A Unified Credit and Interest Rate Arbitrage-Free Contingent Claim Model

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Many financial instruments have embedded interest rate and credit risk options. Consider a callable corporate bond that provides the right to the issuing corporation to buy back the bond at a specified price on the option expiration date. On the one hand, if market interest rates fall, the corporation may exercise the option to refinance the bond at a prevailing market interest rate that is lower than the coupon rate of the bond. On the other hand, if the corporation's creditworthiness improves after the bond issuance, the bond value would rise as the credit spread tightens, and the corporation may buy back the bond by exercising the option to refinance the debt, again, at a lower cost of funds. Therefore, the call option embedded in the corporate bond is an interest rate and credit option.

In essence, these instruments have embedded "chooser" options where the option holder has the right to exercise one of the two options. The credit and interest rate contingent claims are not just options on two specific marketable securities. They are contingent claims on the yield curve and the hazard curve. Therefore, the valuation of such an instrument requires a model of a joint arbitrage-free movement of the term structures of interest rates and hazard rates; such a model is called a *unified model*.

A unified model can be used to analyze complex effects of the interest rate and credit risks on the interest rate and credit contingent claims. For example, when the hazard rates increase for a callable bond, the default risk lowers both the duration and the value of the bond. Both effects affect the embedded interest-rate option value of the bond. Consider the tranches of a collateralized loan obligation that form the capital structure of a bond portfolio. The equity tranche in essence has an embedded credit call option, whose value rises with significant gamma, or convexity, when the hazard rate falls. The mezzanine tranche supported by the junior tranches has, in essence, two credit call options embedded in its structure (i.e., the investor has sold (bought) a credit call option when the hazard rate is low (high)). But these credit options cannot be isolated from the interest rate risks because the credit option, in turn, affects the duration of the tranches. The interrelationship of the credit and interest rate options requires a unified model to provide the risk measures, hedge ratios, and portfolio strategies. For these reasons, a unified model and its associated analytics are important to securities analysis.

The purpose of this article is to provide such a model, a unified model incorporating both interest rate and credit risks. To the best of the authors' knowledge no paper has yet to present a unified model, although many papers have shown that credit derivatives can be valued using a model analogous to an interest rate arbitrage-free model. For example, Das and Sundaram [2000]; Das, Duffi, Kapadia, and Saita [2007]; Longstaff and Rajan [2008], and Ho and Lee [2008] have proposed valuation models for credit options. This article provides a valuation model for a joint distribution of interest rates and hazard rates.

Furthermore, this article provides two specific applications of the unified model. First, the model can be used to formulate dynamic hedging strategies based on the key rate durations and credit key rate durations. Second, the model can be used to analyze the complex interactions of interest rate and credit risks on the behavior of the instruments, particularly extending the standard convexity measure to principal curvatures to analyze options subject to two sources of risks.

More generally, in valuing the credit and interest-rate contingent claims, the model can identify the contributions of the interest rate and credit volatilities to the valuation of the contingent claim. As a result, this unified model enables portfolio managers to identify the values of these options and risk managers to manage credit risk and interest rate risk in one coherent framework. Therefore, there are many potential applications of the model.

The next section describes the unified model. The third section describes dynamic hedging using credit key rate durations and key rate durations of a credit and interest-rate contingent claim. The fourth section presents the generalized parametric measures of these contingent claims. And finally, the last section concludes.

THE UNIFIED MODEL

The unified model extends from the generalized Ho–Lee model [2007] for interest rate movements. The model is a discrete-time two-factor binomial recombining lattice model for which the number of time steps is given by $T = 0, 1, 2, \ldots, N$. The model follows the standard perfect capital market assumptions. The market term structure of interest rates is represented by the spot yield curve $r^*(T)$, and the discount function $P(T)$ is related to the spot yield curve given by

$$P(T) = \exp(-r^*(T)T) \tag{1}$$

The yield curve $r^*(T)$ can often depict interest rate behavior; therefore, we tend to study the movement of the yield curve and not the discount function, even though both contain the same information about the time value of money. For the purpose of valuation, using the discrete-time model, we further define the one-period discount factor at time T to be $p(T)$, where

$$p(T) = \frac{P(T+1)}{P(T)} \tag{2}$$

We extend this bond arithmetic to analyze credit risk. Credit risk is represented by the survival function $S(T)$, which is defined as the survival probability of the reference name until time T. Therefore, the function $S(T)$ is analogous to the discount function $P(T)$ in that both functions have a value of one initially (when $T = 0$) and both functions decline monotonically, as the interest rate and the default rate are always positive.

We define the hazard curve $h^*(T)$ to be:

$$S(T) = \exp(-h^*(T)T) \tag{3}$$

Again, the hazard curve, analogous to the yield curve, is useful in depicting the term structure of the credit risk of a bond. The hazard curve $(h^*(T))$ is the default rate for a bond to survive until time T. Given the survival function $S(T)$, we can define the one-period survival factor at time T, $s(T)$, as

$$s(T) = \frac{S(T)}{S(T-1)} \tag{4}$$

The one-period survival factor at time T is the marginal probability of survival, conditional on the bond surviving to period $T-1$ and, therefore, $0 < s(T) < 1$ for all $T > 0$.

We define the hazard rate $h(T)$ to be the marginal probability of an event of default during the period T,

$$h(T) = 1 - s(T) \tag{5}$$

which is also called the conditional default rate (CDR) at time T.

Let the T-period discount factor of the joint arbitrage-free movements of the survival function $S(T)$ and the discount function $P(T)$ in the two-factor binomial recombining lattice be denoted by $d^n_{i,j}(T)$, where n is the number of time steps, and i and j are the numbers of ith and jth up-steps of the survival function and discount function, respectively. The unified model provides a specification of this discount factor.

We assume that the term structure of interest rate volatilities, $\sigma_r(n)$, and the term structure of credit volatilities, $\sigma_h(n)$ are given by the following functions:

$$\sigma_r(n) = (a' + b'n)\exp(-c'n) + d' \tag{6a}$$

$$\sigma_h(n) = (a + bn)\exp(-cn) + d \tag{6b}$$

The parameters $a(a')$, $b(b')$, $c(c')$, and $d(d')$ are constant coefficients. This specification of the implied volatility functions allows for a broad range of shapes including downwardly sloping or dumped implied-volatility functions.

Let $\delta^n_{i,h}(T)$ and $\delta^n_{i,r}(T)$ be the T-period volatility factors at time n and state i for the hazard rate and interest rate arbitrage-free movements. In particular, let $\delta^n_{i,h} = \delta^n_{i,h}(1)$ and $\delta^n_{i,r} = \delta^n_{i,r}(1)$ be the one-period volatility factors for the hazard rate and the interest rate, respectively. These volatility factors are specified by the following six recursive equations:

$$\begin{aligned} \delta^n_{i,h}(T) &= \delta^n_{i,h}\delta^{n+1}_{i,h}(T-1)\left(\frac{1+\delta^{n+1}_{i+1,h}(T-1)}{1+\delta^{n+1}_{i,h}(T-1)}\right) \\ \delta^n_{i,r}(T) &= \delta^n_{i,r}\delta^{n+1}_{i,r}(T-1)\left(\frac{1+\delta^{n+1}_{i+1,r}(T-1)}{1+\delta^{n+1}_{i,r}(T-1)}\right) \end{aligned} \tag{7}$$

$$\begin{aligned} \delta^m_{i,h}(1) &= \delta^m_{i,h} = \exp(-2\cdot\sigma_h(m)\min(h^m_i, H)\Delta t^{3/2}) \\ \delta^m_{j,r}(1) &= \delta^m_{j,r} = \exp(-2\cdot\sigma_r(m)\min(r^m_j, R)\Delta t^{3/2}) \end{aligned} \tag{8}$$

$$\begin{aligned} h^m_i\Delta t &= -\ln\left(\frac{S(m)}{S(m-1)}\right) - \sum_{k=1}^{m}\ln\left(\frac{1+\delta^{k-1}_{0,h}(m-k)}{1+\delta^{k-1}_{0,h}(m-k+1)}\right) \\ &\quad - \sum_{k=0}^{i-1}\ln(\delta^{m-1}_{k,h}(1)) \\ r^m_j\Delta t &= -\ln\left(\frac{P(m+1)}{P(m)}\right) - \sum_{k=1}^{m}\ln\left(\frac{1+\delta^{k-1}_{0,r}(m-k)}{1+\delta^{k-1}_{0,r}(m-k+1)}\right) \\ &\quad - \sum_{k=0}^{j-1}\ln(\delta^{m-1}_{k,r}(1)) \end{aligned} \tag{9}$$

where the h^m_i and r^m_i are the one-period hazard rate and interest rate, respectively, at time m and state i, respectively. Equation (7) recursively relates the T-period to the $T-1$ term structure of volatilities. Equation (8) defines the one-period term structure of volatilities using the implied volatility functions and the threshold rates R and H. The threshold rates R and H specify the levels of the one-period interest rates and hazard rates, respectively, at which the stochastic movements switch from lognormal to normal. Finally, Equation (9) defines the one-period interest rates and hazard rates given the term structure of volatilities. These equations determine the unified model that follows.

The T-period discount rate is given by

$$\begin{aligned} d^n_{i,j}(T) &= 4\frac{S(n-1+T)}{S(n-1)}\frac{P(n+T)}{P(n)}\prod_{k=1}^{n}\left(\frac{1+\delta^{k-1}_{0,h}(n-k)}{1+\delta^{k-1}_{0,h}(n-k+T)}\right) \\ &\quad\times\prod_{k=1}^{n}\left(\frac{1+\delta^{k-1}_{0,r}(n-k)}{1+\delta^{k-1}_{0,r}(n-k+T)}\right)\prod_{k=0}^{i-1}\delta^{n-1}_{k,h}(T)\prod_{k=0}^{j-1}\delta^{n-1}_{k,r}(T) \end{aligned} \tag{10}$$

In particular, the one-period discount factor at state (i, j) at time n is given by $d^n_{i,j} = d^n_{i,j}(1)$ in the two-factor binomial recombining lattice,

$$\begin{aligned} d^n_{i,j} &= 4\frac{S(n)}{S(n-1)}\frac{P(n+1)}{P(n)}\prod_{k=1}^{n}\left(\frac{1+\delta^{k-1}_{0,h}(n-k)}{1+\delta^{k-1}_{0,h}(n-k+1)}\right) \\ &\quad\times\prod_{k=1}^{n}\left(\frac{1+\delta^{k-1}_{0,r}(n-k)}{1+\delta^{k-1}_{0,r}(n-k+1)}\right)\prod_{k=0}^{i-1}\delta^{n-1}_{k,h}\prod_{k=0}^{j-1}\delta^{n-1}_{k,r} \end{aligned}$$

For a security with no coupons or other interim payments but only the face value F at maturity T, we can apply the standard recursive procedure to determine the initial value $X(0, 0, 0)$, given the terminal conditions,

$$X^T_{i,j} \quad \text{for } 0 < i, j < T \tag{11}$$

where $X^T_{i,j}$ is defined as the value of the security at time T and for state i and j. Therefore, $X^T_{i,j}$ is equal to $F\cdot s^T_i + F\cdot(1-s^T_i)\cdot R$, where $s^T_i = 2\frac{S(T)}{S(T-1)(1+\delta^T_s)}\delta^i_s$.

If we assume that no correlation exists between the interest rate and credit risk, then the value of the contingent claim would be valued in the recursive equation,

$$X_{i,j}^{n} = 0.25\,[X_{i,j}^{n+1} + X_{i+1,j}^{n+1} + X_{i,j+1}^{n+1} + X_{i+1,j+1}^{n+1}]\,d_{i,j}^{n} + F\cdot(1-s_{i}^{n})\cdot R \qquad (12)$$

More generally, when the interest rate risk and the credit risk are correlated, with a constant one-period correlation ρ, then the rollback calculation is

$$X_{i,j}^{n} = \left[\frac{1+\rho}{4}\cdot X_{i,j}^{n+1} + \frac{1-\rho}{4}\cdot X_{i+1,j}^{n+1} + \frac{1-\rho}{4}\cdot X_{i,j+1}^{n+1} + \frac{1+\rho}{4}\cdot X_{i+1,j+1}^{n+1}\right] d_{i,j}^{n} + F\cdot(1-s_{i}^{n})\cdot R \qquad (13)$$

where ρ is the correlation coefficient and $s_{i}^{n} = 2\frac{S(n)}{S(n-1)(1+\delta_{s}^{n})}\delta_{s}^{i}$.

The unified model specifies the movements of the hazard rates and the interest rates; furthermore, the model provides a methodology to value a bond by the rollback process.

In the special case when the one-period hazard rate and interest rate volatility factors are constant, denoted by δ_{s}^{n} and δ_{r}^{n}, respectively, and the distributions are normal without switching to lognormal, then the model reduces to a model analogous to the Ho–Lee [1986] model,

$$d_{i,j}^{n} = 4\frac{S(n)}{S(n-1)(1+\delta_{s}^{n})}\frac{P(n+1)}{P(n)(1+\delta_{r}^{n})}\delta_{s}^{i}\delta_{r}^{j} \qquad (14)$$

Equations (12) and (13) can then be used to determine the value of a contingent claim.

When there is no correlation between interest rate risk and credit risk, $\rho = 0$, or when there is no recovery rate, the value of a zero-coupon bond with maturity T is independent of the specification of the interest rate risk and credit risk processes and is given by

$$V(T) = P(T)S(T) \qquad (15)$$

Similarly, when the recovery rate, R, is not zero, but the correlation is zero, then the zero-coupon bond value is

$$V(T) = R\{(S(0)-S(1))P(1) + (S(1)-S(2))P(2) + \cdots + (S(T-1)-S(T))P(T)\} + S(T)P(T) \qquad (16)$$

Appendix A shows that the unified model indeed values the defaultable bond identically to the standard valuation model of Equations (15) and (16).

However, when there is a non-zero correlation between interest rate risk and credit risk, and the recovery rate is non-zero as well, then a zero-coupon defaultable bond cannot be valued by the static model of Equations (15) and (16) even though the bond has no explicitly stated embedded option, such as a call provision. This is because the recovery amount at default is a "prepayment" that is correlated to the interest rate uncertainty. Therefore, the recovery payments are in fact a valuable embedded interest-rate option and the bond can only be valued by the unified model and not by the static model of Equations (15) and (16).

Typically, the embedded interest-rate option value is relatively low, as illustrated by the simulation that follows. We assume an upwardly sloping yield curve with rates of 0.01, 0.02, 0.03, 0.04, 0.05, 0.06, and 0.065 for 0.25, 0.5, 1, 2, 5, 10, and 30 years, respectively. The hazard rate curve is assumed to be downwardly sloping, such that the rates are 0.05, 0.045, 0.04, 0.035, 0.02, 0.015, and 0.01 for the same year intervals, respectively. For example, assume that the face value is $100; the coupon rate and the recovery rate are zero, for simplicity; and the bond price is 81.8732 regardless of the correlation coefficient. However, when 0.2 and 0.1 are given for the volatility of the interest rate and the hazard rate, respectively, the defaultable bond price changes from 81.8732 to 82.0692 with the corresponding correlation coefficient changing from 0.0 to 0.8.[1]

VALUATION OF BOND, CALLABLE BONDS, AND MAKE-WHOLE BONDS

We use the unified model to simulate the value of a callable bond under a parallel shift of the yield curve and the hazard curve. The swap curve and hazard curve are assumed to be constant at 5% and 1%, respectively. The interest rate volatility and hazard rate volatilities are assumed to be 5% and 10%, respectively. We simulate a surface of value, the performance profile, by shifting the yield curve in a parallel fashion from 4% to 6% with a

0.1% increment and the hazard curve from 0% to 2% with a 0.1% increment.

Performance Profile of a Coupon Bond

The performance profile of a fixed-rate coupon bond with a 10-year maturity, 6% coupon rate, and 40% recovery rate is provided in Exhibit 1. The impact of the shifts of the yield curve and the hazard curve are similar in their effects on bond value.

Performance Profile of a Callable Bond

We also use the model to simulate a callable bond's value. The bond is immediately callable at par and, therefore, the value is capped at the call price. Once again, the impacts on bond value of the shift in the hazard curve and the yield curve are similar.

A callable bond gives the issuer the right, but not the obligation, to buy back the bond at a pre-specified schedule of prices, the call schedule, k_n, $n = 1, \ldots, T$. The bond can then be valued by the recursive equations,

$$X_c(n,i,j) = 0.25[B_c(n+1,i+1,j+1) + B_c(n+1,i,j+1) + B_c(n+1,i+1,j) + B_c(n+1,i,j)]\, d^n_{i,j} \tag{17}$$

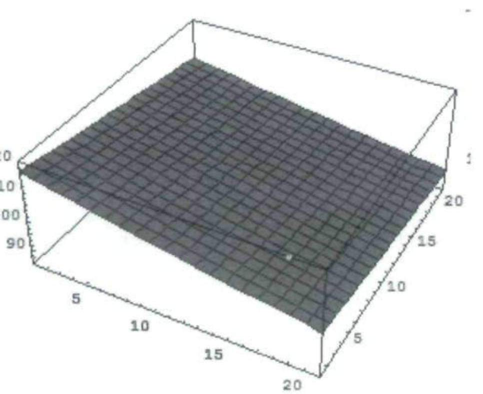
where $n = 1,\ldots,T$ and $i, j = 0,\ldots, n$, with the boundary condition,

$$B_c(n,i,j) = \text{Min}[X_c(n,i,j), k_n] + c \tag{18}$$

where c is the fixed coupon rate per binomial period.

Exhibit 1

Performance Surfaces of Fixed-Rate Coupon Bond



By way of contrast, we simulate the bond value of a make-whole bond. A make-whole bond gives the issuer (the borrower) the right to retire the bond by exchanging the bond for a Treasury bond with a pre-specified spread or a fixed make-whole premium (MWP). The call option of the make-whole bond, therefore, does not offer any benefits in refinancing when interest rates fall. The option does have value, however, when the borrower's credit has improved such that the funding cost is lowered. The results which follow show the impact of the credit spread tightening on the option value.

The bond can be valued by the recursive equations

$$\begin{aligned} X_{mwc}(n,i) = 0.25[&B_{mwc}(n+1,i+1,j+1) \\ &+ B_{mwc}(n+1,i,j+1) + B_{mwc}(n+1,i+1,j) \\ &+ B_{mwc}(n+1,i,j)]d^n_{i,j} \end{aligned} \tag{19}$$

$n = 1,\ldots,T$ and $i,j = 0,\ldots, n$ with the boundary condition,

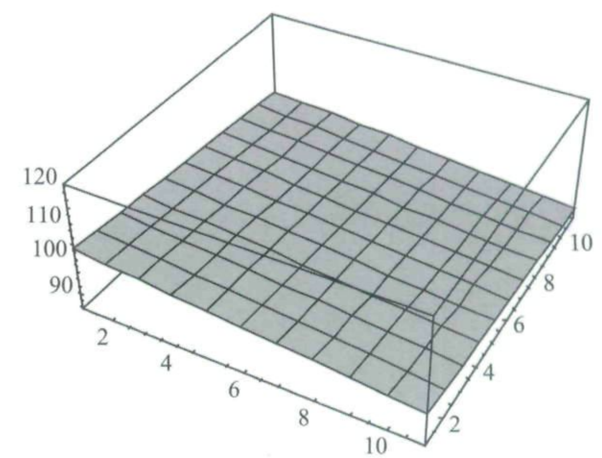
$$B_{mwc}(n,i,j) = \text{Min}[X_{mwc}(n,i,j), kk^n_{i,j}] + c \tag{20}$$

where

$$kk^n_{i,j} = \sum_{k=1}^{T-n} \frac{c}{(1+r^n_{i,j}+MWP)^k} + \frac{F}{(1+r^n_{i,j}+MWP)^{T-n}} \tag{21}$$

Exhibit 2

Performance Surfaces of Callable Bond



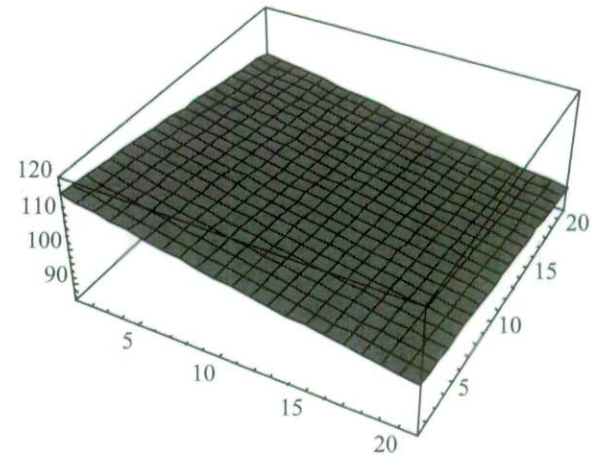
and *MWP* is the make-whole premium; *F* is the face value of the bond, $r^n_{i,j}$ is the appropriate risk-free rate derived from the lattice at the node $(n; i, j)$.

The make-whole redemption price is equal to the sum of the present value of the remaining coupon and principal payments discounted at an Adjusted Treasury Rate (ATR) plus the make-whole premium.[2] We assume that the ATR is the risk-free rate at each node and the make-whole premium equals 45 basis points (bps).

The performance profiles of a coupon bond and bonds with a call option and a make-whole option are presented in Exhibits 1–3. The profiles depict the impact on bond value of shifts in the yield curve and the hazard curve and, thus, the combined effect of the shifts of interest rate and hazard rate on bond value. These surfaces motivate the use of the geometry of surfaces to analyze the behavior of the interest rate and credit contingent claims, which are addressed later in the article.

EXHIBIT 3

Performance Surfaces of Make-Whole Bond



HEDGING WITH CREDIT DERIVATIVE SWAPS AND INTEREST RATE SWAPS

In this simulation, we assume the same market conditions assumed in the previous section—a refinancing call protection period of five years and a first call price of par. The call protection period for the make-whole callable bond is five years and the make-whole premium is 45 bps.

The *credit key rate duration* measures the price sensitivity of a bond to each key rate change in the hazard curve. Ho and Lee [2008] provided a detailed description of the credit key rate duration measures, which are defined analogously to the key rate durations that measure the price sensitivity of a bond to the shift in each key rate of the yield curve (see Ho [1992]). Note that the sum of the credit key rate durations equals the *credit duration*, the proportional change in price to a small parallel shift of the hazard curve. Likewise, the sum of the key rate durations equals the duration, the proportional change in price to a small parallel shift of the yield curve.

The key rate and credit key rate durations of the coupon bond, refinancing callable bond, and the make-whole callable bonds are shown below.

The results provide several interesting observations. First, the credit durations are lower than the durations for all three types of bonds, because the credit risk of these bonds results in a partial "prepayment" of the principal, as mentioned earlier. In the extreme case, when the recovery ratio is 100%, an increase in credit risk essentially leads to early payments of principal; in these numerical examples the impact on price can be minimal, thus credit duration is lower.

Key Rate Duration								
Key Year	0.25	1	2	3	5	7	10	Parallel
Coupon Bond	0.0127	0.0553	0.1106	0.2548	0.4608	0.7344	5.8610	7.4895
Make-Whole Bond	0.0127	0.0553	0.1106	0.2548	0.4598	0.7320	5.8657	7.4909
Callable Bond	0.0137	0.5985	0.1200	0.3471	2.4080	0.7483	0.7987	4.4956

Credit Key Rate Duration[3]								
Key Year	0.25	1	2	3	5	7	10	Parallel
Coupon Bond	0.0079	0.0345	0.0690	0.1590	0.2876	0.4585	3.5243	4.5409
Make-Whole Bond	0.0079	0.0345	0.0690	0.1625	0.3267	0.4892	3.3937	4.4834
Callable Bond	0.0085	0.0373	0.0746	0.3186	0.9422	0.4271	1.4342	3.2325

Second, the make-whole option has a negligible impact on the key rate durations of the bonds when the key rate durations of the coupon bond and the make-whole bond are compared. The credit key rate durations are affected, however, in particular, lowering the price sensitivity and confirming the discussion in this section. Note that the credit key rate durations in the call protection period are unaffected. But for the callable bond, both the key rate durations and credit key rate durations are affected because the option is a contingent claim on both the credit and interest rate risks.

Many uses are suggested by the combination of key rate duration and credit key rate duration. First, the methodology enables portfolio managers and traders to dynamically hedge interest rate risk and credit risk in a coherent manner. Dynamic hedging uses credit derivative swaps and interest rate swaps to simultaneously match the dollar key rate durations and credit key rate durations of a bond, whereas dollar key rate durations and dollar credit key rate durations are defined as the mark-to-market change in value with respect to a key rate change in the yield curve and the hazard curve. The unified model can appropriately separate the interest rate risk and the credit risk of a bond, resulting in an effective hedging strategy.

In contrast, the traditional approach to managing interest rate and credit risk separately may lead to overhedging or underhedging when the credit model is inconsistent with the interest rate option model.

Second, the unified model provides a tool to value credit options within the interest-rate risk framework. Thus, the model provides a consistent risk measure to manage credit risk. To date, credit risks are often measured, stressed scenarios to determine the potential loss in promised payments. These methods fail to value embedded credit options, such as those embedded in callable or make-whole bonds. The combined used of key rate durations and credit key rate durations can provide a delta-normal Value-at-Risk measure, for example. Such a measure would enable risk managers to manage risks in a more coherent framework.

DIRECTIONAL CURVATURE

Defining Directional Curvature

Thus far, we have analyzed the deltas of contingent claims that are subject to both interest rate risk and credit risk. The deltas are subject to small shifts in the risk factors and do not capture the nonlinearity of the embedded option value under larger movements of the risk factors. To extend this analysis to the study of the nonlinearity of option behavior, we use the convexity measures.

Duration and convexity are used to study the behavior of an interest rate contingent claim. For an interest rate and credit contingent claim, we have to evaluate the second-order term of the Taylor Expansion of the valuation model subject to small shifts in the spot curve and the hazard curve. Unlike the convexity measures of an interest rate option, however, the nonlinearity of an interest rate and credit option is measured by the behavior of the surface of the performance profile, as depicted in the previous section, and a way is needed to describe the behavior of the performance profile surface.

To analyze the behavior of the performance profile surface, we use the principal curvature and the principal direction. The principal direction is the combination of a small shift in the yield curve and the hazard curve that results in the maximum or minimum of nonlinearity. The principal curvature is the measure of the nonlinearity for this combination.

Consider a second-order Taylor Expansion of the valuation of an interest rate and credit contingent claim,

$$\Delta P = \frac{\partial P}{\partial r}\Delta r + \frac{\partial P}{\partial h}\Delta h + 0.5\frac{\partial^2 P}{\partial r^2}\Delta r^2 + \frac{\partial^2 P}{\partial h \Delta r}\Delta h \Delta r + 0.5\frac{\partial^2 P}{\partial h^2}\Delta h^2 \tag{22}$$

If we further define the durations with respect to x and y to be

$$D_r = -\frac{\partial P}{\partial r} / P \tag{23a}$$

$$D_y = -\frac{\partial P}{\partial y} / P \tag{23b}$$

and convexity with respect to r and h to be

$$C_{rr} = \frac{\partial^2 P}{\partial r^2} / P \tag{24a}$$

$$C_{hh} = \frac{\partial^2 P}{\partial h^2} / P \tag{24b}$$

$$C_{rh} = \frac{\partial^2 P}{\partial r \partial h} / P \qquad (24c)$$

then,

$$\frac{\Delta P}{P} = -D_r \Delta r - D_h \Delta r + 0.5 C_{rr} \Delta r^2 + C_{rh} \Delta r \Delta h + 0.5 C_{hh} \Delta h^2 \qquad (25)$$

The results show that we can determine a linear transformation of the interest rate and credit shifts such that we can identify the principal directions to measure the nonlinearity of the price surface. In other words, the principal direction is the combination of Δr and Δh that maximizes or minimizes Equation (26) with respect to Δr and Δh per unit length of the shifts in the combined interest rate and hazard rate,

$$L = \frac{0.5 C_{rr} \Delta r^2 + C_{rh} \Delta r \Delta h + 0.5 C_{hh} \Delta h^2}{\Delta r^2 + \Delta h^2 + (D_r \Delta r + D_h \Delta h)^2} \qquad (26)$$

Equation (26) shows that the numerator is the change amount of the convexity and the denominator is the length of the shift of the interest rate and the hazard rate.[4] The maximal and minimal values are called the directional curvatures. The product of the two directional curvatures, a scalar, is the Gaussian curvature of the surface that measures the extent of the value surface that deviates from the plane defined by the durations and, therefore, measures the effectiveness of the use of delta hedging.

The principal directions, curvature, and Gaussian curvature specify the ratio of the interest rate shift to the hazard rate shift that leads to the maximum and minimum nonlinear price impact (the principal curvatures). Thus, they identify the potential convexity, or gamma, risks of an interest rate and credit contingent claim, which may be significantly higher than the interest rate risk or the credit risk when measured separately.

Directional Curvatures of Coupon Bonds, Callable Bonds, and Make-Whole Bonds

In this section, we will numerically simulate the principal curvatures of several interest-rate credit contingent claims to illustrate the use of these risk analytics. In our simulations, we assume a constant yield curve of 5% and a hazard curve of 1%. The volatility of the yield curve and the hazard curve is assumed to be 20% and 10%, respectively. The face value of the bond is \$100 with a 6% coupon rate. We assume a recovery ratio of 40% and the bond matures in 10 years.

The refinancing call protection period is five years and the first call price is \$102. The call price linearly decreases to \$100 at maturity. The call protection period for the make-whole callable bond is five years and the make-whole premium is 45 bps. The make-whole redemption price is equal to the sum of the present value of the remaining coupon and principal payments discounted at an ATR plus the make-whole premium.[5] We assume that the ATR is the risk-free rate at each node.

We calculate the principal curvature and principal direction for the coupon bond at the interest rate of 5% and the hazard rate of 1%. The maximum and minimum principal curvatures are 0.0327881 and –0.0000667766, respectively. The principal directions are defined as the change in the interest rate to one unit change in the hazard rate. The corresponding principal directions are −1.56976 and −1.64956, respectively. The results show that when the hazard rate rises (falls) a unit with a corresponding fall (rise) of 1.56976 units would result in a maximum nonlinear change in bond value. This result can be intuitively explained as follows. It is well known that a bond with amortization has a higher convexity than a vanilla bond. An increase of default risk effectively results in early involuntary prepayments of principal. When the loss of value due to the increase in default risk is balanced by a fall in interest rates, then the principal curvature shows that the bond attains a maximum curvature of 0.0327881, a positive convexity, as expected.

Next, we calculate the principal curvature and principal direction for the callable bond using the market assumptions made previously. The maximum principal curvature and the minimum principal curvature are 0.000007263 and –6.4334, respectively. The associated principal direction is –2.22277 and −1.39071, respectively. The result shows that while a fall in either the interest rate or the hazard rate would increase the embedded option value, the principal direction is an increase of 1.39071 units of interest rate to a unit fall of hazard rate that leads to the maximal curvature. Since the minimal curvature is negligibly small, the result shows that the bond value basically has negative convexity in all directions.

IMPLICATIONS OF A UNIFIED MODEL ON RISK MANAGEMENT

To date, much of financial research has been devoted to risk management. Most financial institutions tend to separately measure and manage credit risk and interest rate risk. This risk management organizational structure is further reinforced by regulatory standards, such as Basel II in which market risks and credit risks are measured separately using different methodologies. To the extent the interest rate risk and credit risk are intricately related, this silo approach to risk management can be problematic.

The unified model presented in this article provides a tool to measure credit and market risks and a framework to manage the two risks in a coherent way. From the unified model perspective, credit risk can be defined as the potential loss of interest and principal in the event of default. The potential losses are in fact the cash flows of the "guarantees" of the interest and principal, called defaulted dollar (D$). In particular, at the event of default, the bond pays the accrued interest and the principal in full, which is a recovery ratio of 100%. Therefore, the guaranteed value is the value of the bond based on a 100% recovery ratio net of the bond value based on a realistic recovery ratio.

Let B_f denote the guaranteed bond value, when the recovery ratio is 100%. The bond value can be rewritten as

$$B = B_f - D\$ \tag{27}$$

Given Equation (27), this suggests that market risk management should focus on the value and risk of B_f and credit risk management should focus on managing the value and risk of $D\$$. The stress tests on defaulted dollar can be analyzed taking concentration risks and the severity of loss given default can be simulated. Credit derivatives can be used to manage the risks of $D\$$.

CONCLUSION

This article provides a valuation model of an interest rate and credit contingent claim. The model can value bonds and derivatives with embedded options that have both underlying credit risk and interest rate risk. The model is arbitrage free in terms of both the credit risk and interest rate risk. The model takes the yield curve and the credit derivative curve as given. The term structures of volatilities for interest rate risk and credit risk can take on a broad range of functional forms and can be calibrated from market prices of interest rate and credit options.

We showed that the model can be used to specify the key rate duration and credit key rate durations that can be used for dynamic hedging, using interest rate swaps and credit derivative swaps. We further introduced principal directions and principal curvatures of interest risk and credit contingent claims. These measures extend the convexity measures to securities with both credit and interest rate risks.

Finally, we showed that the model can separate credit risk and market risk, enabling risk managers to manage credit risk and market risk separately, but in a coherent framework.

APPENDIX A

PROOF OF PROPOSITION

Proposition: Under default risk represented by survival function stochastic movements and interest rate risk represented by discount function stochastic movements, the T-period discount bond, which is subject to default risk and interest rate risk, should satisfy Equation (A1) to avoid arbitrage opportunities,

$$\begin{aligned} X^n_{i,j}(T) = X^n_{i,j}(1)\Bigg[& \frac{1+\rho}{4}\cdot X^{n+1}_{i,j}(T-1) + \frac{1-\rho}{4}\cdot X^{n+1}_{i+1,j}(T-1) \\ & + \frac{1-\rho}{4}\cdot X^{n+1}_{i,j+1}(T-1) + \frac{1+\rho}{4}\cdot X^{n+1}_{i+1,j+1}(T-1)\Bigg] \end{aligned} \tag{A1}$$

where ρ is the correlation coefficient.

$X^n_{i,j}(T)$ is the T-period bond price under default risk and interest rate risk at time n and state i and j, where i refers to default risk and j refers to interest rate risk.

It should satisfy the initial condition

$$\begin{aligned} & S(T-1)P(T) \\ & = X^0_{0,0}(1)\Bigg[\frac{1+\rho}{4}\cdot X^1_{0,0}(T-1) + \frac{1-\rho}{4}\cdot X^1_{0,1}(T-1) \\ & \qquad + \frac{1-\rho}{4}\cdot X^1_{1,0}(T-1) + \frac{1+\rho}{4}\cdot X^1_{1,1}(T-1)\Bigg] \end{aligned}$$

To satisfy Equation (A1), the T-period bond price subject to default risk and interest risk should follow Equation (A2),[6]

$$X^n_{i,j}(T) = 2\frac{S(T+n-1)}{S(n-1)}\frac{[(1+\delta_s^{n-1})(1+\delta_s^{n-2})\cdots(1+\delta_s^{1})]}{(1+\delta_s^{T+n-1})\cdots(1+\delta_s^{T})}\delta_s^{iT}$$
$$\times 2\frac{P(T+n)}{P(n)}\frac{[(1+\delta_r^{n-1})(1+\delta_r^{n-2})\cdots(1+\delta_r^{1})]}{(1+\delta_r^{T+n-1})\cdots(1+\delta_r^{T})}\delta_s^{jT}$$
$$+F\cdot(1-s_i^n)\cdot R \tag{A2}$$

where $S(T)$ is the survival function and $P(T)$ is the discount function.

First, we show that $X^0_{0,0}(T) = S(T-1)P(T)$, which means that the T-period bond price under default risk and interest risk at the node in Time 0 and State 0 and State 0 is consistent with the initial survival function and the initial discount function.

$$X^0_{0,0}(T) = 0.25\cdot X^0_{0,0}(1)\cdot\{X^1_{0,0}(T-1)+X^1_{1,0}(T-1) + X^1_{0,1}(T-1)+X^1_{1,1}(T-1)\}$$

$$= 0.25\cdot X^0_{0,0}(1) \times \left\{\begin{array}{l}
2\dfrac{S(T-1)}{S(0)}\dfrac{1}{1+\delta_s^{T-1}}\delta_s^{0\times(T-1)}\cdot 2\dfrac{P(T)}{P(1)}\dfrac{1}{1+\delta_r^{T-1}}\delta_r^{0\times(T-1)} \\
+2\dfrac{S(T-1)}{S(0)}\dfrac{1}{1+\delta_s^{T-1}}\delta_s^{1\times(T-1)}\cdot 2\dfrac{P(T)}{P(1)}\dfrac{1}{1+\delta_r^{T-1}}\delta_r^{0\times(T-1)} \\
+2\dfrac{S(T-1)}{S(0)}\dfrac{1}{1+\delta_s^{T-1}}\delta_s^{0\times(T-1)}\cdot 2\dfrac{P(T)}{P(1)}\dfrac{1}{1+\delta_r^{T-1}}\delta_r^{1\times(T-1)} \\
+2\dfrac{S(T-1)}{S(0)}\dfrac{1}{1+\delta_s^{T-1}}\delta_s^{1\times(T-1)}\cdot 2\dfrac{P(T)}{P(1)}\dfrac{1}{1+\delta_r^{T-1}}\delta_r^{1\times(T-1)}
\end{array}\right\}$$

$$= 0.25\cdot X^0_{0,0}(1)\cdot 4\cdot S(T-1)\cdot\frac{P(T)}{P(1)}$$

$$\because \frac{1}{1+\delta_s^{T-1}}\frac{1}{1+\delta_r^{T-1}}(1+\delta_s^{T-1}+\delta_r^{T-1}+\delta_s^{T-1}\delta_r^{T-1}) = 1$$

$$= 2\frac{S(0)}{S(-1)}\frac{1}{1+\delta_s^0}\delta_s^0\cdot 2\frac{P(1)}{P(0)}\frac{1}{1+\delta_r^0}\delta_r^0\cdot S(T-1)\cdot\frac{P(T)}{P(1)}$$

$$= S(T-1)P(T) \quad \because S(0)=S(-1)=1,\ P(0)=1,\ \delta_s^0=\delta_r^0=1$$

Second, we have

$$X^{n+1}_{i,j}(T-1) = 2\frac{S(T-1+n)}{S(n)}\frac{[(1+\delta_s^{n})(1+\delta_s^{n-1})\cdots(1+\delta_s^{1})]}{(1+\delta_s^{T+n-1})\cdots(1+\delta_s^{T})(1+\delta_s^{T-1})}\delta_s^{i(T-1)}$$
$$\times 2\frac{P(T-1+n+1)}{P(n+1)}\frac{[(1+\delta_r^{n})(1+\delta_r^{n-1})\cdots(1+\delta_r^{1})]}{(1+\delta_r^{T+n-1})\cdots(1+\delta_r^{T})(1+\delta_r^{T-1})}\delta_r^{j(T-1)} \tag{A3}$$

$$X^{n+1}_{i+1,j}(T-1) = 2\frac{S(T-1+n)}{S(n)}\frac{[(1+\delta_s^{n})(1+\delta_s^{n-1})\cdots(1+\delta_s^{1})]}{(1+\delta_s^{T+n-1})\cdots(1+\delta_s^{T})(1+\delta_s^{T-1})}\delta_s^{(i+1)(T-1)}$$
$$\times 2\frac{P(T-1+n+1)}{P(n+1)}\frac{[(1+\delta_r^{n})(1+\delta_r^{n-1})\cdots(1+\delta_r^{1})]}{(1+\delta_r^{T+n-1})\cdots(1+\delta_r^{T})(1+\delta_r^{T-1})}\delta_r^{j(T-1)} \tag{A4}$$

$$X^{n+1}_{i,j+1}(T-1) = 2\frac{S(T-1+n)}{S(n)}\frac{[(1+\delta_s^{n})(1+\delta_s^{n-1})\cdots(1+\delta_s^{1})]}{(1+\delta_s^{T+n-1})\cdots(1+\delta_s^{T})(1+\delta_s^{T-1})}\delta_s^{i(T-1)}$$
$$\times 2\frac{P(T-1+n+1)}{P(n+1)}\frac{[(1+\delta_r^{n})(1+\delta_r^{n-1})\cdots(1+\delta_r^{1})]}{(1+\delta_r^{T+n-1})\cdots(1+\delta_r^{T})(1+\delta_r^{T-1})}\delta_r^{(j+1)(T-1)} \tag{A5}$$

$$X^{n+1}_{i+1,j+1}(T-1) = 2\frac{S(T-1+n)}{S(n)}\frac{[(1+\delta_s^{n})(1+\delta_s^{n-1})\cdots(1+\delta_s^{1})]}{(1+\delta_s^{T+n-1})\cdots(1+\delta_s^{T})(1+\delta_s^{T-1})}\delta_s^{(i+1)(T-1)}$$
$$\times 2\frac{P(T-1+n+1)}{P(n+1)}\frac{[(1+\delta_r^{n})(1+\delta_r^{n-1})\cdots(1+\delta_r^{1})]}{(1+\delta_r^{T+n-1})\cdots(1+\delta_r^{T})(1+\delta_r^{T-1})}\delta_r^{(j+1)(T-1)} \tag{A6}$$

$$X^n_{i,j}(1) = 2\frac{S(n)}{S(n-1)}\frac{1}{(1+\delta_s^n)}\delta_s^i \times 2\frac{P(n+1)}{P(n)}\frac{1}{(1+\delta_r^n)}\delta_r^j \tag{A7}$$

To prove that the arbitrage condition holds in the stochastic movements of the survival function and the discount function, we show that the following equation holds:

$$X^n_{i,j}(T) = X^n_{i,j}(1)\frac{1}{4}[X^{n+1}_{i,j}(T-1)+X^{n+1}_{i+1,j}(T-1) + X^{n+1}_{i,j+1}(T-1)+X^{n+1}_{i+1,j+1}(T-1)] \tag{A8}$$

Substituting Equations (A3), (A4), (A5), (A6), and (A7) into the right side of Equation (A8) and simplifying, we have

$$X^n_{i,j}(T) = 2\frac{S(T+n-1)}{S(n-1)}\frac{[(1+\delta_s^{n-1})(1+\delta_s^{n-2})\cdots(1+\delta_s^{1})]}{(1+\delta_s^{T+n-1})\cdots(1+\delta_s^{T})}\delta_s^{iT}$$
$$\times 2\frac{P(T+n)}{P(n)}\frac{[(1+\delta_r^{n-1})(1+\delta_r^{n-2})\cdots(1+\delta_r^{1})]}{(1+\delta_r^{T+n-1})\cdots(1+\delta_r^{T})}\delta_s^{jT}$$

Then, we add the default cash flow $F\cdot(1-S_i^n)\cdot R$, which is Equation (A2).

We used the following calculation in the simplification:

$$\delta_s^{i(T-1)}\delta_r^{j(T-1)} + \delta_s^{(i+1)(T-1)}\delta_r^{j(T-1)} + \delta_s^{i(T-1)}\delta_r^{(j+1)(T-1)} + \delta_s^{(i+1)(T-1)}\delta_r^{(j+1)i(T-1)}$$
$$= (\delta_s^{-i}\delta_r^{-j})(\delta_s^{iT}\delta_r^{jT})(1+\delta_s^{T-1}+\delta_r^{T-1}+\delta_s^{T-1}\delta_r^{T-1})$$
$$= (\delta_s^{-i}\delta_r^{-j})(\delta_s^{iT}\delta_r^{jT})(1+\delta_s^{T-1})(1+\delta_r^{T-1})$$

$(\delta_s^{-i}\delta_r^{-j})$ is canceled out by $(\delta_s^{i}\delta_r^{j})$ in Equation (A7) and $(1+\delta_s^{T-1})(1+\delta_r^{T-1})$ is cancelled out by $(1+\delta_s^{T-1})(1+\delta_r^{T-1})$ in the denominator of Equations (A3), (A4), (A5), and (A6).

APPENDIX B

DERIVATION OF PRINCIPAL CURVATURE AND DIRECTION

Let **p** be a point of $M \subset R^3$. The maximum and minimum values of the normal curvature $k(u)$ of M at **p** are called the principal curvature of M at **p**, and are denoted by k_1 and k_2. The directions in which these extreme values occur are called principal directions of M at **p**. The principal curvature can be interpreted as the maximum bend and the minimum bend at **p**.

Here, we present a method to calculate the principal curvature and the principal direction.

When $v = du \cdot x_u + dv \cdot x_v = v_1x_u + v_2x_v$ is principal direction, then, $k(\theta) = S(v)\cdot v$ is principal curvature where S is called the shape operator of M at **p**

$$\Rightarrow k(\theta) = S(v)\cdot v$$
$$= v'\cdot U = -v\cdot U'$$
$$= -\frac{dx}{ds}\cdot\frac{dU}{ds}$$
$$= -\frac{(v_1x_u + v_2x_v)\cdot(v_1U_u + v_2U_v)}{(v_1x_u + v_2x_v)\cdot(v_1x_u + v_2x_v)}$$
$$= -\frac{v_1^2(x_u\cdot U_u) + v_1v_2(x_u\cdot U_v + x_v\cdot U_u) + v_2^2(x_v\cdot U_v)}{v_1^2(x_u\cdot x_u) + 2v_1v_2(x_u\cdot x_v) + v_2^2(x_v\cdot x_v)}$$

$$(\mathrm{I} = v_1^2(x_u\cdot x_u) + 2v_1v_2(x_u\cdot x_v) + v_2^2(x_v\cdot x_v),$$
$$\mathrm{II} = v_1^2(x_u\cdot U_u) + v_1v_2(x_u\cdot U_v + x_v\cdot U_u) + v_2^2(x_v\cdot U_v))$$

$$= -\frac{v_1^2(x_u\cdot U_u) + 2v_1v_2(x_u\cdot U_v) + v_2^2(x_v\cdot U_v)}{v_1^2(x_u\cdot x_u) + 2v_1v_2(x_u\cdot x_v) + v_2^2(x_v\cdot x_v)}$$

$$(\because\ x_u\cdot U_v = x_v\cdot U_u) \qquad \text{(B1)}$$

APPENDIX C

RELATIONSHIP BETWEEN PRINCIPAL CURVATURE AND ΔP (PRICE CHANGE)

From Equation (B1), we can show that

$$k(\theta) = -\frac{v_1^2(-x_{uu}\cdot U) + 2v_1v_2(-x_{uv}\cdot U) + v_2^2(-x_{vv}\cdot U)}{v_1^2(x_u\cdot x_u) + 2v_1v_2(x_u\cdot x_v) + v_2^2(x_v\cdot x_v)}$$

$$\left\{\begin{array}{l}\because\ 0 = \dfrac{\partial(U\cdot x_u)}{\partial v} = x_u\cdot U_v + x_{uv}\cdot U \\ x_u\cdot U_v = -x_{uv}\cdot U\end{array}\right\}$$

$$= \frac{v_1^2(x_{uu}\cdot U) + 2v_1v_2(x_{uv}\cdot U) + v_2^2(x_{vv}\cdot U)}{v_1^2(x_u\cdot x_u) + 2v_1v_2(x_u\cdot x_v) + v_2^2x_v^2}$$

$$= \frac{(v_1^2x_{uu} + 2v_1v_2x_{uv} + v_2^2x_{vv})\cdot U}{v_1^2(x_u\cdot x_u) + 2v_1v_2(x_u\cdot x_v) + v_2^2(x_v\cdot x_v)}$$

$$= \frac{(v_1^2x_{uu} + 2v_1v_2x_{uv} + v_2^2x_{vv})\cdot U}{v_1^2\|x_u\|^2 + v_2^2\|x_v\|^2}\left\{\begin{array}{l}\because\ x_u\cdot x_u = \|x_u\|^2 \\ \because\ x_u\cdot x_v = 0\ \text{(Othorgonal)}\end{array}\right\}$$

$$= \frac{(v_1^2x_{uu} + 2v_1v_2x_{uv} + v_2^2x_{vv})\cdot U}{\|v\|^2}$$

$$\therefore\ \|v\|^2\cdot k(\theta) = (v_1^2x_{uu} + 2v_1v_2x_{uv} + v_2^2x_{vv})\cdot U \qquad \text{(C1)}$$

$k(\theta)$ can be interpreted as follows:

The numerator is the dot product of $(v_1^2x_{uu} + 2v_1v_2x_{uv} + v_2^2x_{vv})$ to U, which is a projection of $(v_1^2x_{uu} + 2v_1v_2x_{uv} + v_2^2x_{vv})$ to U. Recall that $(v_1^2x_{uu} + 2v_1v_2x_{uv} + v_2^2x_{vv})$ is the amount of change in the Hessian matrix (i.e., second derivatives) and U is a normal vector at **p**. In other words, the curvature is the length of the normal vector from **p** to the point where the $(v_1^2x_{uu} + 2v_1v_2x_{uv} + v_2^2x_{vv})$ projects on the normal vector. The denominator is the length of $v_1x_u + v_2x_v$.

The price change by the Taylor expansion is

$$\Delta \mathrm{x} = \begin{bmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v}\end{bmatrix}\begin{bmatrix}du \\ dv\end{bmatrix} + \frac{1}{2}\begin{bmatrix}du & dv\end{bmatrix}\begin{bmatrix}\dfrac{\partial^2 x}{\partial u^2} & \dfrac{\partial^2 x}{\partial v\partial u} \\ \dfrac{\partial^2 x}{\partial u\partial v} & \dfrac{\partial^2 x}{\partial v^2}\end{bmatrix}\begin{bmatrix}du \\ dv\end{bmatrix}$$

$$= \begin{bmatrix}x_u & x_v\end{bmatrix}\begin{bmatrix}v_1 \\ v_2\end{bmatrix} + \frac{1}{2}\begin{bmatrix}v_1 & v_2\end{bmatrix}\begin{bmatrix}x_{uu} & x_{uv} \\ x_{vu} & x_{vv}\end{bmatrix}\begin{bmatrix}v_1 \\ v_2\end{bmatrix}$$

$$= v_1 x_u + v_2 x_v + \frac{1}{2}(v_1{}^2 x_{uu} + 2v_1 v_2 x_{uv} + v_2{}^2 x_{vv}) \qquad \text{(C2)}$$

From Equations (C1) and (C2), we have

$$\Delta x \cdot U = \left(v_1 x_u + v_2 x_v + \frac{1}{2}(v_1^2 x_{uu} + 2v_1 v_2 x_{uv} + v_2^2 x_{vv}) \right) \cdot U$$

$$\Delta x \cdot U = (v_1 x_u + v_2 x_v) \cdot U + \frac{1}{2}(v_i^2 x_{uu} + 2v_1 v_2 x_{uv} + v_2^2 x_{vv}) \cdot U$$

$$2 \cdot \Delta x \cdot U = (v_1^2 x_{uu} + 2v_1 v_2 x_{uv} + v_2^2 x_{vv}) \cdot U$$

$$\because \quad (v_1 x_u + v_2 x_v) \cdot U \ = \ v \cdot U = 0 \ \text{(othorgonal)}$$

$$\therefore \quad 2 \cdot \Delta x \cdot U = (v_1^2 x_{uu} + 2v_1 v_2 x_{uv} + v_2^2 x_{vv}) \cdot U = \|v\|^2 \cdot k(\theta)$$

$$\therefore \quad 2 \cdot \Delta x \cdot U = \|v\|^2 \cdot k(\theta) \qquad \text{(C3)}$$

By Equation (C3), a relation exists between principal curvature and Δx.

Proposition: By maximizing or minimizing $k(\theta)$, we can calculate the principal curvatures and the principal directions.

Proof:

$$\kappa_n = \frac{\mathrm{II}}{\mathrm{I}} = \frac{v_1^2(x_u \cdot U_u) + v_1 v_2 (x_u \cdot U_v + x_v \cdot U_u) + v_2^2 (x_v \cdot U_v)}{v_1^2 (x_u \cdot x_u) + 2v_1 v_2 (x_u \cdot x_v) + v_2^2 (x_v \cdot x_v)}$$

$$= \frac{Lv_1^2 + 2Mv_1 v_2 + Nv_2^2}{Ev_1^2 + 2Fv_1 v_2 + Gv_2^2}$$

If we have the maximum and minimum value at (v_1, v_2)

$$\left.\frac{\partial \kappa_n}{\partial v_1}\right|_{(v_1,v_2)} = 0, \ \left.\frac{\partial \kappa_n}{\partial v_2}\right|_{(v_1,v_2)} = 0 \text{ or } \left.\frac{\mathrm{I} \cdot \mathrm{II}_{v_1} - \mathrm{I}_{v_1} \cdot \mathrm{II}}{\mathrm{I}^2}\right|_{(v_1,v_2)} = 0,$$

$$\left.\frac{\mathrm{I} \cdot \mathrm{II}_{v_2} - \mathrm{I}_{v_2} \cdot \mathrm{II}}{\mathrm{I}^2}\right|_{(v_1,v_2)} = 0$$

Multiply by I, then

$$\mathrm{II}_{v_1} - \tfrac{\mathrm{II}}{\mathrm{I}} \cdot \mathrm{I}_{v_1}\Big|_{(v_1,v_2)} = 0, \ \mathrm{II}_{v_2} - \tfrac{\mathrm{II}}{\mathrm{I}} \cdot \mathrm{I}_{v_2}\Big|_{(v_1,v_2)} = 0$$

and $\left.\frac{\mathrm{II}}{\mathrm{I}}\right|_{(v_1,v_2)} = \kappa_n\big|_{(v_1,v_2)} = \kappa_0$.

So,

$$\mathrm{II}_{v_1} - \kappa_0 \cdot \mathrm{I}_{v_1}\Big|_{(v_1,v_2)} = 0, \ \mathrm{II}_{v_2} - \kappa_0 \cdot \mathrm{I}_{v_2}\Big|_{(v_1,v_2)} = 0$$

and, $\mathrm{I}_{v_1} = 2Ev_1 + 2Fv_2$, $\mathrm{II}_{v_1} = 2Lv_1 + 2Mv_2$. Then,

$$(Lv_1 + Mv_2) - \kappa_0(Ev_1 + Fv_2) = 0,$$
$$(Mv_1 + Nv_2) - \kappa_0(Fv_1 + Gv_2) = 0 \qquad \text{(C4)}$$

To let the simultaneous equations have a solution,

$$\Rightarrow \begin{vmatrix} L - \kappa E & M - \kappa F \\ M - \kappa F & N - \kappa G \end{vmatrix} = 0 \quad \text{or}$$

$$(EG - F^2)\kappa^2 - (EN - 2FM + GL)\kappa + (LN - M^2) = 0$$

Thus, κ_n is the maximum and minimum curvature. From Equation (C4),

$$(L - \kappa E)v_1 + (M - \kappa F)v_2 = 0,$$
$$(M - \kappa F)v_1 + (N - \kappa G)v_2 = 0 \quad \text{or}$$

$$(Lv_1 + Mv_2) - \kappa(Ev_1 + Fv_2) = 0,$$
$$(Mv_1 + Nv_2) - \kappa(Fv_1 + Gv_2) = 0$$

For any κ,

$$\Rightarrow \begin{vmatrix} Lv_1 + Mv_2 & Ev_1 + Fv_2 \\ Mv_1 + Nv_2 & Fv_1 + Gv_2 \end{vmatrix} = 0 \quad \text{or}$$

$$(EM - LF)v_1^2 + (EN - LG)v_1 v_2 + (FN - MG)v_2^2 = 0$$

We can show that maximizing (or minimizing) $k(\theta) = \frac{(v_1^2 x_{uu} + 2v_1 v_2 x_{uv} + v_2^2 x_{vv}) \cdot U}{\|v\|^2}$ is equal to maximizing (or minimizing) $k(\theta) = \frac{(v_1^2 x_{uu} + 2v_1 v_2 x_{uv} + v_2^2 x_{vv})}{\|v\|^2}$.

We can further show that maximizing (or minimizing) $k(\theta) = \frac{(v_1^2 x_{uu} + 2v_1 v_2 x_{uv} + v_2^2 x_{vv})}{\|v\|^2}$ is equal to maximizing (or minimizing) $k(\theta) = (v_1^2 x_{uu} + 2v_1 v_2 x_{uv} + v_2^2 x_{vv})$ subject to $\|v\|^2$ = a small number.

From the preceding argument, we can understand that the principal direction is a direction to which the amount of change in the Hessian matrix $(v_1^2 x_{uu} + 2v_1 v_2 x_{uv} + v_2^2 x_{vv})$ is the maximum (or minimum).

ENDNOTES

[1]Specifically, the defaultable bond prices are 81.8731, 81.8976, 81.9221, 81.9466, 81.9711, 81.9956, 82.0202, 82.0441, and 82.0692 for 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, and 0.8 correlation coefficients.

[2]The Adjusted Treasury Rate (ATR) is calculated by selecting a U.S. Treasury security having a maturity comparable to the remaining maturity of the bond to be redeemed. An average price over multiple primary dealers is used to calculate the bond equivalent yield that becomes the ATR.

[3]The credit key rate durations of the CDS are –0.2812, –1.2276, –2.4561, –5.6585, and –90.5017 for the key year 0.25, 1, 2, 3, and 5, respectively. The effective duration is –100.125.

[4]For a formal proof, see Appendix C.

[5]The Adjusted Treasury Rate (ATR) is calculated by selecting a U.S. Treasury security having a maturity comparable to the remaining maturity of the bond to be redeemed. An average price over multiple primary dealers is used to calculate the bond equivalent yield that becomes the ATR.

[6]Without loss of generality, we assume that the correlation coefficient is zero.

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