frac{\partial s(T)}{\partial \lambda_0}\right)^2} \quad (20)$$

The volatility has a closed form solution which we do not present here for brevity. It can be found in Appendix B. We calculate the volatility as implied by the model (20) since one can get a good fit for the spread data when there are very large cancelations between various terms that depend on different parameters. In such cases the model is over-specified and some of its parameters should be eliminated. One signal of this potential problem is that the volatility in (20) is very large.[2] We will calculate, for each day, the maximum annualized volatility[3] where the maximization is over all maturities for that day's volatility,

$$\max_{T} \sigma_s(T) \tag{21}$$

In fitting the model to the data we distinguish between the two kinds of parameters: slow and fast varying. The values of the fast-varying parameter are refit daily such that the value of the goodness of fit ψ (19) is minimized for the data corresponding to that day, with the slow-varying parameters held fixed. The slow-varying parameters will be fit over the entire period; they are taken as constant throughout the period such that the resulting sum of the daily values of ψ is minimized. When changing the values of the slow-varying parameters the fast-varying parameters are refit for each day and ψ is recalculated. In fitting the model to the datasets the parameter volatilities σ_x, σ_y, and σ_λ were given constant values and were not fitted for as was done for the rest of the slow-varying parameters. We use annualized absolute units for which an annual yield of 10% is set equal to 0.1.

We now proceed to fit the model to the five-year data described for each of the countries U.S., Japan and U.K. and present the results and discussions separately for each of the countries. The fitting was carried out as described previously.

U.S. Data

The fitting of the U.S. dataset differs from that of Japan and the U.K. due to the special nature of the 30-year sector. The U.S. Treasury discontinued the issuance of 30-year bonds in the early part of this decade when faced with surpluses and brought them back a few years ago. Thus we fit the model twice, once without the 30-year swap and then again with 30-year swaps (described in the appendix). Recently, 30-year U.S. swap spreads went negative, so this is an important test for the model.

Exhibit 1

The values of the slow-varying parameters as fitted for each of the three datasets: U.S., U.K., and Japan. The values are given in annualized absolute units such that 0.1 corresponds to 10% annual rate.

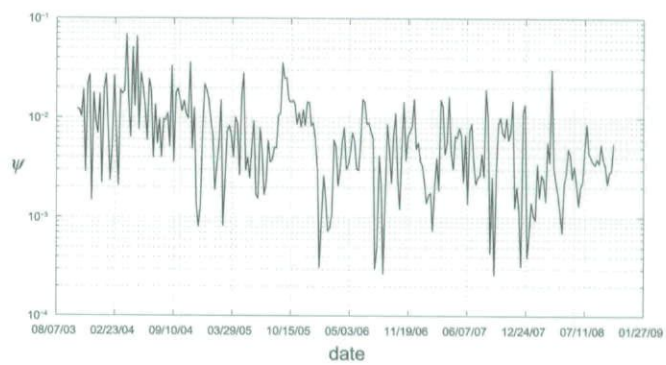
Dataset	μ_λ	θ_λ	σ_λ	μ_x	μ_y	σ_x	σ_y	k
U.S.	2.05	0.01	0.002	0.01	0.4	0.009	0.015	3
U.K.	3.05	0.002	0.002	0.012	0.5	0.009	0.015	1.2
Japan	0.55	0.0005	0.0007	0.03	0.11	0.001	0.002	0.3

The resulting goodness of fit ψ (19) and the maximum annualized volatility (21) is plotted for the different trading days and given in Exhibit 2. The volatility in the model is relatively low compared to the model volatility for the U.K. and Japan datasets.

The values of the slow-varying parameters resulting from the best fit for the data are given in Exhibit 1 while the values of the fast-varying parameters are plotted in Exhibits 3 and 4 for each of the days

Exhibit 2

U.S. goodness of fit; the goodness of fit ψ as given in (19), and the maximum annualized volatility (21) are plotted for each of the days in the dataset.



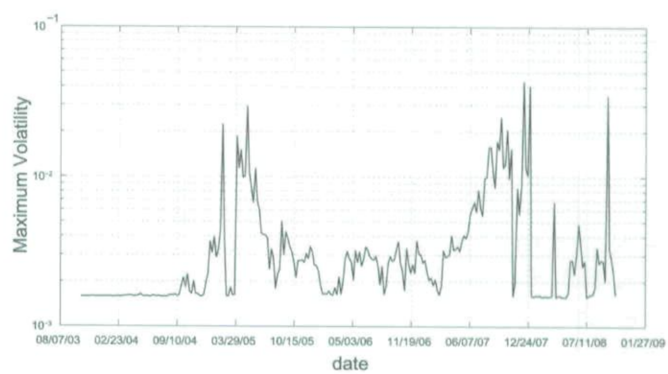
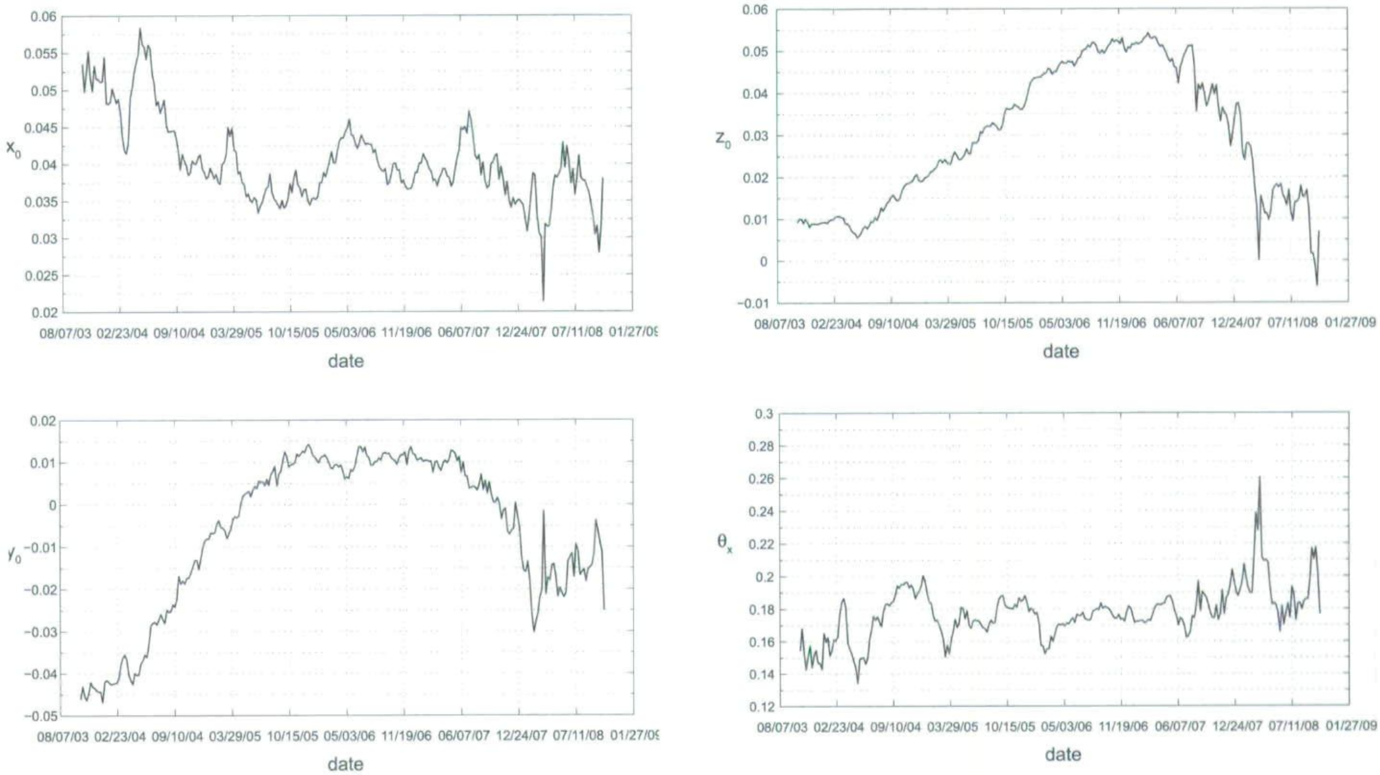


EXHIBIT 3

U.S. parameter values; the daily values as derived from fitting the term structure model for each of the trading days in the dataset.



in the dataset. By visually inspecting the parameters we see here, as well as in the U.K. dataset, transitions around July 2007 and August 2008 where λ_0 increases dramatically. At the time of the second increase in August 2008 the LIBOR rate became very high and volatile as a consequence of the liquidity and banking crisis. This can be seen in Exhibit 6 where we present the spread curve for four different days within the timeframe of the dataset. For each of the days we compare the curve as resulting from the best fit to the empirical data. It can be seen that the short-term swap spreads increase and that for October 3, 2008 the three-month spread is approximately 400 bps. Moreover, the spread curve has changed its structure from an almost monotonically increasing shape as can be seen in April 4, 2005 to the recent monotonically decreasing shape as can be seen in October 3, 2008. This is also the case when one considers the 30-year swap as can be seen in Appendix C. We added a comparison of the fit to the data including the 30-year swap plotted for four different days in Exhibit 7. Another similarity with the U.K. dataset is the occasional large values of the parameter γ_2. These large fluctuations coincide with the periods in which the parameter y is small and since the contribution to the spread (3) scales as $\gamma_2 y$, the parameter's contributions is several orders of magnitude less.

As another test of the model we examine the economic validity of the parameter values resulting from the fit. Since our model depends on the drivers of the term structure model we can verify that our model captures the dependence of the swap spread on the yield curve structure. It has been shown that one of the drivers of the swap spread is the slope of the yield curve. In [1, 2] the value of the 10-year swap spread was regressed versus the yield curve slope for data spanning January 1997 to August 2003. As the slope of the yield curve the author used the difference between the 10-year and 2-year rates, and the linear regression coefficient was calculated to be −0.19. The negative value of the coefficient implies that the swap spread tightens as the curve steepens. This coincides with the intuition that as the yield curve steepens market players prefer to be the fixed-rate receiver in a swap and commercial banks are less inclined to hedge their short-term variable rate

Exhibit 4

U.S. parameter values; the daily values as derived from fitting the model for each of the trading days in the dataset.

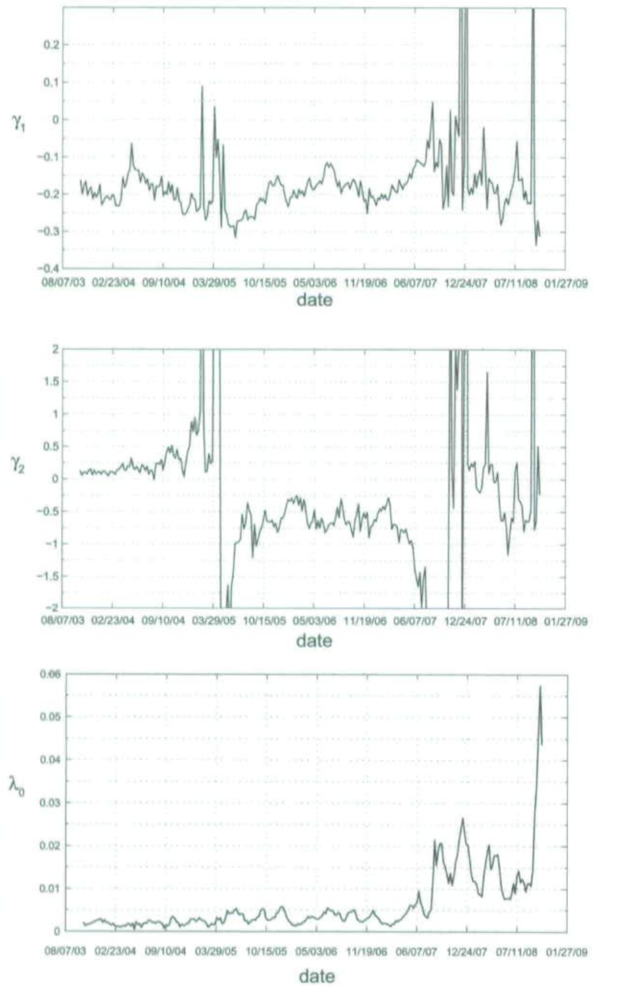
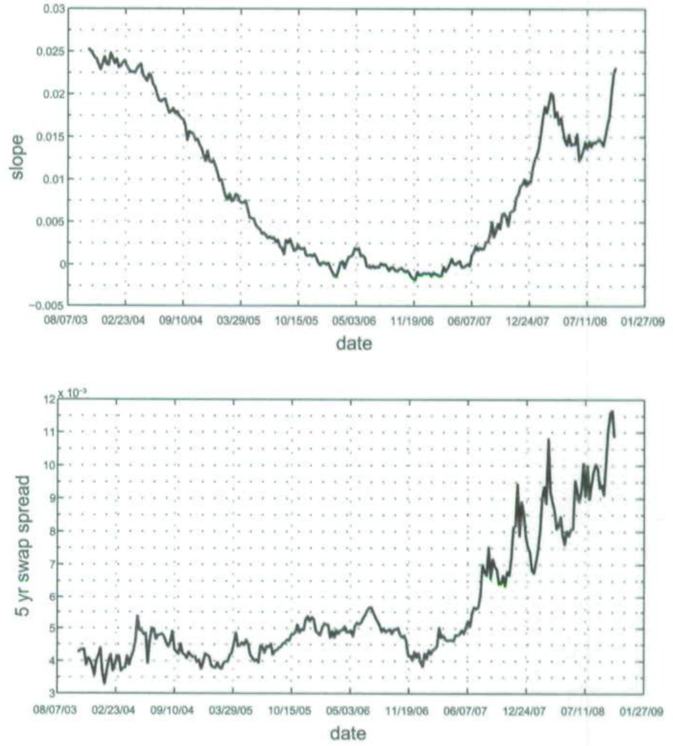


Exhibit 5

Top: The yield curve slope calculated as the difference between the 10-year and the 2-year rates. Bottom: The 5-year swap spread.



deposits, reducing the demand to be a fixed rate payer. This will reduce the demand for swaps and as a result will decrease the swap rates and tighten the spread. The instantaneous slope of the spot rate is given by the instantaneous interest rate $z(t)$ which depends on $x(t)$ and $y(t)$ through (5). The parameter y mean reverts to zero with the rate $\mu_y = 0.4$ which is large compared to $\mu_x = 0.01$ for both the 2-year rate and especially for the 10-year rate. As such, the slope as defined in [1, 2], is mainly determined by the value of x. The contribution of x to the swap spread is determined by the parameter γ_1 which should be negative to coincide with a negative relation between the slope and the curve. As can be seen in Exhibit 4 the value of γ_1 is usually negative which coincides with such a negative dependence.

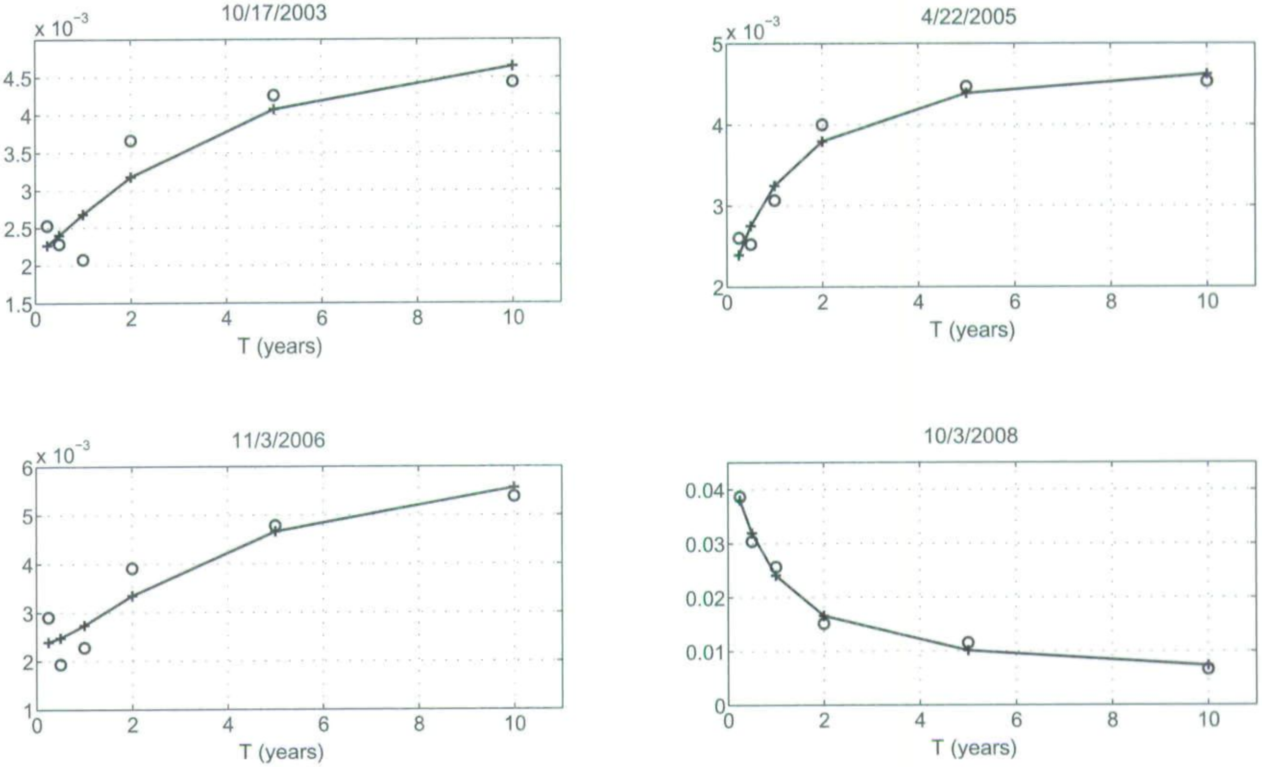
Interestingly, the negative dependence of the swap spread on the slope of the yield curve breaks down around the summer of 2007 as can be seen in Exhibit 5 where we plot both the 5 year swap spread and the slope of the yield curve. As can be seen in the figure the negative dependence of the swap spread on the curve seems to be valid for the first period whereas the slope decreased the swap spread widened by approximately an order of magnitude less which corresponds to a coefficient of approximately −0.1. Starting at around June 2007 the swap spread widens as the slope of the yield curve steepens. During that period the value of γ_1 oscillates wildly. This behavior seems to be associated with a breaking of the usual dependence of the swap spread on the slope of the yield curve.

U.K. Data

For the U.K. dataset, the goodness of fit ψ (19) and the maximum annualized volatility (21) are plotted for the different trading days and given in Exhibit 8. Relative to the other datasets, Japan and the U.S., the maximum annualized

EXHIBIT 6

U.S. spread curve; the spread curve from fitting the model is compared to the empirical spread for four different days throughout the five-year time period. The empirical data is given by (∘) while the spread resulting from the model is given by (+) and is connected with a full line.



volatility from the model is high and is around 100 basis points. To understand how that reflects on the daily curve we turn the annualized volatility to daily volatility by dividing the value by the square root of the number of trading days in a year. Thus the daily volatility is of order $\approx 100/16 = 6.25$ basis points. This is smaller than the swap spread which is on the order of magnitude of tens of basis points throughout the five-year period.

The values of the slow-varying parameters resulting from the best fit for the data are given in Exhibit 1 while the values of the fast-varying parameters are plotted in Exhibits 9 and 10 for each of the days in the dataset. By examining the parameter values one can see that around July 2007 there is a transition in which the parameters become more volatile and especially interesting is the parameter λ_0 which increases by an order of magnitude from 5 bps to around 50 bps. Even though the value of γ_2 is relatively large, as can be seen in Exhibit 10, the contribution to the spread (13) scales as $\sim\gamma_2 y$ which is several orders of magnitude smaller especially since the parameter y takes on small values for these days. For the U.K. as opposed to the U.S. the dependence of the swap spread on the slope was not observed to be as significant relative to the other factors, and as such we do not expect the parameter γ_1 to be negative as is the U.S. case.

In Exhibit 11 we present the spread curve for four different days within the timeframe of the dataset. For each of the days we compare the curve as resulting from the best fit to the empirical data. By comparing the swap spread curve between the four different days throughout the five-year period one notices that the swap spread curve has changed its structure considerably. The short-term spread is sharply peaked with the peak around 40 basis points throughout most of the period while lately it has leaped to over 200 basis points as can be seen in the fit for October 3, 2008 in accord with the large shift in the value of λ_0 observed. Another interesting change in the curve is the long-term rate, which has shifted recently from around 20 to 40 bps to negative values.

Exhibit 7

U.S. spread curve; the spread curve as resulting from fitting the model to the data set including the 30 year swap, is compared to the empirical spread for four different days throughout the five year time period. The empirical data is given by (∘) while the spread resulting from the model is given by (+) and is connected with a full line.

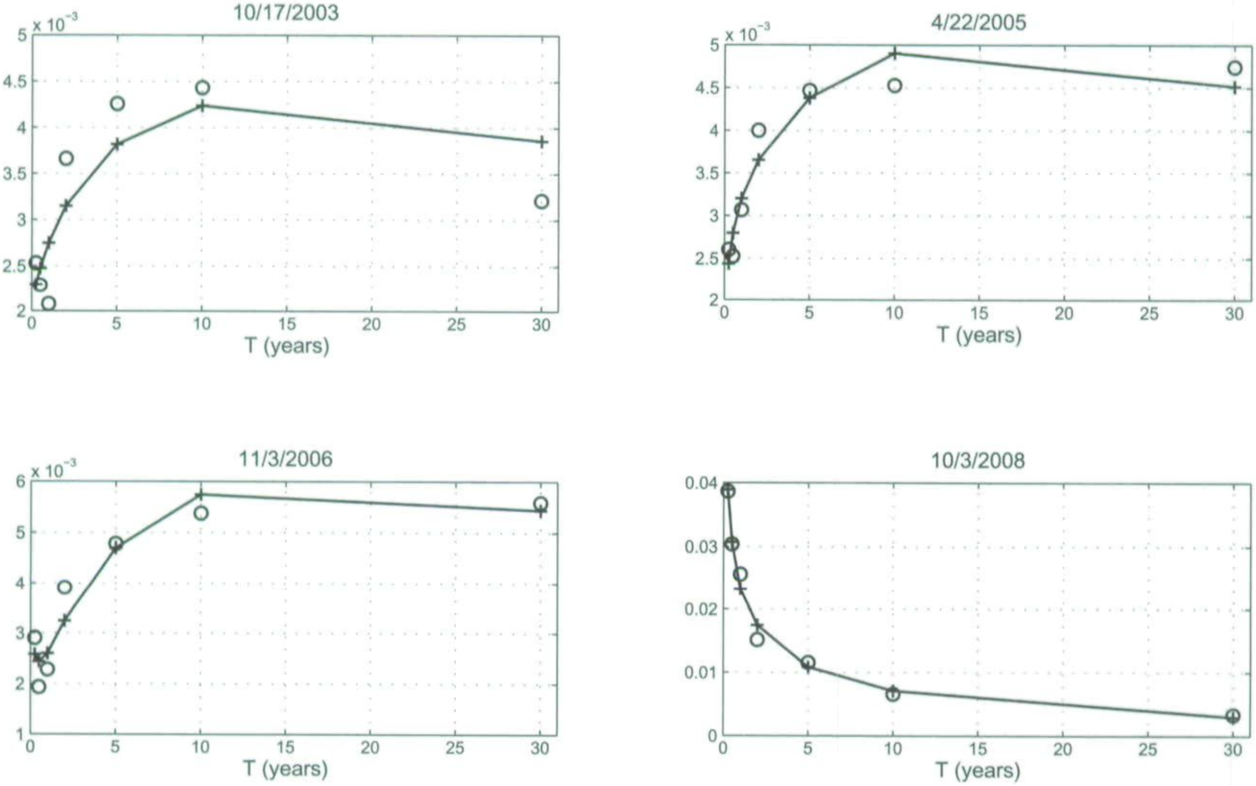
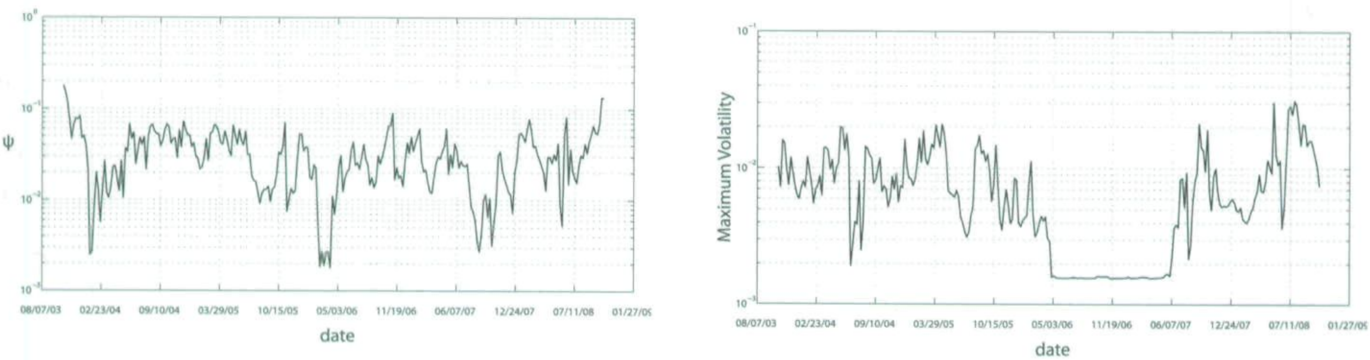


Exhibit 8

U.K. goodness of fit; the goodness of fit ψ as given in (19) and the maximum annualized volatility (21) are plotted for each of the days in the dataset.



Even though the curve as fitted from the model does not pass through the points, it adapts to the change with some points above and some below, which might allow for cheap/rich analysis.

Japan Data

For the Japan dataset, the goodness of fit ψ (19) and the maximum annualized volatility (21) are plotted for the different trading days and given in Exhibit 12.

EXHIBIT 9

U.K. parameter values; the daily values as derived from fitting the term structure model for each of the trading days in the dataset.

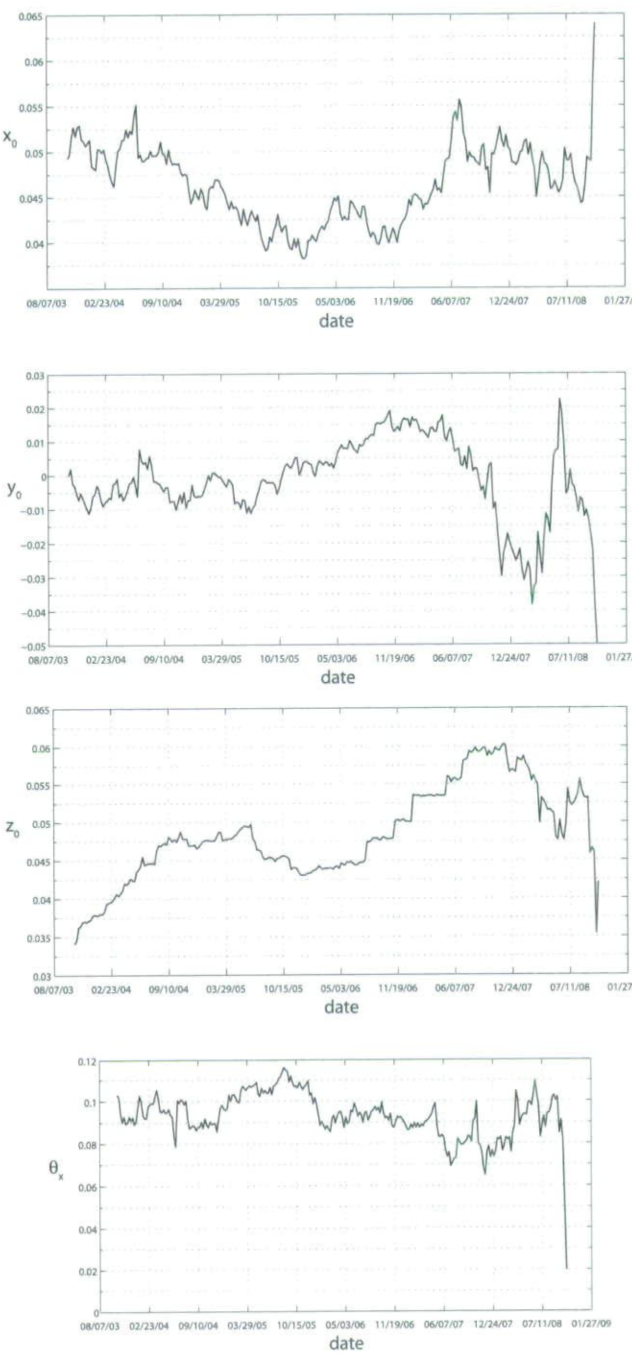
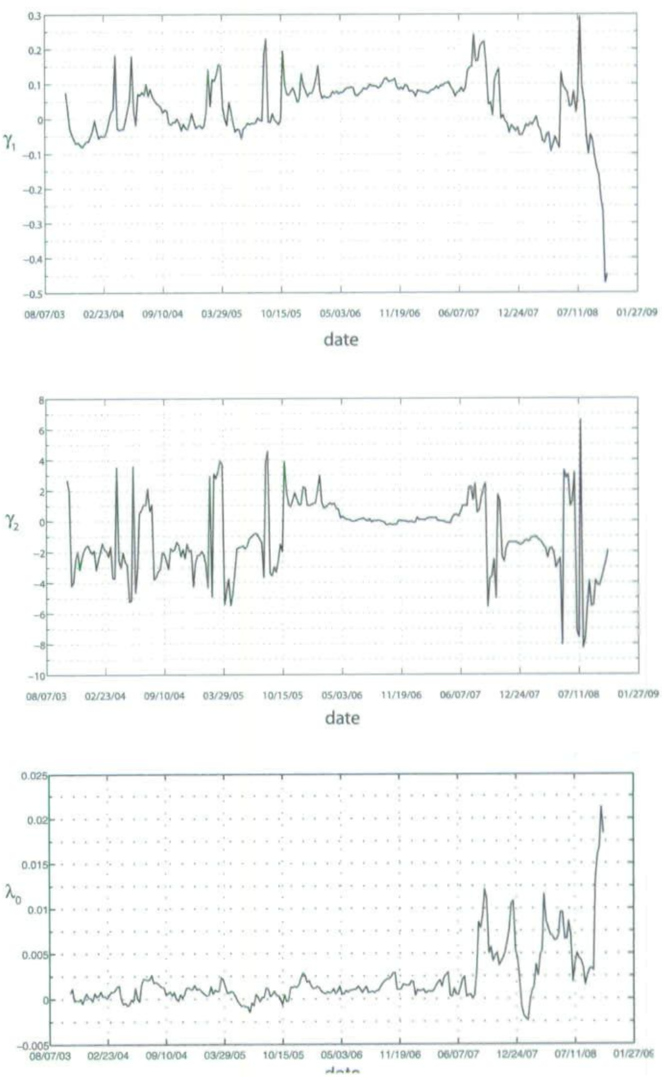


EXHIBIT 10

U.K. parameter values; the daily values as derived from fitting the model for each of the trading days in the dataset.



The annualized volatility of the model for this dataset is lower than that of the other two datasets, which can be attributed to the relatively smaller values of the swap spread itself. The values of the slow varying parameters resulting from the best fit for the data are given in Exhibit 1. The Japan dataset is different from that of the U.S. and the U.K. in that the short-term rates: 3 m,

EXHIBIT 11

U.K. spread curve; the spread curve as rresulting from fitting the model is compared to the empirical spread for four different days throughout the five-year period. The empirical data is given by (∘) while the spread resulting from the model is given by (+) and is connected with a full line.

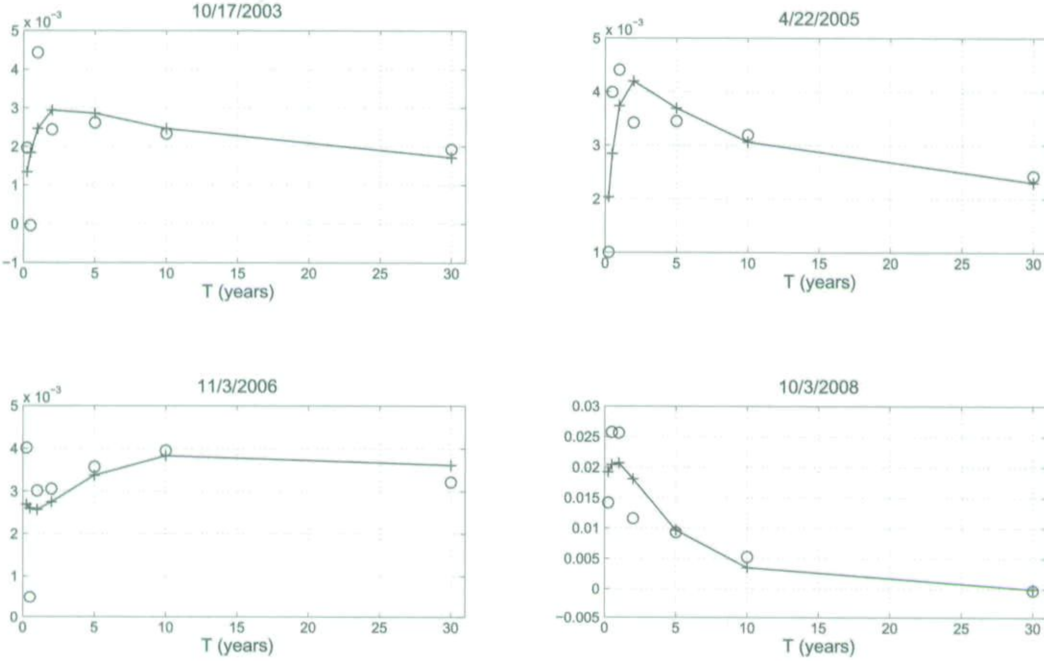
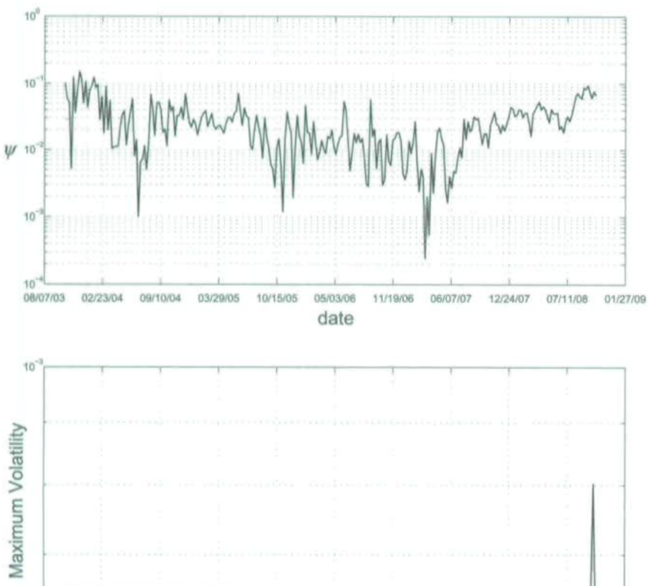


EXHIBIT 12

Japan goodness of fit; the goodness of fit ψ as given in (19) and the maximum annalized volatility (21) are plotted for each of the days in the dataset.



6 m, and 1 yr were essentially zero for the first half of the dataset up to May 2006, when the rates increased by almost an order of magnitude. To fit the whole term structure we have chosen to turn the parameter θ_x into a slow-varying parameter. It was fitted accordingly and found to have the value $\theta_x = 0.02$. Due to the unique nature of the Japanese yield curve, if we allow θ_x to vary daily then the fit parameters x_0 and y_0 are unreasonably large. The fits look good because there are large cancelations between the terms that involve x_0 and y_0. The values of the fast-varying parameters are plotted in Exhibits 13 and 14 for each of the days in the dataset. Not surprisingly here too we observe a transition in the parameter values around July 2007 where the parameter λ_0 has an order of magnitude increase.

In Exhibit 15 we present the spread curve for four different days within the timeframe of the dataset. For each of the days we compare the curve as resulting from the best fit to the empirical data. The Japanese swap spread is the tightest of the three datasets with values ranging mostly at the single digit level. Here too, as was the case for the U.S. and the U.K., around July 2007 there is a recent order of magnitude increase in the short-term swap spread, though

EXHIBIT 13

Japan parameter values; the daily values as derived from fitting the term structure model for each of the trading days in the dataset.

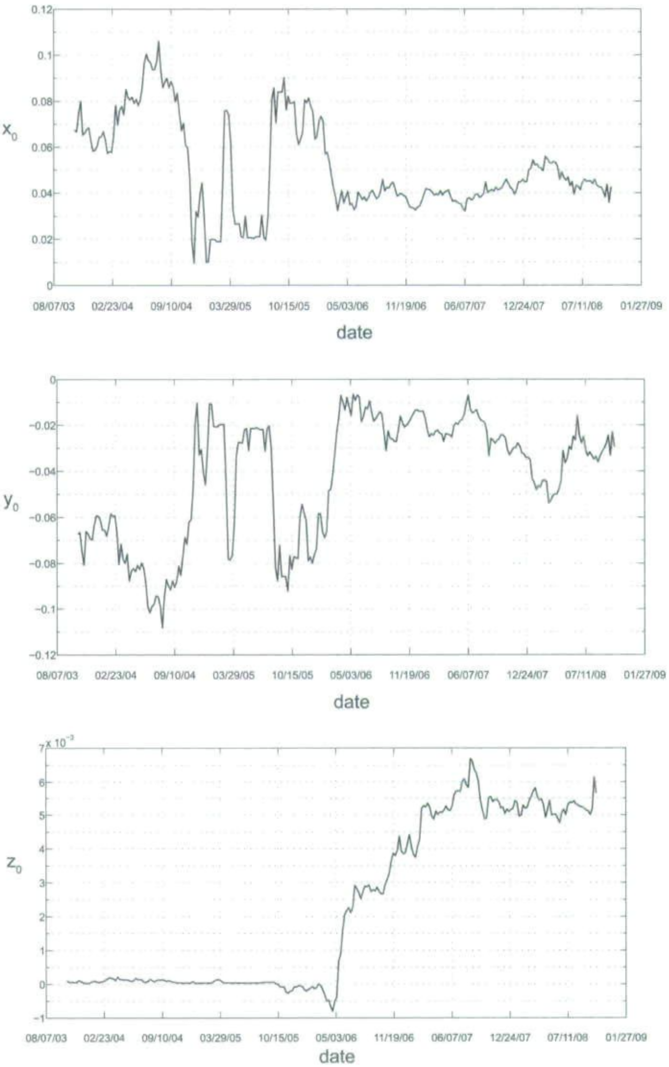
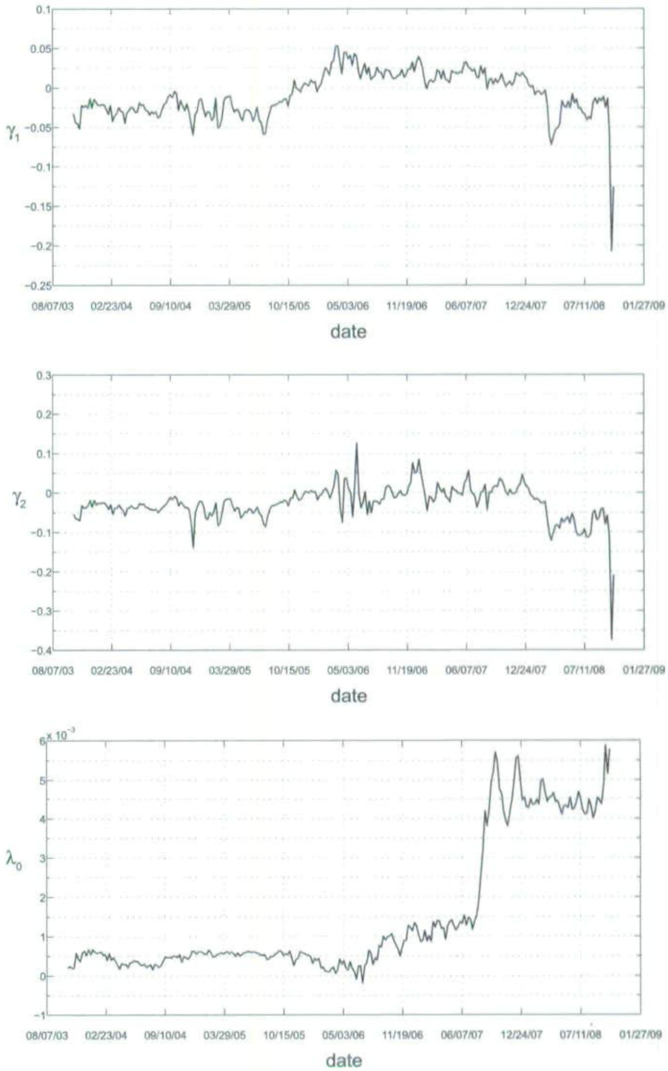


EXHIBIT 14

Japan parameter values; the daily values as derived from fitting the model for each of the trading days in the dataset.



here it increases to around 50 bps as opposed to several hundred bps in the case of the U.K. and the U.S. An interesting feature in the Japanese dataset is that the long-term swap spread has been negative throughout the dataset except for a period, as can be seen for November 3, 2006, in which the swap spread curve was increasing with a positive long-term spread.

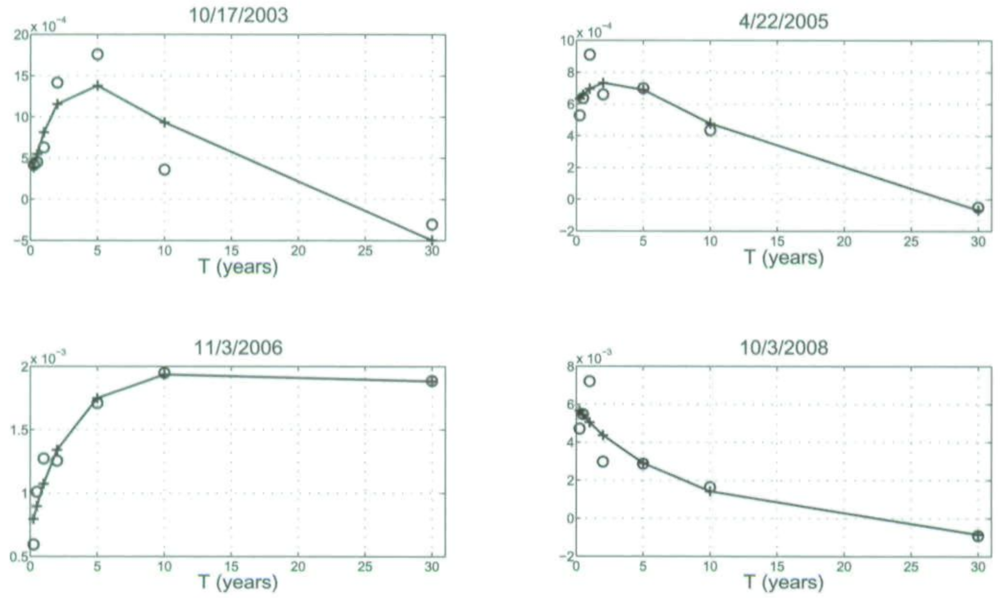
For Japan the dependence of the swap spread on the slope of the yield curve was not tested. Still, the value of γ_1 which determines the impact of the parameter x is mostly negative, implying a negative contribution for an increasing slope while for a period around the end of 2006 the parameter becomes positive coinciding with the increase of the short-term rate rise.

CONCLUSIONS

We have introduced a simple integrated model for the term structure of the swap spread. The model builds on a two-factor model for the treasury curve by adding on an implied hazard rate for interest rate swaps. The hazard rate was constructed with an economic intuition in mind, using known empirical results for the main drivers of the swap spread as a guideline. We have succeeded

EXHIBIT 15

Japan spread curve; the spread curve as resulting from fitting the model is compared to the empirical spread for four different days throughout the five-year time period. The empirical data is given by (∘) while the spread resulting from the model is given by (+) and is connected with a full line.



in constructing a parsimonious model with a relatively small number of parameters that can be attributed explicitly to the interest rate term structure and implicitly to the other drivers of the swap spread. Using the model we obtained a closed form for the swap spread term structure and the term structure of the volatilities.

The framework was applied for three swap markets: the U.S., the U.K., and Japan. For each of the swap markets the data spanned five years, from October 2003 to October 2008, which includes the recent liquidity crisis. During this period the swap spread term structure has varied dramatically; for the U.S. and the U.K. the term structure changed from an increasing function to a decreasing function, and for both the U.K. and Japan the swap spread was negative for a fraction of the dataset. This allowed us to test the model not only for different countries but for both normal and stressed periods. Remarkably this simple model provides a good fit for the data for all three markets.

The liquidity crisis offered a good testing ground for the framework not only because the term structures changed their shape but also because there seems to be a change in the drivers themselves. We have shown that for the U.S. (and the U.K. to a lesser extent) the usual dependence of the swap spread on the slope has broken. The recent stressed period offers a remarkable opportunity for testing the model under fundamentally different market dynamics and the simple model presented in this article was shown to be quite robust.

Appendix A

Analytical Solution of the Term Structure Model

The instantaneous interest rate in the term structure model, follows the evolution

$$
\begin{aligned}
dz &= k(x+y-z)dt \\
dx &= -\mu_x(x-\theta_x)dt + \sigma_x dW_x \\
dy &= -\mu_y y dt + \sigma_y dW_y
\end{aligned} \tag{A1}
$$

where both dW_x and dW_y are uncorrelated Wiener processes and the parameters in the model are constants independent of time. The solution for the time evolution of the parameters is given by

$$
\begin{aligned}
x(t) &= x_0 e^{-\mu_x t} + \theta_x(1-e^{-\mu_x t}) + \sigma_x e^{-\mu_x t}\int_0^t e^{\mu_x s} dW_x(s) \\
y(t) &= y_0 e^{-\mu_y t} + \sigma_y e^{-\mu_y t}\int_0^t e^{\mu_y s} dW_y(s) \\
z(t) &= z_0 e^{-kt} + \frac{k(x_0-\theta_x)}{k-\mu_x}(e^{-\mu_x t}-e^{-kt}) + \theta_x(1-e^{-kt}) + \frac{ky_0}{k-\mu_y}(e^{-\mu_y t}-e^{-kt}) + \frac{k\sigma_x}{k-\mu_x}\int_0^t (e^{\mu_x(s-t)}-e^{k(s-t)})dW_x(s) \\
&\quad + \frac{k\sigma_y}{k-\mu_y}\int_0^t (e^{\mu_y(s-t)}-e^{k(s-t)})dW_y(s)
\end{aligned} \tag{A2}
$$

The martingale condition (2) is written as

$$
m\left[P_t dt + P_x dx + P_y dy + P_z dz + \frac{1}{2}(P_{xx}\sigma_x^2 + P_{yy}\sigma_y^2) - zP\right] = 0 \tag{A3}
$$

where we have used the notation $P_\alpha \equiv \partial P/\partial\alpha$ to denote a partial derivative. For a price of the form

$$
P(x,y,z,t) = \exp[A(t)x + B(t)y + C(t)z + D(t)] \tag{A4}
$$

the solution is given by solving the following ODEs

$$
\begin{aligned}
0 &= A' + kC - \mu_x A \\
0 &= B' + kC - \mu_y B \\
1 &= C' - kC \\
0 &= D' + \mu_x\theta_x A + \frac{1}{2}(A\sigma_x)^2 + \frac{1}{2}(B\sigma_y)^2
\end{aligned} \tag{A5}
$$

with the boundary condition that the functions A, B, C, and D vanish for $T = 0$. The solution for each of the above is given by

$$
\begin{aligned}
A &= \frac{k}{k-\mu_x}\left(\frac{1-e^{-kT}}{k}+\frac{-1+e^{-T\mu_x}}{\mu_x}\right)\\
B &= \frac{k}{k-\mu_y}\left(\frac{1-e^{-kT}}{k}+\frac{-1+e^{-T\mu_y}}{\mu_y}\right)\\
C &= \frac{1}{k}(e^{-kT}-1)
\end{aligned}
\tag{A6}
$$

and

$$
\begin{aligned}
D = \frac{1}{4k}\Bigg(& \frac{4\theta_x(k+(1-kT)\mu_x)}{\mu_x}+\frac{4e^{-kT}\theta_x(-e^{T(k-\mu_x)}k^2+\mu_x^2)}{(k-\mu_x)\mu_x}+\frac{(-3k^2+k(-5+2kT)\mu_x+(-3+2kT)\mu_x^2)\sigma_x^2}{\mu_x^3(k+\mu_x)}-\frac{1}{(k-\mu_x)^2\mu_x^3(k+\mu_x)}\\
& \times e^{-2T(k+\mu_x)}(-e^{2kT}(-1+4e^{T\mu_x})k^4+e^{2kT}k^3\mu_x+4e^{T(k+\mu_x)}(-1+e^{kT}+e^{T\mu_x})k^2\mu_x^2+e^{2T\mu_x}k\mu_x^3+e^{2T\mu_x}(1-4e^{kT})\mu_x^4)\sigma_x^2\\
& +\frac{(-3k^2+k(-5+2kT)\mu_y+(-3+2kT)\mu_y^2)\sigma_y^2}{\mu_y^3(k+\mu_y)}-\frac{1}{(k-\mu_y)^2\mu_y^3(k+\mu_y)}e^{-2T(k+\mu_y)}(-e^{2kT}(-1+4e^{T\mu_y})k^4+e^{2kT}k^3\mu_y\\
& +4e^{T(k+\mu_y)}(-1+e^{kT}+e^{T\mu_y})k^2\mu_y^2+e^{2T\mu_y}k\mu_y^3+e^{2T\mu_y}(1-4e^{kT})\mu_y^4)\,\sigma_y^2\Bigg)
\end{aligned}
\tag{A7}
$$

The zero yield for a horizon T is given by plugging in the above solutions to A, B, C, and D into

$$
i(T) = -\frac{1}{T}(Ax_0+By_0+Cz_0+D) \tag{A8}
$$

The volatility of the term structure model defined as

$$
V_i(T) = \sqrt{\left(\frac{\partial i(T)}{\partial x_0}\sigma_x\right)^2+\left(\frac{\partial i(T)}{\partial y_0}\sigma_y\right)^2} \tag{A9}
$$

is given by

$$
V_i(T) = \sqrt{\frac{(k-e^{-T\mu_x}k+(-1+e^{-kT})\mu_x)^2\sigma_x^2}{T^2(k-\mu_x)^2\mu_x^2}+\frac{(k-e^{-T\mu_y}k+(-1+e^{-kT})\mu_y)^2\sigma_y^2}{T^2(k-\mu_y)^2\mu_y^2}} \tag{A10}
$$

Appendix B

Analytical solution of the hazard rate model

The evolution for the hazard is given by

$$
d\lambda = -\mu_\lambda(\lambda-\theta_\lambda)dt+\gamma_1 x dt+\gamma_2 y dt+\sigma_\lambda dW_\lambda \tag{B1}
$$

The solution for the instantaneous hazard rate is given by

$$
\begin{aligned}
\lambda(t) = {} & \lambda_0 e^{-\mu\lambda t} + \theta_\lambda(1-e^{-\mu_\lambda t}) + \frac{\gamma_1\theta_x}{\mu_\lambda}(1-e^{-\mu_\lambda t}) + \frac{\gamma_1(x_0-\theta_x)}{\mu_\lambda-\mu_x}(e^{-\mu_x t}-e^{-\mu_\lambda t}) + \frac{\gamma_2 y_0}{\mu_\lambda-\mu_y}(e^{-\mu_y t}-e^{-\mu_\lambda t}) \\
& + \frac{\gamma_1\sigma_x}{\mu_\lambda-\mu_x}\int_0^t (e^{\mu_x(s-t)}-e^{\mu_\lambda(s-t)})\,dW_x(s) + \frac{\gamma_2\sigma_y}{\mu_\lambda-\mu_y}\int_0^t (e^{\mu_y(s-t)}-e^{\mu_\lambda(s-t)})dW_y(s) + \sigma_\lambda \mathrm{e}^{-\mu_\lambda t}\int_0^t e^{\mu_\lambda s}dW_\lambda(s)
\end{aligned}
\tag{B2}
$$

We solve for the swap spread in a similar fashion as was done for the interest rate by demanding that the price of a swap is a martingale

$$\mathrm{E}[d(mP)] = 0 \tag{B3}$$

Since

$$dm = -[z(t)+\lambda(t)]mdt \tag{B4}$$

the martingale condition (B3) is written as

$$m\left[P_t dt + P_x dx + P_y dy + P_z dz + P_\lambda d\lambda + \frac{1}{2}(P_{xx}\sigma_x^2 + P_{yy}\sigma_y^2 + P_{\lambda\lambda}\sigma_\lambda^2) - (z+\lambda)P\right] = 0 \tag{B5}$$

for which we guess a solution

$$P(x,y,z,t) = \exp[A(t)x + B(t)y + C(t)z + F(t)\lambda + D(t)] \tag{B6}$$

The solution is given by solving the following ODEs

$$
\begin{aligned}
0 &= A' + kC - \mu_x A + \gamma_1 F \\
0 &= B' + kC - \mu_y B + \gamma_2 F \\
1 &= C' - kC \\
1 &= F' - \mu_\lambda F \\
0 &= D' + \mu_x\theta_x A + \mu_\lambda A\theta_\lambda + \frac{1}{2}(A\sigma_x)^2 + \frac{1}{2}(B\sigma_y)^2 + \frac{1}{2}(F\sigma_\lambda)^2
\end{aligned}
\tag{B7}
$$

with the boundary condition that the functions A, B, C, D and F vanish for $T=0$. The solution for each of the above is given by

$$
\begin{aligned}
A = {} & (e^{-T\mu_x}(e^{-kT}(e^{kT}(-1+e^{T\mu_x})k - e^{T\mu_x}(-1+e^{kT})\mu_x)(\mu_x-\mu_\lambda)\mu_\lambda \\
& + \gamma_1(-k+\mu_x)(e^{T\mu_x}(-1+e^{-T\mu_\lambda})\mu_x + (-1+e^{T\mu_x})\mu_\lambda)))/(\mu_x(-k+\mu_x)(\mu_x-\mu_\lambda)\mu_\lambda) \\
B = {} & (e^{-T\mu_y}(e^{-kT}(e^{kT}(-1+e^{T\mu_y})k - e^{T\mu_y}(-1+e^{kT})\mu_y)(\mu_y-\mu_\lambda)\mu_\lambda \\
& + \gamma_2(-k+\mu_y)(e^{T\mu_y}(-1+e^{-T\mu_\lambda})\mu_y + (-1+e^{T\mu_y})\mu_\lambda)))/(\mu_y(-k+\mu_y)(\mu_y-\mu_\lambda)\mu_\lambda) \\
C = {} & \frac{1}{k}(\mathrm{e}^{-kT}-1) \\
F = {} & \frac{1}{\mu_\lambda}(\mathrm{e}^{-\mu_\lambda T}-1)
\end{aligned}
\tag{B8}
$$

Since the solution for D is quite long we will decompose it for the sake of clarity as

$$D = a_x\theta_x + a_\lambda\theta_\lambda + (b_0 + b_1\gamma_1 + b_2\gamma_1^2)\sigma_x^2 + (c_0 + c_1\gamma_2 + c_2\gamma_2^2)\sigma_y^2 + (d_0 + d_1\gamma_1 + d_2\gamma_2)\sigma_\lambda^2 \tag{B9}$$

given by

$$a_x = -\frac{kT}{k-\mu_x} + \frac{k - e^{-T\mu_x}k}{k\mu_x - \mu_x^2} - \frac{\gamma_1(1 - e^{-T\mu_x} - T\mu_x)}{\mu_x(\mu_x - \mu_\lambda)} + \mu_x\left(-\frac{1 - e^{-kT} - kT}{k^2 - k\mu_x} + \frac{\gamma_1(1 - e^{-T\mu_\lambda} - T\mu_\lambda)}{(\mu_x - \mu_\lambda)\mu_\lambda^2}\right) \tag{B10}$$

$$a_\lambda = -T + \frac{1}{\mu_\lambda}(1 - e^{-T\mu_\lambda}) \tag{B11}$$

$$\begin{aligned} b_0 = {} & (e^{-2T(k+\mu_x)}(-e^{2kT}(1 - 4e^{T\mu_x} + 3e^{2T\mu_x})k^4 + e^{2kT}k^3(-1 + e^{2T\mu_x}(1 + 2kT))\mu_x - 2e^{T(k+\mu_x)}k^2 \\ & \times (-2 + 2e^{kT} + 2e^{T\mu_x} + e^{T(k+\mu_x)}(-2 + kT))\mu_x^2 - e^{2T\mu_x}k(1 + e^{2kT}(-1 + 2kT))\mu_x^3 \\ & + e^{2T\mu_x}(-1 + 4e^{kT} + e^{2kT}(-3 + 2kT))\mu_x^4))/(4k(k-\mu_x)^2\mu_x^3(k+\mu_x)) \end{aligned} \tag{B12}$$

$$\begin{aligned} b_1 = {} & (e^{-T(5k+7\mu_x+4\mu_\lambda)}(2e^{T(4k+7\mu_x+3\mu_\lambda)}(-1 + e^{kT})(-1 + e^{T\mu_\lambda})k\mu_x^4\mu_\lambda(k+\mu_\lambda) + e^{T(5k+5\mu_x+4\mu_\lambda)}(1 - 4e^{T\mu_x} + 3e^{2T\mu_x})k^3\mu_\lambda^3(k+\mu_\lambda) \\ & + e^{T(5k+5\mu_x+4\mu_\lambda)}k^2\mu_x\mu_\lambda^2(k+\mu_\lambda)((-1 + e^{T\mu_x})^2k + (1 - 2e^{T\mu_x} + e^{2T\mu_x}(1 - 2kT))\mu_\lambda) - e^{T(4k+5\mu_x+3\mu_\lambda)}(-1 + e^{T\mu_x})k\mu_x^2\mu_\lambda(k+\mu_\lambda) \\ & \times (2e^{T(k+\mu_x)}(-1 + e^{T\mu_\lambda})k^2 + e^{T(k+\mu_\lambda)}(1 + e^{T\mu_x})k\mu_\lambda + 2e^{T(\mu_x+\mu_\lambda)}(-1 + e^{kT})\mu_\lambda^2) - 2e^{T(4k+7\mu_x+3\mu_\lambda)}\mu_x^5(-e^{kT}(-1 + e^{T\mu_\lambda})k^2 \\ & + k(-1 + e^{kT} + e^{T\mu_\lambda} + e^{T(k+\mu_\lambda)}(-1 + kT))\mu_\lambda + e^{T\mu_\lambda}(1 + e^{kT}(-1 + kT))\mu_\lambda^2) + 2\mu_x^3(-e^{T(5k+7\mu_x+3\mu_\lambda)}(-1 + e^{T\mu_\lambda})k^4 \\ & + e^{T(5k+6\mu_x+3\mu_\lambda)}k^3(-1 + e^{T\mu_x} + e^{T\mu_\lambda} + e^{T(\mu_x+\mu_\lambda)}(-1 + kT))\mu_\lambda - e^{T(4k+6\mu_x+3\mu_\lambda)}k^2(e^{kT} - e^{T\mu_x} + e^{T\mu_\lambda} - 2e^{T(k+\mu_\lambda)} \\ & + e^{T(k+\mu_x+\mu_\lambda)}(1 - kT))\mu_\lambda^2 + e^{2T(3\mu_x+2(k+\mu_\lambda))}k(-1 + e^{kT} + e^{T\mu_x} + e^{T(k+\mu_x)}(-1 + kT))\mu_\lambda^3 \\ & + e^{T(7\mu_x+4(k+\mu_\lambda))}(1 + e^{kT}(-1 + kT))\mu_\lambda^4)))/(2k\mu_x^3(k^2 - \mu_x^2)\mu_\lambda^2(k+\mu_\lambda)(\mu_x^2 - \mu_\lambda^2)) \end{aligned} \tag{B13}$$

$$\begin{aligned} b_2 = {} & (e^{-3T(\mu_x+\mu_\lambda)}(-e^{T(\mu_x+3\mu_\lambda)}(1 - 4e^{T\mu_x} + 3e^{2T\mu_x})\mu_\lambda^4 + e^{T(\mu_x+3\mu_\lambda)}\mu_x\mu_\lambda^3(-1 + e^{2T\mu_x} + 2e^{2T\mu_x}T\mu_\lambda) \\ & + e^{T(3\mu_x+\mu_\lambda)}\mu_x^4(-1 + 4e^{T\mu_\lambda} - 3e^{2T\mu_\lambda} + 2e^{2T\mu_\lambda}T\mu_\lambda) - e^{T(3\mu_x+\mu_\lambda)}\mu_x^3\mu_\lambda(1 - e^{2T\mu_\lambda} + 2e^{2T\mu_\lambda}T\mu_\lambda) \\ & - 2e^{2T(\mu_x+\mu_\lambda)}\mu_x^2\mu_\lambda^2(-2(-1 + e^{T\mu_x})(-1 + e^{T\mu_\lambda}) + e^{T(\mu_x+\mu_\lambda)}T\mu_\lambda)))/(4\mu_x^3(\mu_x - \mu_\lambda)^2\mu_\lambda^3(\mu_x + \mu_\lambda)) \end{aligned} \tag{B14}$$

$$\begin{aligned} c_0 = {} & (e^{-3T(k+\mu y)}(-e^{T(3k+\mu_y)}(1 - 4e^{T\mu_y} + 3e^{2T\mu_y})k^4 + e^{T(3k+\mu_y)}k^3(-1 + e^{2T\mu_y}(1 + 2kT))\mu_y \\ & - 2e^{2T(k+\mu_y)}k^2(-2 + 2e^{kT} + 2e^{T\mu_y} + e^{T(k+\mu_y)}(-2 + kT))\mu_y^2 - e^{T(k+3\mu_y)}k(1 + e^{2kT}(-1 + 2kT))\mu_y^3 \\ & + e^{T(k+3\mu_y)}(-1 + 4e^{kT} + e^{2kT}(-3 + 2kT))\mu_y^4))/(4k(k-\mu_y)^2\mu_y^3(k+\mu_y)) \end{aligned} \tag{B15}$$

$$
\begin{aligned}
c_1 = {} & (e^{-T(4\mu_y+3(k+\mu_\lambda))}(-2e^{2T(k+2\mu_y+\mu_\lambda)}(-1+e^{kT})(-1+e^{T\mu_\lambda})k\mu_y^4\mu_\lambda(k+\mu_\lambda)+e^{T(2\mu_y+3(k+\mu_\lambda))}(-1+4e^{T\mu_y}-3e^{2T\mu_y}) \\
& \times k^3\mu_\lambda^3(k+\mu_\lambda)+e^{T(2\mu_y+3(k+\mu_\lambda))}k^2\mu_y\mu_\lambda^2(k+\mu_\lambda)(-(-1+e^{T\mu_y})^2k+(-1+2e^{T\mu_y}+e^{2T\mu_y}(-1+2kT))\mu_\lambda) \\
& +e^{2T(k+\mu_y+\mu_\lambda)}(-1+e^{T\mu_y})k\mu_y^2\mu_\lambda(k+\mu_\lambda)(2e^{T(k+\mu_y)}(-1+e^{T\mu_\lambda})k^2+e^{T(k+\mu_\lambda)}(1+e^{T\mu_y})k\mu_\lambda+2e^{T(\mu_y+\mu_\lambda)}(-1+e^{kT})\mu_\lambda^2) \\
& +2e^{2T(k+2\mu_y+\mu_\lambda)}\mu_y^5(-e^{kT}(-1+e^{T\mu_\lambda})k^2+k(-1+e^{kT}+e^{T\mu_\lambda}+e^{T(k+\mu_\lambda)}(-1+kT))\mu_\lambda+e^{T\mu_\lambda}(1+e^{kT}(-1+kT))\mu_\lambda^2) \\
& -2\mu_y^3(-e^{T(3k+4\mu_y+2\mu_\lambda)}(-1+e^{T\mu_\lambda})k^4+e^{T(3k+3\mu_y+2\mu_\lambda)}k^3(-1+e^{T\mu_y}+e^{T\mu_\lambda}+e^{T(\mu_y+\mu_\lambda)}(-1+kT))\mu_\lambda-e^{T(3\mu_y+2(k+\mu_\lambda))} \\
& \times k^2(e^{kT}-e^{T\mu_y}+e^{T\mu_\lambda}-2e^{T(k+\mu_\lambda)}+e^{T(k+\mu_y+\mu_\lambda)}(1-kT))\mu_\lambda^2+e^{T(2k+3\mu_y+3\mu_\lambda)}k(-1+e^{kT}+e^{T\mu_y}+e^{T(k+\mu_y)}(-1+kT))\mu_\lambda^3 \\
& +e^{T(2k+4\mu_y+3\mu_\lambda)}(1+e^{kT}(-1+kT))\mu_\lambda^4)))/(2k\mu_y^3(-k^2+\mu_y^2)\mu_\lambda^2(k+\mu_\lambda)(\mu_y^2-\mu_\lambda^2))
\end{aligned}
\tag{B16}
$$

$$
\begin{aligned}
c_2 = {} & -(e^{-3T(\mu_y+\mu_\lambda)}(e^{T(\mu_y+3\mu_\lambda)}(1-4e^{T\mu_y}+3e^{2T\mu_y})\mu_\lambda^4-e^{T(\mu_y+3\mu_\lambda)}\mu_y\mu_\lambda^3(-1+e^{2T\mu_y}+2e^{2T\mu_y}T\mu_\lambda) \\
& +e^{T(3\mu_y+\mu_\lambda)}\mu_y^4(1-4e^{T\mu_\lambda}+3e^{2T\mu_\lambda}-2e^{2T\mu_\lambda}T\mu_\lambda)+e^{T(3\mu_y+\mu_\lambda)}\mu_y^3\mu_\lambda(1-e^{2T\mu_\lambda}+2e^{2T\mu_\lambda}T\mu_\lambda) \\
& +2e^{2T(\mu_y+\mu_\lambda)}\mu_y^2\mu_\lambda^2(-2(-1+e^{T\mu_y})(-1+e^{T\mu_\lambda})+e^{T(\mu_y+\mu_\lambda)}T\mu_\lambda)))/(4\mu_y^3(\mu_y-\mu_\lambda)^2\mu_\lambda^3(\mu_y+\mu_\lambda))
\end{aligned}
\tag{B17}
$$

$$d_0 = -\frac{3+\mathrm{e}^{-2T\mu_\lambda}-4\mathrm{e}^{-T\mu_\lambda}-2T\mu_\lambda}{4\mu_\lambda^3} \tag{B18}$$

$$d_1 = 0 \tag{B19}$$

and

$$d_2 = 0 \tag{B20}$$

The swap rate for a horizon T is given by plugging in the above solutions to A, B, C, F and D into

$$i(T)+s(T) = -\frac{1}{T}(Ax_0+By_0+Cz_0+F\lambda_0+D) \tag{B21}$$

The spread $s(T)$ can then be calculated by subtracting the yield $i(T)$ for which the solution is given in Appendix A. As for the terms structure, the spread volatility as given by the model is defined as

$$\sigma_s(T) = \sqrt{\left(\sigma_x\frac{\partial s(t)}{\partial x_0}\right)^2+\left(\sigma_y\frac{\partial s(T)}{\partial y_0}\right)^2+\left(\sigma_\lambda\frac{\partial s(T)}{\partial \lambda_0}\right)^2} \tag{B22}$$

which is given by

$$\sigma_s(T) = \frac{1}{T\mu_\lambda}\left[\sigma_\lambda^2(1-e^{-T\mu_\lambda})^2+\frac{\sigma_x^2\gamma_1^2(\mu_x(1-e^{-T\mu_\lambda})-\mu_\lambda(1-e^{-T\mu_x}))^2}{\mu_x^2(\mu_x-\mu_\lambda)^2}+\frac{\sigma_y^2\gamma_2^2(\mu_y(1-\mathrm{e}^{-T\mu_\lambda})-\mu_\lambda(1-e^{-T\mu_y}))^2}{\mu_y^2(\mu_y-\mu_\lambda)^2}\right]^{1/2} \tag{B23}$$

APPENDIX C

Fitting the U.S. Data Set with the 30-year Swap

We add for completeness an analysis of the U.S. data-set with the inclusion of the 30-year swap. The comparison of the fit for the data is given in Exhibit 7. The parameter values as calculated from fitting the data are given in C1. One can see that there is some change in the fits for some periods, October 10, 2003 as an example, where the inclusion of the 30-year spread changes the overall structure of the curve.

EXHIBIT C1

The values of the slow varying parameters as fitted for the U.S. data including the 30-year swap. The values are given in annualized absolute units such that 0.1 corresponds to 10% annual rate.

Dataset	μ_λ	θ_n	σ_λ	μ_x	μ_y	σ_x	σ_y	k
U.S. (with 30 year spread)	5.0	0.001	0.002	0.003	0.5	0.009	0.015	1.2

ENDNOTES

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[1]By time-independent parameters we do not mean that the values of the parameters are constant throughout the different trading days but that at a given day their value does not depend on the time horizon we consider.

[2]This is what occurs if, for example, a term $\gamma_3 z dt$ is added to the right hand side of (13).

[3]The volatility we quote is annualized since the rates we choose to work with are yearly rates.

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