Duration and Pricing of TIPS

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Treasury inflation-protected securities (TIPS) were first issued in the U.S. fixed-income market in January 1997. According to an August 2005 report prepared by the Office of Debt Management at the Department of the Treasury, as of July 2005 there are 17 outstanding TIPS issues, with maturities ranging from 2007 to 2032. The U.S. Treasury is the largest issuer of inflation-linked bonds worldwide, with market capitalization of $306 billion (which is close to 8% of the market capitalization of nominal Treasury securities) and average daily volume exceeding $8 billion.[1] The introduction and rapid growth of the TIPS market has directed academic and practitioner interest to these instruments.

A TIPS bond pays a constant coupon rate that applies to principal that is fully adjusted to inflation, based on a consumer price index (CPI).[2] Thus, coupons will be adjusted upwards in case of inflation and downwards following deflation. The treatment of the principal payment is somewhat different. The inflation-protection scheme guarantees that the value of the inflation-adjusted principal is never below its original value. This means that, at redemption, bondholders receive a principal payment higher than the original principal if, in total, the inflation rate is positive over the life of the bond. On the other hand, if in total the CPI decreases over the life of the bond (deflation), then bondholders receive the original (fixed) principal amount.

In terms of pricing, the inflation-adjustment scheme implies that a long position in a pure-discount TIPS bond is equivalent to a long position in an unadjusted (nominal) Treasury bond and a long position in a European call option written on a fully adjusted (real) pure-discount riskless bond. This implies that, if in total the CPI increases over the life of the bond, at maturity the bondholder will exercise the call option and swap the nominal bond for the upward-adjusted principal of the real bond. On the other hand, in case of deflation over the life of the bond, the call will expire worthless, leaving the bondholder with the unadjusted principal payment of the nominal bond.

Alternatively, a long position in a pure-discount TIPS bond is also equivalent to a long position in a fully adjusted (real) riskless bond and a long position in a European put option written on a real pure-discount riskless bond. In case of deflation over the life of the bond, the bondholder will exercise the put option and swap the real bond for the unadjusted principal payment of the nominal bond. Otherwise, the put option will expire worthless.

Most research in this area assumes, explicitly or implicitly, that the value of the embedded option that offers protection against deflation is trivial. Given recent inflation history in most major economies, experiencing deflation in total over the life of a bond appears to be an unlikely event. However, the recent experience of Japan puts the validity of this assumption in question.[3] Even for the U.S. market, Richard Roll claims that, in 2004 deflation "... no longer seems such an unlikely event" (Roll [2004, p. 31]). In this article we use option-pricing theory (following Black and Scholes [1973]) to model the value of a TIPS bond accounting for the value of the embedded option. To the best of our knowledge, this is the first and only attempt to price TIPS while considering the embedded option.

Given our TIPS valuation model, we proceed to examine whether traditional measures of elasticity, such as Macaulay duration, apply for TIPS. We provide a theoretical methodology for adjusting Macaulay duration to enable proper use of duration analysis for TIPS. Finally, we provide empirical evidence supporting the need for this adjustment.

Overall, this article provides an improved pricing model and an adjusted duration measure for TIPS bonds by properly considering the embedded option. This enhances our understanding of TIPS bonds and provides bond portfolio managers with adequate tools for dealing with TIPS. We also conduct a numerical simulation, based on a sample of Treasury bonds, which demonstrates that the embedded option is nontrivial.

Two studies in the existing literature look at the issue of pricing inflation-protected bonds. Jarrow and Yildirim [2003] use the foreign currency analogy of Jarrow and Turnbull [1998] and Heath, Jarrow, and Morton [1992] create a model to price the evolution of real and nominal zero-coupon bonds prices through time. They confirm the validity of their model by testing its hedging performance. Brown and Schafer [1994] fit the Cox, Ingersoll, and Ross (CIR [1985]) model to the real term structure obtained from British government index-linked bond prices. They find that the CIR model approximates the shape of the estimated real term structure closely. Both of these studies do not address the embedded option.

Siegel and Waring [2004] examine the elasticity of TIPS with respect to expected inflation (inflation duration) and with respect to the real rate (real duration). Demonstrating that these two duration measures are different for TIPS, they apply a dual duration analysis and show how a portfolio combining TIPS and nominal bonds can be created to hedge a liability stream. Specifically, the portfolio is constructed so that its weighted average inflation and real durations are identical to the weighted average inflation and real durations of the liability stream, respectively. Similar to other work in this area, Siegel and Waring [2004] do not address the embedded option in their analysis.

THE VALUE OF A TIPS BOND

Recall that, in nominal terms, a pure-discount TIPS bond is equivalent to a portfolio of an unprotected (nominal) pure-discount bond and a European call option written on the value of a fully adjusted (real) bond with exercise price equal to the face value of the bond. We adopt this interpretation in order to price a pure-discount TIPS bond. Note that, using put–call parity, one can easily show that this approach must produce the same price as that given when the analogous definition with a put option is used instead.[4]

The payoff at maturity (T) of a zero-coupon TIPS bond with an initial face value of F dollars is given by:

$$B_{TIPS,T} = \max[Fe^{\pi(T-t)} - F, 0] + F$$

where π is the continuously compounded stochastic inflation rate. The term $Fe^{\pi(T-t)}$ is the stochastic value of a fully adjusted (real) bond at time T.

We solve for the current value of the TIPS bond under two separate sets of assumptions. We first solve under the assumption that the nominal and real interest rates are deterministic.[5] Alternatively, we solve for the value of the TIPS bond allowing for stochastic interest rates.

For the derivation under deterministic interest rates, we assume that the stochastic value of the real bond follows a geometric Brownian motion process.[6] Then, using risk-neutral valuation, we show that the time-t value ($t < T$) of the TIPS bond carrying an unadjusted face value of F is given by:[7]

$$B_{TIPS,t} = Fe^{-i(T-t)}N(-d_2) + Fe^{-r(T-t)}N(d_1) \tag{1}$$

where

$$d_1 = \frac{\left(i - r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \text{ and } d_2 = \frac{\left(i - r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}$$

r is the continuously compounded real riskless rate, i is the continuously compounded nominal riskless rate, $N(\cdot)$ is the cumulative standard normal probability, and σ is the volatility of the continuously compounded return on the fully adjusted (real) pure-discount bond.

Next, we solve for the value of the TIPS bond allowing for stochastic interest rates. Specifically, now we assume that the price of the fully adjusted (real) pure-discount bond, $P(r, t, T)$, and the price of the nominal pure-discount bond $P(i, t, T)$, follows geometric Brownian motion processes. Following Merton [1973], and analogous to Chance's [1990] model for the valuation of defaultable bonds, we can express the value of the zero-coupon TIPS bond in a stochastic interest rate environment as:

$$B_{TIPS,t} = FP(i,T)N(-d_2) + FP(r,T)N(d_1) \tag{2}$$

where

$$d_1 = \frac{\ln P(r,T) - \ln P(i,T) + (\sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

and

$$\sigma^2 T = \int_0^T \{\sigma_{p(r)}(t) + \sigma_{p(i)}(t) - 2\rho\sigma_{p(r)}\sigma_{p(i)}(t)\}dt,\ \rho = corr(i,r)$$

Although Equation (2) is derived under a more realistic case, where interest rates are stochastic, in the remainder of this article we focus on the model in Equation (1). This is mainly because it leads to more tractable results when we examine the comparative statics of the model. Note that, unlike model (1), model (2) has the correlation between the nominal rate and the real rate as one of the price determinants. Below we show that the relationship between the two rates is an important factor also under model (1).

Comparative Statics

In our analysis below, we show that many of the results depend on the sign of partial derivatives based on the Fisher relation. Signing these derivatives is an empirical, rather than a theoretical, question. Recall that the continuous-time version of the Fisher effect is given by $i = r + \pi^*$, where π^* is the expected continuously compounded inflation rate. In practice, the real interest rate and the inflation rate are not independent. The relationship between the two will vary across time and across countries, depending on the implemented monetary policy.

Voluminous empirical research exists in the extant literature testing this relationship in different countries and different time periods (for a good discussion see Van Horne [2001]). Recent U.S. studies by Roll [2004], Evans [1998], Crowder and Hoffman [1996], and Crowder [2003] generally agree that 1) the relationship between changes in expected inflation rates and changes in nominal rates is positive (see, for example, Jarrow and Yildirim [2003], Crowder [2003], McCulloch and Kochin [2000], and Crowder and Hoffman [1996]), 2) the relationship between changes in expected inflation rates and changes in real rates is positive (see Roll [2004], Evans [1998]), and 3) the relationship between changes in nominal and real rates is positive (see Jarrow and Yildirim [2003] and McCulloch and Kochin [2000]).

These recent empirical results are in agreement with the monetary policy of the U.S. Federal Reserve Bank since the early 1990s. The main policy objective of the Federal Reserve is to achieve price stability to attain maximum sustainable economic growth. To achieve this goal, when inflation expectations decline and the expected growth in the economy decelerates, the Federal Reserve will try to ease the monetary policy and stimulate the economy by lowering real interest rates. On the other hand, when inflation expectations rise and the expected growth in the economy accelerates, the Federal Reserve will try to tighten the monetary policy by raising real rates. With this in mind, we now proceed with the comparative statics for TIPS bond pricing Equation (1).

Proposition 1: *As the volatility of the return on the fully adjusted (real) bond increases, the value of the TIPS bond increases.*

Proof: All proofs of propositions are in Appendix B.

Because the fully adjusted (real) bond is the underlying asset for the embedded option, a higher volatility of its return implies a higher probability that the option will be in the money. In other words, a higher volatility implies that there is a higher chance of deflation and, therefore, the protection against deflation (provided by the embedded option) becomes more valuable.

Proposition 2: *As the nominal interest rate increases, the value of the TIPS bond can increase or decrease, depending on the monetary policy practiced.*

To see the intuition behind this result, consider the derivative of the value of the inflation-protected security with respect to the nominal rate:

$$\frac{dB_{TIPS,t}}{di} = -\tau\left[\frac{dr}{di}Fe^{-r\tau}N(d_1) + Fe^{-i\tau}N(-d_2)\right] \leq \geq 0$$

To sign this derivative, one has to take a closer look at the term dr / di. Rearranging the Fisher relation, we get $r = i - p^*$. Thus, we have $\frac{dr}{di} = (1 - \frac{d\pi^*}{di})$. Recall that recent empirical evidence suggests that the relationship between changes in expected inflation rates and changes in nominal rates, $d\pi^*/di$, is positive but lower than one (see Jarrow and Yildirim [2003], Crowder [2003], McCulloch and Kochin [2000], and Crowder and Hoffman [1996]). Also recall that the empirical relation between changes in nominal and real rates is positive (see Jarrow and Yildirim [2003] and McCulloch and Kochin [2000]). This implies that empirically dr/di is positive. This is consistent with the current monetary policy practiced by the U.S. Federal Reserve.

Thus, under the current monetary policy, the value of the TIPS bond decreases as the nominal interest rate increases. This is due to: 1) the direct negative impact of the nominal interest rate shift on the value of the TIPS bond (this is captured by the second term in the square brackets in the above derivative); and 2) the indirect negative impact of the nominal interest rate increase on the value of the bond through its impact on the real interest rate (this is captured by the first term in the square brackets in the above derivative).

Proposition 3: *As the real interest rate increases, the value of the TIPS bond can increase or decrease, depending on the monetary policy practiced.*

To better understand this result, consider the derivative of the value of the inflation-protected security with respect to the real rate:

$$\frac{dB_{TIPS,t}}{dr} = -\tau\left[Fe^{-r\tau}N(d_1) + \frac{di}{dr}Fe^{-i\tau}N(-d_2)\right] \leq \geq 0$$

To sign this derivative, one has to examine di/dr. Based on the Fisher relation, we get $\frac{di}{dr} = (1 + \frac{d\pi^*}{dr})$. Recent empirical evidence suggests that the relation between changes in expected inflation rates and changes in real rates is positive (see Roll [2004] and Evans [1998]). Recall that the U.S. Federal Reserve uses the real rate to control expectations of future inflation. This means that empirically we have $di/dr > 0$, and the value of the TIPS bond decreases in the real rate.

Thus, under the current monetary policy in the U.S., the value of the TIPS bond decreases as the real interest rate increases. This is because of 1) the direct negative impact of the real interest rate increase on the value of the bond (this is captured by the second term in the square brackets in the above derivative) and 2) the indirect negative impact of the real interest rate increase on the value of the bond through its impact on the nominal interest rate (this is captured by the first term in the square brackets in the above derivative).

Proposition 4: *As the expected inflation rate increases, the value of the TIPS bond can increase or decrease, depending on the monetary policy practiced.*

Once again, to see the intuition behind this proposition, consider the derivative of the value of the inflation-protected security with respect to the expected inflation rate:

$$\frac{dB_{TIPS,t}}{d\pi^*} = -\tau\left[\frac{dr}{d\pi^*}Fe^{-r\tau}N(d_1) + \frac{di}{d\pi^*}Fe^{-i\tau}N(-d_2)\right] \leq \geq 0 \tag{3}$$

To sign this derivative, we have to examine $di/d\pi^*$ and $dr/d\pi^*$. Based on the Fisher relation, we get $\frac{di}{d\pi^*} = (\frac{dr}{d\pi^*} + 1)$ Given this result, we rewrite the above derivative as

$$\frac{dB_{TIPS,t}}{d\pi^*} = -\tau\left[\frac{dr}{d\pi^*}B_{TIPS,t} + Fe^{-i\tau}N(-d_2)\right] \leq \geq 0$$

This result shows that therelationship between the value of the TIPS bond and expected inflation depends on the sign of $dr/d\pi^*$. As previously noted, when expectations for inflation decline (increase) and the growth in the economy decelerates (accelerates), the Federal Reserve stimulates (tightens) the economy by lowering (increasing) real interest rates. Given recent empirical evidence and the monetary policy applied by the Federal Reserve, this relationship is currently positive in the U.S.

This means that under the current monetary policy in the U.S., the value of the TIPS bond decreases as the expected inflation rate increases. This is because of 1) the direct positive impact an increase in the expected inflation rate has on the nominal rate, which in turn has a negative impact on the value of the bond (captured by the first term in the square brackets in Equation (3)) and 2) the positive impact an increase in the expected inflation rate has on the real rate, which in turn has a negative impact on the value of the bond (captured by the second term in the square brackets in Equation (3)).

Next, we consider the impact of a potential change in the Federal Reserve's policy on our results. If the Federal Reserve decides to pursue a neutral policy with respect to changes in expected inflation, the value of $dr/d\pi^*$. will be zero and the value of the TIPS bond will still decline with a higher expected inflation rate. Note that it does not make sense for the Federal Reserve to apply a policy that stimulates (slows down) the economy when there are expectations for inflation (deflation). Therefore, in reality $dr/d\pi^*$ will not take a negative sign.

Proposition 5: *As the maturity of the TIPS bond increases, its value can increase or decrease.*

To explain the intuition behind this proposition, consider the derivative of the value of the inflation-protected security with respect to the bond's time to maturity:

$$\frac{dB_{TIPS,t}}{d\tau} = Fe^{-r\tau}N'(d_1)\frac{\sigma}{2\sqrt{\tau}} - rFe^{-r\tau}N(d_1) - iFe^{-i\tau}N(-d_2) \leq \ \geq 0$$

There are two opposing effects of the time to maturity on the value of the bond. On one hand, as τ increases the value of the TIPS bond drops due to the time value of money impact. On the other hand, as τ increases the value of the embedded option in the TIPS bond increases.

NUMERICAL SIMULATION

In this section we run a simulation based on parameter values observed in the real world in order to estimate the magnitude of the error that results from ignoring the embedded option. We use a sample of daily returns on Treasury inflation-protected securities, obtained from the U.S. Department of Treasury. We then calculate the annualized historical volatility of TIPS returns with $\sigma = \sqrt{252} \times \sqrt{\frac{1}{N-1} \times \Sigma_{t=1}^{N}(R_{ti} - \bar{R})^2}$, where R_t is the daily return on the TIPS bond, $\bar{R}$ is the mean of daily returns, and N is the number of observations in the sample.

Panel A of Exhibit 1 reports that the annualized return volatility on TIPS bond return data ranges between 2.55% and 10.80%.[8] Thus, in the ensuing numerical simulation, we allow the volatility to range between 0% and 15%. For the continuously compounded nominal and real interest rates, we use the averages of the 5-year daily Treasury nominal and real spot rates for November 2006, taken from the United States Department of Treasury website (based on daily estimated yield curves using a cubic spline model). These rates are 4.58% and 2.41%, respectively.

Panel A of Exhibit 2 illustrates the calculated model values of a TIPS bond as a function of the return volatility of the fully adjusted (real) bond for different maturities. The results show a positive TIPS value–real bond return volatility relation. This is consistent with Proposition 1. Recall that because the fully adjusted (real) bond is the underlying asset for the embedded option, a higher volatility of its return implies a higher probability of deflation and, therefore, the protection against deflation provided by the option becomes more valuable. The exhibit also shows that, everything else being equal, as the maturity of the TIPS bond increases, its value decreases due to the time value of money.

In Panel B of Exhibit 2, we compare calculated model values of the TIPS bond with the value of a comparable real bond. The latter ignores the embedded option and is calculated by discounting the face value of the bond with the real interest rate. We then calculate the error of ignoring the embedded option as the difference between the model price and the price of the real bond. Note that this difference gives the value of the embedded option. For a volatility ranging from 0% to 15%, the pricing error for the 5-year (30-year) TIPS bond is between 0 and \$7.22 (\$3.34) for a \$100 face value bond. However, for the 5.3% average return volatility of the sampled TIPS, the pricing error for the 5-year (30-year) TIPS bond amounts to \$0.98 (\$0.05) for a \$100 face value bond. Thus, the magnitude of the simulated error is economically significant. It is evident that the pricing error is smaller for longer maturity TIPS bonds. This result is intuitive. The probability that, in total, the CPI will decrease over the life of the bond (deflation) is higher over shorter horizons (shorter maturity).

EXHIBIT 1
TIPS Bonds and Their Characteristics

Panel A: TIPS Data

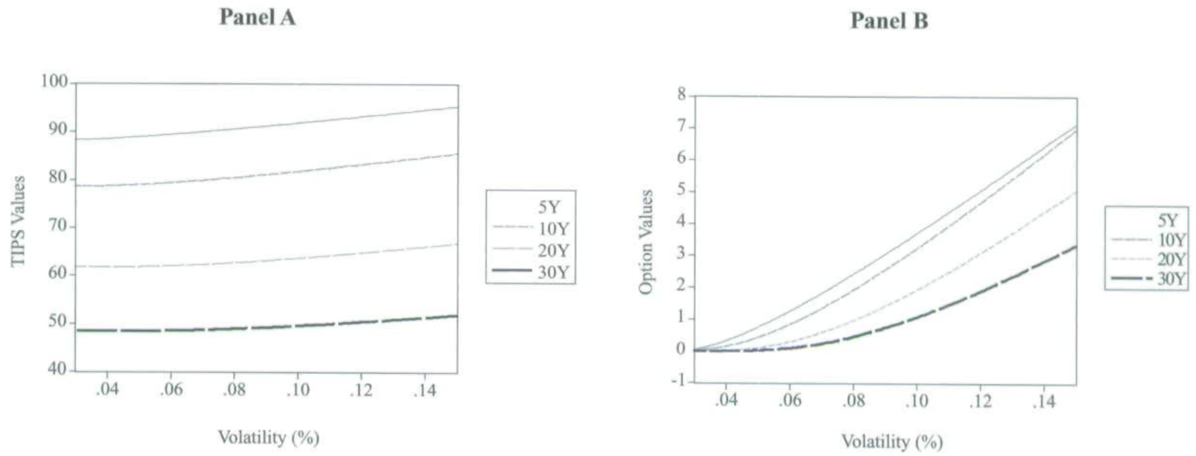
Coupon Rate	Issue Date	Maturity Date	Years to Maturity	Daily Data Begin	Daily Data End	Number of Observations	Mean Return	Standard Dev. of Returns
2.375	15/04/2006	15/04/2011	5	25/04/2006	11/1/2007	187	–0.87%	2.74%
0.875	15/10/2004	15/04/2010	5.5	1/11/2004	11/1/2007	571	–2.34%	2.92%
3.375	01/01/1997	15/01/2007	10	30/01/1997	24/02/2004	1836	1.24%	2.55%
3.625	01/01/1998	15/01/2008	10	09/01/1998	11/1/2007	2346	0.19%	3.04%
3.875	01/01/1999	15/01/2009	10	07/01/1999	11/1/2007	2087	0.24%	3.21%
4.25	01/01/2000	15/01/2010	10	13/01/2000	11/1/2007	1822	0.81%	3.69%
3.5	01/01/2001	15/01/2011	10	10/01/2001	11/1/2007	1562	0.65%	4.38%
3.375	01/01/2002	15/01/2012	10	10/01/2002	11/1/2007	1033	0.82%	4.93%
3	01/07/2002	15/07/2012	10	12/07/2002	11/1/2007	1172	0.47%	5.34%
1.875	01/07/2003	15/07/2013	10	10/07/2003	11/1/2007	913	–0.90%	5.75%
2	15/01/2004	15/01/2014	10	09/01/2004	11/1/2007	782	–1.36%	5.28%
2	15/07/2004	15/07/2014	10	09/07/2004	11/1/2007	652	–1.21%	4.96%
1.625	15/01/2005	15/01/2015	10	14/01/2005	11/1/2007	517	–2.66%	5.01%
1.875	15/07/2005	15/07/2015	10	15/07/2005	11/1/2007	388	–2.53%	4.90%
2	15/01/2006	15/01/2016	10	12/01/2006	11/1/2007	260	–3.75%	4.86%
2.5	15/07/2006	15/07/2016	10	13/07/2006	11/1/2007	130	0.76%	4.53%
2	15/01/2006	15/01/2026	20	24/01/2006	11/1/2007	252	–7.35%	8.32%
2.375	15/07/2004	15/01/2025	20.5	28/07/2004	11/1/2007	639	0.04%	8.66%
3.625	01/04/1998	15/04/2028	30	10/04/1998	11/1/2007	2277	2.23%	8.00%
3.875	01/04/1999	15/04/2029	30	09/04/1999	11/1/2007	2016	2.78%	8.37%
3.375	01/10/2001	15/04/2032	30.5	11/10/2001	11/1/2007	1367	3.65%	10.80%
Mean							–0.43%	5.34%

Panel B: Current Spot Rates

Average daily Treasury real spot rates for 11:2006	2.41 (5Y)	2.35 (7Y)	2.29 (10Y)	2.23 (20Y)
Average daily Treasury nominal spot rates for 11:2006	4.58 (5Y)	4.58 (7Y)	4.60 (10Y)	4.78 (20Y)

Note: Panel A of this exhibit reports the bond-specific characteristics of the Treasury inflation-protected securities in our sample. The annualized historical mean return and its associated annualized historical volatility of each TIPS bond over the specified sample period are also reported in Panel A. We calculate the annualized mean return with: $\mu = 252 \times (\frac{1}{N}\sum_{t=1}^{N} R_t)$*. The annualized historical volatility of TIPS returns is calculated with:* $\sigma = \sqrt{252} \times \sqrt{\frac{1}{N-1} \times \sum_{t=1}^{N}(R_t - \overline{R})^2}$*, where* R_t *is the daily return on the TIPS bond,* $\overline{R}$ *is the mean of daily returns, and N is the number of observations in the sample. Panel B of this exhibit reports average daily nominal and real spot rates for maturities of 5, 7, 10, and 20 years, calculated for November 2006, taken from the U.S. Department of Treasury website (based on daily estimated yield curves).*

EXHIBIT 2
The Value of the Embedded Option



Note: This numerical exercise assumes continuously compounded nominal and real interest rates of 4.58 and 2.41%, respectively. These rates are the averages of the 5-year daily Treasury nominal and real spot rates for November 2004, taken from the U.S. Department of Treasury website (based on estimated yield curves). The assumed volatility of the fully adjusted (real) bond ranges between 0% and 15%, whereas the face value is maintained at $100. We simulate values for 5, 10, 20, and 30-year bonds. In Panel A we plot the calculated model values of a TIPS bond as a function of the return volatility for different maturities. In Panel B we plot the error of ignoring the embedded option as a function of the return volatility for different maturities.

DURATION FOR TIPS BONDS

The practice of bond portfolio managers, using duration as a measure of risk or as an immunization tool, is to calculate the elasticity of the TIPS bond with respect to its own (real) yield to maturity. By doing so, they implicitly assume that the value of the embedded put option is trivial, and therefore the value of the TIPS bond is equal to the value of the fully adjusted (real) bond: $B_t^* = Fe^{-r\tau}$. This duration measure is the bond's standard Macaulay duration, which is equal to the time to maturity of the bond for a pure-discount TIPS bond (τ). When the embedded option is valuable, Macaulay duration may be a biased estimator for the elasticity of the TIPS bond. If a portfolio manager uses nominal benchmarks, such as a Lehman bond index, to evaluate the performance of a TIPS bond, then the manager needs to use a duration measure calculated with respect to the nominal interest rate. We show that when the embedded option is taken into consideration, the effective duration of a TIPS bond is different from its Macaulay duration. We consider the elasticity with respect to both the nominal rate (i) and the real rate (r).

"Nominal" Duration

The use of unadjusted Macaulay duration to compare the elasticity of nominal bonds and TIPS bonds is inadequate. Unadjusted duration is a valid elasticity measure only for bonds that are priced off the same yield curve. This is because the reference rate for nominal bond Macaulay elasticity is the nominal rate and the reference rate for TIPS bond Macaulay elasticity is the TIPS bond yield to maturity. Therefore, to obtain a more meaningful measure of TIPS bond elasticity in the context of a portfolio containing both nominal and TIPS bonds, there is a need to develop an elasticity measure for TIPS with the reference rate being the nominal rate. We apply the standard price-elasticity definition of duration on bond-pricing Equation (1), when the nominal interest rate, i, is continuously compounded, $D_i = -\frac{1}{B_{TIPS,t}} \frac{\partial B_{TIPS,t}}{\partial r}$. This yields the following "nominal" duration:[9]

$$D_i = \tau \left(1 - \frac{\partial \pi^*}{\partial i} \frac{Fe^{-r\tau} N(d_1)}{B_{TIPS,t}} \right) \tag{4}$$

Recall that a pure-discount TIPS bond is equivalent to a portfolio of an unprotected (nominal) pure-discount bond and the embedded call option. Given this equivalence, one can show that the nominal duration of the TIPS bond in Equation (4) is equal to the weighted average of the durations of the unprotected (nominal) bond and that of the embedded call option on the real bond.[10]

As noted previously, recent empirical evidence for the U.S. suggests that the relationship between changes in the expected inflation rate and changes in the nominal rate is positive. This means that empirically $\partial\pi^* / \partial i$ is positive. Given this evidence and the duration in Equation (4), one may expect the nominal duration of a U.S. TIPS (D_i) to be lower than that of an unprotected nominal Treasury bond (τ). This is because of the protection provided to the TIPS holder against loss resulting from inflation risk that unprotected (nominal) bondholders face.

Equation (4) also shows that the nominal duration of the TIPS bond converges to the duration of the nominal Treasury security when the embedded call option is deep out-of-the-money, then the option's delta, $N(d_1)$, approaches zero. In this case a shift in the nominal rate will have a trivial effect on the value of the embedded call.

"Real" Duration

We apply the standard price-elasticity definition of duration on bond-pricing Equation (1), when the real interest rate, r, is continuously compounded:

$$D_i = -\frac{1}{B_{TIPS,t}} \frac{\partial B_{TIPS,t}}{\partial r}$$

This yields the following "real" duration:

$$D_r = \tau \left(1 + \frac{\partial \pi^*}{\partial r} \frac{F e^{-r\tau} N(-d_2)}{B_{TIPS,t}} \right) \quad (5)$$

When one uses the real rate as the reference rate, a pure-discount TIPS bond is equivalent to a portfolio of a fully adjusted (real) pure-discount bond and an embedded European put option written on the value of a fully adjusted (real) bond. Given this equivalence, one can show that the real duration of the TIPS bond in Equation (5) is equal to the weighted average of the durations of the fully adjusted (real) bond and the embedded put option on the real bond.

As noted previously, empirical evidence for the U.S. suggests that the relationship between changes in expected inflation rates and changes in real rates is positive. In other words, empirically $\partial\pi^* / \partial r$ is positive. In the context of Equation (5), this evidence means that the real duration of a U.S. TIPS (D_r) is higher than that of a fully adjusted (real) Treasury bond (τ). This is due to the added sensitivity of the embedded put option.

Equation (5) also shows that the real duration of the inflation-protected security approaches the duration of the real Treasury security when the embedded put option is deep out of the money and that the probability of exercising the option, $N(-d_2)$, is close to zero. In this case, a shift in the real rate will have a marginal effect on the value of the embedded put.

Comparing the Two Duration Measures

The relative size of the duration measures derived above is in agreement with Wilcox (1998) who argues that nominal interest rate shocks will have a small effect on the prices of TIPS bonds, but a significant effect on the prices of nominal bonds. At the same time, real interest rate shocks will have a considerable effect on the prices of TIPS bonds, but a lower effect on the prices of nominal bonds. This is consistent with our nominal duration being lower than Macaulay duration and our real duration being greater than Macaulay duration. The question of whether these differences in elasticity measures are significant is an empirical question that will be addressed in the empirical test reported in the following section.

EMPIRICAL TEST

Equation (4), along with evidence in the extant literature on the sign of the expected inflation rate-nominal rate relation, implies that the nominal duration of a TIPS bond (D_i) is lower than its Macaulay duration. In this section we test this prediction. Because fully adjusted real bonds do not exist in the U.S. fixed-income market, we cannot empirically test the implications of Equation (5) with respect to the real duration.

Data and Methodology

We use weekly indices of market yield on U.S. Treasury securities at 5-, 7-, 10-, and 20-year constant maturities. These data are obtained from the U.S. Federal

Reserve Board. The sample consists of yields for both nominal and inflation-indexed (TIPS) yields. It covers the 01/01/2003 to 01/12/2006 period (except for the 20-year yields for which the sample starts on 30/07/2004).[11]

The nominal elasticity of a TIPS bond is given by $D_i = -\frac{1}{B_{TIPS,t}} \frac{\partial B_{TIPS,t}}{\partial i}$. The Macaulay elasticity of the TIPS bond is given by $D_m = -\frac{1}{B_{TIPS,t}} \frac{\partial B_{TIPS,t}}{\partial y}$, where y is the yield to maturity on the TIPS bond. It can be easily shown that the following relationship holds:

$$\frac{D_i}{D_m} = \frac{\partial y}{\partial i} \tag{6}$$

Given Equation (6), we now directly estimate the ratio between the nominal duration for TIPS bonds and its Macaulay counterpart with the following regression model:

$$\Delta y_{kt} = \gamma_k + \delta_k \Delta i_{kt} + \varepsilon_{kt} \tag{7}$$

where

Δy_{kt} = weekly changes in the k-year constant maturity TIPS yield index, $k = 5, 7, 10, 20$

Δi_{kt} = weekly changes in the k-year constant maturity nominal yield index, $k = 5, 7, 10, 20$

δ_k = the slope coefficient that measures the ratio between the nominal duration and its Macaulay counterpart ($\frac{D_i}{D_m}$)

γ_k = intercept

ε_{kt} = error term

Results

In Exhibit 3, we report the estimates for regression model (7). Using a stepwise autoregression method, we find that our dataset is characterized by the first-order autoregressive nature of the OLS residuals of regression model (7) for all indices. In a number of cases the OLS residuals also exhibit a nonconstant volatility consistent with a GARCH (1,1) process. In those instances, we apply a maximum-likelihood estimation procedure for a combined first-order autoregressive model and a GARCH (1,1) model. When the OLS residuals only follow a first-order autoregressive process, we apply the Yule–Walker method. In analyzing the results we note that one's conclusions are insensitive to whether one uses OLS or the AR-GARCH estimation method. Therefore, we focus our discussion on the OLS results.

For every constant-maturity index, Panel A of Exhibit 3 shows the OLS estimates of the regression coefficients; the standard error of the slope estimator; the adjusted R^2; a Durbin–Watson statistic; and the t-statistic for the null hypothesis that the estimated slope coefficient is lower than one (which means that TIPS nominal duration is lower than Macaulay duration). If the null hypothesis is not rejected, we may conclude that the magnitude of the error caused by failing to adjust the Macaulay duration for TIPS bonds is nontrivial. The results for the AR-GARCH estimation are reported in Panel B of Exhibit 3.

Both panels of Exhibit 3 report that the estimated slope coefficient of all constant-maturity indices is lower than one. In all cases this result is statistically significant at the 1% level. Furthermore, the relative adjustment for Macaulay duration, estimated with $(1 - \hat{\delta}_k)$, is economically significant for all TIPS indices. This adjustment ranges from 24.28% for the 5-year constant-maturity index to 34.23% of Macaulay for the 20-year constant-maturity index (Panel A). This means that the adjusted duration is 24% to 34% lower than its Macaulay counterpart. In general, the required adjustment for TIPS duration increases with maturity.

The reported results lend strong support for the implication of Equation (4) that the nominal duration of a TIPS bond (D_i) is lower than its Macaulay Duration. These results are robust with respect to whether regression model (7) is estimated using OLS or an AR-GARCH specification.

SUMMARY AND CONCLUSION

Most research on inflation-protected securities assumes, explicitly or implicitly, that the value of the put option that offers protection against deflation is trivial. In this article we use option-pricing theory to model the value of TIPS, accounting for the value of the embedded option. A numerical simulation, based on a sample of Treasury bonds, demonstrates that ignoring the embedded option is costly.

We examine several determinants of TIPS. Specifically, our model predicts that the pricing of TIPS is determined by the volatility of the return of a real bond, the nominal rate, the real rate, the expected inflation rate, and the time to maturity. We examine the relationship between the price of a TIPS bond and these parameters.

EXHIBIT 3

Estimation of the Relationship between TIPS Nominal Duration and Its Corresponding Macaulay Duration

Panel A: OLS Results

Index (*k*)	*N*	γ_k	δ_k	*SE*	*Adj* R^2	*DW*	$t(\delta_k<1)$
5	205	-0.0034	0.7572***	0.0363	0.68	1.51	6.69***
7	205	-0.0036	0.7124***	0.0325	0.70	1.50	8.84***
10	205	-0.0031	0.7158***	0.0321	0.71	1.60	8.86***
20	123	0.0009	0.6577***	0.0371	0.72	1.57	9.22***

Panel B: AR-GARCH Results

Index (*k*)	*N*	γ_k	δ_k	*SE*	*m*	*p*	*q*	*Adj* R^2	LM	$t(\delta_k<1)$
5	205	-0.0038	0.7657***	0.0373	1	-	-	0.70	-	6.28***
7	205	-0.0038	0.7196***	0.0252	1	1	1	0.72	0.0355	11.13***
10	205	-0.0022	0.7390***	0.0257	1	1	1	0.72	0.0105	10.16***
20	123	0.0008	0.6641***	0.0375	1	-	-	0.73	-	8.96***

Note: This exhibit reports the results of both OLS and AR-GARCH estimations of regression model (7): $\Delta ykt = \gamma_k + \delta_k \Delta i_{kt} + \varepsilon_{kt}$ where Δy_{kt} is the weekly changes in the k-year constant maturity TIPS yield index; Δi_{kt} is the weekly changes in the k-year constant maturity nominal yield index; δ_k is the slope coefficient that measures the ratio between the nominal duration and its Macaulay counterpart $(\frac{D_i}{D_m})$; γ_k is the intercept; and ε_{kt} is the error term (k = 5, 7, 10, 20 years to maturity). Panel A shows the OLS estimates of the regression coefficients; the standard error of the slope estimator (SE); the adjusted R^2; a Durbin–Watson statistic; and the t-statistic for the null hypothesis that the estimated slope coefficient is lower than one. Panel B reports AR-GARCH estimates for the regression intercept and slope coefficients; m gives the degree of the autoregressive process as determined by the stepwise autoregression method; p and q are the GARCH(p,q) parameters; and finally LM gives the p-value for the Lagrange multiplier test.

In the context of a portfolio containing both nominal and TIPS bonds there is a need to adjust the TIPS bond Macaulay elasticity. We develop an elasticity measure for TIPS with the reference rate being the nominal rate, which is different from the traditional Macaulay measure for TIPS with the reference rate being the TIPS bond yield to maturity. The need for this adjustment is strongly supported by the data.

To the best of our knowledge, this is the only attempt to price TIPS while considering the embedded option. We note that the protection against deflation provided by coupon-bearing TIPS applies only to the face value paid at maturity, not to the coupon payments. Thus, one cannot price coupon-bearing TIPS by individually pricing strips and aggregating these prices to obtain the value of the TIPS coupon bond. Because in our model we price pure-discount TIPS bonds, one can view this study as a first step in this line of research. Future research should focus on the pricing of coupon TIPS bonds.

APPENDIX A

Derivation of the Bond-Pricing Equation and the Yield Spread Equation

Note that because we need to discount expected nominal cash flows, we use the nominal riskless rate (i) under risk-neutral valuation. The payoff at maturity (T) of a zero-coupon TIPS bond carrying \$1 face value is given by:

$$B_{TIPS,T} = \max[e^{\pi(T-t)} - 1, 0] + 1$$

Under risk-neutral valuation, the time-t value of the call option written on the real bond is given by:

$$\begin{aligned} C(B_t^*, t, T) &= e^{-i(T-t)} E_q\left(\max\left[B_T^* - 1, 0\right]\right) \\ &= e^{-i(T-t)} E_q\left(\max\left[e^{\pi(T-t)} - 1, 0\right]\right) \end{aligned}$$

To calculate the value of the call, we assume that the stochastic value of the real bond, B^*, is well described under the risk-neutral measure Q, by the following geometric Brownian motion process:

$$dB^* = B^*idt + B^*\sigma dZ$$

Let $G = \ln B^*$. Applying Ito's Lemma, we get:

$$dG = \left[\frac{dG}{dB^*}B^*i + \frac{dG}{dt} + \frac{1}{2}\frac{d^2G}{dB^{*2}}\sigma^2 B^{*2}\right]dt + \frac{dG}{dB^*}B^*\sigma dZ$$

and

$$d\ln B^* = \left[\frac{d\ln B^*}{dB^*}B^*i + \frac{d\ln B^*}{dt} + \frac{1}{2}\frac{d^2\ln B^*}{dB^{*2}}\sigma^2 B^{*2}\right]dt + \frac{d\ln B^*}{dB^*}B^*\sigma dZ \quad \text{(A1)}$$

Taking the derivatives in the above stochastic differential equation, we get:

$$\frac{d\ln B^*}{dB^*} = \frac{1}{B^*},\ \frac{d\ln B^*}{dt} = 0,\ \text{and}\ \frac{d^2\ln B^*}{dB^{*2}} = -\frac{1}{B^{*2}}$$

Substituting these values back into Equation (A1) we get:

$$d\ln B^* = \left(i - \frac{\sigma^2}{2}\right)dt + \sigma dZ$$

This implies the following risk-neutral distribution for the continuously compounded return on the real bond:

$$\ln\left[\frac{B_T^*}{B_t^*}\right] \sim N\left[\left(i - \frac{\sigma^2}{2}\right)(T-t),\ \sigma^2(T-t)\right]$$

Or we can write that:

$$\ln B_T^* \sim N\left[\ln B^* + \left(i - \frac{\sigma^2}{2}\right)(T-t),\ \sigma^2(T-t)\right]$$

Noting that $\ln B^* = -r(T-t)$, we get:

$$\ln B_T^* \sim N\left[\left(i - r - \frac{\sigma^2}{2}\right)(T-t),\ \sigma^2(T-t)\right]$$

Following Black and Scholes [1973], the time-t value of the call option is given by:

$$C(B_t^*, t, T) = e^{-r(T-t)}N(d_1) - e^{-i(T-t)}N(d_2)$$

where

$$d_1 = \frac{\left(i - r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \text{ and } d_2 = \frac{\left(i - r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

Recall that the value of a TIPS bond is equivalent to the value of a portfolio consisting of a corresponding unadjusted riskless bond and the above call option. Given the value of the call, the time-t value of the TIPS bond is given by:

$$B_{TIPS,t} = e^{-i(T-t)} + \left[e^{-r(T-t)}N(d_1) - e^{-i(T-t)}N(d_2)\right]$$

Rearranging, we get:

$$B_{TIPS,t} = e^{-i(T-t)}N(-d_2) + e^{-r(T-t)}N(d_1)$$

For a bond with an initial face value of F, we get the TIPS bond-pricing equation:

$$B_{TIPS,t} = Fe^{-i(T-t)}N(-d_2) + Fe^{-r(T-t)}N(d_1) \quad (1)$$

Appendix B

Proofs of Propositions

Proof of Proposition 1. Differentiating the value of the inflation-protected security with respect to the volatility of the return on the fully adjusted (real) bond, we get:

$$\begin{aligned}\frac{dB_{TIPS,t}}{d\sigma} &= Fe^{-r\tau}\frac{\partial N(d_1)}{\partial\sigma} - Fe^{-i\tau}\frac{\partial N(d_2)}{\partial\sigma}\\ &= Fe^{-r\tau}N'(d_1)\frac{\partial d_1}{\partial\sigma} - Fe^{-i\tau}N'(d_2)\frac{\partial d_2}{\partial\sigma}\\ &= Fe^{-r\tau}N'(d_1)\frac{\partial d_1}{\partial\sigma} - Fe^{-i\tau}N'(d_1)e^{-(r-i)\tau}\frac{\partial d_2}{\partial\sigma}\\ &= Fe^{-r\tau}N'(d_1)\left[\frac{\partial d_1}{\partial\sigma} - \frac{\partial d_2}{\partial\sigma}\right]\\ &= Fe^{-r\tau}N'(d_1)\sqrt{\tau} > 0.\end{aligned}$$

Q.E.D.

Proof of Proposition 2. Differentiating the value of the inflation-protected security with respect to the nominal rate, we get:

$$\frac{dB_{TIPS,t}}{di} = -\tau Fe^{-i\tau} - \tau\frac{dr}{di}Fe^{-r\tau}N(d_1) + Fe^{-r\tau}\frac{\partial N(d_1)}{\partial i}$$
$$+\tau Fe^{-i\tau}N(d_2) - Fe^{-i\tau}\frac{\partial N(d_2)}{\partial i}$$
$$= -\tau Fe^{-i\tau} - \tau\frac{dr}{di}Fe^{-r\tau}N(d_1) + Fe^{-r\tau}N'(d_1)\frac{\partial d_1}{\partial i}$$
$$+\tau Fe^{-i\tau}N(d_2) - Fe^{-i\tau}N'(d_2)\frac{\partial d_2}{\partial i}$$
$$= -\tau Fe^{-i\tau} - \tau\frac{dr}{di}Fe^{-r\tau}N(d_1) + Fe^{-r\tau}N'(d_1)\frac{\partial d_1}{\partial i}$$
$$+\tau Fe^{-i\tau}N(d_2) - Fe^{-i\tau}N'(d_1)e^{(i-r)\tau}\frac{\partial d_2}{\partial i}$$
$$= -\tau Fe^{-i\tau} - \tau\frac{dr}{di}Fe^{-r\tau}N(d_1) + \tau Fe^{-i\tau}N(d_2)$$
$$= -\tau\left[\frac{dr}{di}Fe^{-r\tau}N(d_1) + Fe^{-i\tau}N(-d_2)\right] \leq\geq 0.$$

Q.E.D.

Proof of Proposition 3. Similar to the differentiation for Proposition 2, we get:

$$\frac{dB_{TIPS,t}}{dr} = -\tau\left[\frac{di}{dr}Fe^{-i\tau}N(-d_2) + Fe^{-r\tau}N(d_1)\right] \leq\geq 0.$$

Q.E.D.

Proof of Proposition 4. The expected inflation rate is given by: $\pi^* = i - r$. Differentiating the value of the inflation-protected security with respect to π^*, we get:

$$\frac{dB_{TIPS,t}}{d\pi^*} = -\tau\frac{di}{d\pi^*}Fe^{-i\tau}N(-d_2) - Fe^{-i\tau}\frac{\partial N(d_2)}{\partial\pi^*}$$
$$-\tau\frac{dr}{d\pi^*}Fe^{-r\tau}N(d_1) + Fe^{-r\tau}\frac{\partial N(d_1)}{\partial\pi^*}$$
$$= -\tau\frac{di}{d\pi^*}Fe^{-i\tau}N(-d_2) - Fe^{-i\tau}N'(d_2)\frac{\partial d_2}{\partial\pi^*}$$
$$-\tau\frac{dr}{d\pi^*}Fe^{-r\tau}N(d_1) + Fe^{-r\tau}N'(d_1)\frac{\partial d_1}{\partial\pi^*}$$
$$= -\tau\frac{di}{d\pi^*}Fe^{-i\tau}N(-d_2) - Fe^{-i\tau}N'(d_1)e^{(i-r)\tau}\frac{\partial d_1}{\partial\pi^*}$$
$$-\tau\frac{dr}{d\pi^*}Fe^{-r\tau}N(d_1) + Fe^{-r\tau}N'(d_1)\frac{\partial d_1}{\partial\pi^*}$$
$$= -\tau\left[\frac{di}{d\pi^*}Fe^{-i\tau}N(-d_2) + \frac{dr}{d\pi^*}Fe^{-r\tau}N(d_1)\right] \leq\geq 0$$

Q.E.D.

Proof of Proposition 5. Differentiating the value of the inflation-protected security with respect to τ, we get:

$$\frac{dB_{TIPS,t}}{d\tau} = -iFe^{-i\tau} - rFe^{-r\tau}N(d_1) + Fe^{-r\tau}\frac{\partial N(d_1)}{\partial\tau}$$
$$+iFe^{-i\tau}N(d_2) - Fe^{-i\tau}\frac{\partial N(d_2)}{\partial\tau}$$
$$= -iFe^{-i\tau} - rFe^{-r\tau}N(d_1) + Fe^{-r\tau}N'(d_1)\frac{\partial d_1}{\partial\tau}$$
$$+iFe^{-i\tau}N(d_2) - Fe^{-i\tau}N'(d_2)\frac{\partial d_2}{\partial\tau}$$
$$= -iFe^{-i\tau} - rFe^{-r\tau}N(d_1) + Fe^{-r\tau}N'(d_1)\frac{\partial d_1}{\partial\tau}$$
$$+iFe^{-i\tau}N(d_2) - Fe^{-i\tau}N'(d_1)e^{(i-r)\tau}\frac{\partial d_2}{\partial\tau}$$
$$= -iFe^{-i\tau} - rFe^{-r\tau}N(d_1) + iFe^{-i\tau}N(d_2)$$
$$+Fe^{-r\tau}N'(d_1)\left[\frac{\partial d_1}{\partial\tau} - \frac{\partial d_2}{\partial\tau}\right]$$
$$= Fe^{-r\tau}N'(d_1)\left[\frac{\partial d_1}{\partial\tau} - \frac{\partial d_2}{\partial\tau}\right] - rFe^{-r\tau}N(d_1)$$

Q.E.D.

APPENDIX C

Derivations of the Duration Measures

The Nominal Duration. Applying the standard price-elasticity definition of duration to bond-pricing Equation (1), we get:

$$D_i = -\frac{1}{B_{TIPS,t}}\frac{\partial B_{TIPS,t}}{\partial i}$$
$$= \frac{1}{B_{TIPS,t}}\tau\left[\frac{dr}{di}Fe^{-r\tau}N(d_1) + Fe^{-i\tau}N(-d_2)\right]$$
$$= \frac{1}{B_{TIPS,t}}\tau\left[\left(1-\frac{d\pi^*}{di}\right)Fe^{-r\tau}N(d_1) + Fe^{-i\tau}N(-d_2)\right]$$
$$= \frac{\tau B_{TIPS,t} - \tau\frac{\partial\pi^*}{\partial i}Fe^{-r\tau}N(d_1)}{B_{TIPS,t}}$$
$$= \tau\left(1 - \frac{\partial\pi^*}{\partial i}\frac{Fe^{-r\tau}N(d_1)}{B_{TIPS,t}}\right).$$

The Real Duration. Applying the standard price-elasticity definition of duration to bond-pricing Equation (1), we get:

$$
\begin{aligned}
D_r &= -\frac{1}{B_{TIPS,t}}\frac{\partial B_{TIPS,t}}{\partial r} \\
&= \frac{1}{B_{TIPS,t}}\tau\left[Fe^{-r\tau}N(d_1)+\frac{di}{dr}Fe^{-i\tau}N(-d_2)\right] \\
&= \frac{1}{B_{TIPS,t}}\tau\left[Fe^{-r\tau}N(d_1)+\left(1+\frac{d\pi^*}{dr}\right)Fe^{-i\tau}N(-d_2)\right] \\
&= \frac{\tau B_{TIPS,t}+\tau\frac{\partial\pi^*}{\partial r}Fe^{-r\tau}N(-d_2)}{B_{TIPS,t}} \\
&= \tau\left(1+\frac{\partial\pi^*}{\partial r}\frac{Fe^{-r\tau}N(-d_2)}{B_{TIPS,t}}\right)
\end{aligned}
$$

ENDNOTES

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[1]Other countries issuing inflation-linked bonds include the U.K., Israel, Sweden, Canada, Australia, and New Zealand. Wilcox [1998] reports that as of mid-1997, the U.K. issued the highest aggregate volume of inflation-indexed debt (US$71.1 billion). Israel ranked second (US$27.9 billion), then the U.S. (US$15.0 billion), Sweden (US$5.7 billion), Canada (US$4.3 billion), Australia (US$2.7 billion), and New Zealand (US$0.1 billion).

[2]Note that TIPS are adjusted for the CPI with a two-to three-month lag. Wilcox [1998] argues that TIPS bondholders still face a small but certain amount of inflation risk due to this lag. Because this exposure is trivial, in this article we assume that the inflation adjustment is contemporaneous.

[3]Another example is Israel, which ranked second in terms of aggregate volume of inflation-indexed debt (US$27.9 billion), where a prolonged recession prevailed in the beginning of the new millennium.

[4]Note that the protection against deflation provided by coupon-bearing TIPS applies only to the face value paid at maturity, not to the coupon payments. Thus, one cannot price coupon-bearing TIPS by individually pricing strips and aggregating these prices to obtain the value of the inflation-protected coupon bond. In our analysis we price pure-discount TIPS bonds. To price the coupons, one can strip the bond and treat each coupon as pure-discount real bond.

[5]This assumption does not preclude the inflation rate from being stochastic. Recall that under the continuous-time version of the Fisher relation, we have: $i = r + E[\pi]$, where r is the continuously compounded real riskless rate and i is the continuously compounded nominal riskless rate. Therefore, even if the nominal and real rates are deterministic, the difference between them is the expected value of the future inflation rate (which is still stochastic).

[6]This assumption is often criticized because it does not allow for the pull-to-par phenomenon. The geometric Brownian motion process implies a constant volatility of the underlying. However the bond price must converge to par at maturity, and therefore, the price volatility must change over time. In our case, the pull-to-par effect does not apply to the real bond price because at maturity a real bond pays the par value adjusted for stochastic inflation.

[7]See derivation in Appendix A.

[8]Our volatility estimates of daily returns on existing TIPS bonds are consistent with the estimates obtained by Roll [2004]. He uses slightly different samples of daily TIPS bond return data but obtains approximately equivalent annualized standard deviations in the range of 1.08% to 10.70%. Note that, optimally, one needs to use real bonds (unprotected for deflation) rather than TIPS bonds. Unfortunately, only TIPS are issued by the U.S. Treasury. Thus, we use the TIPS volatility to estimate the volatility on real bond returns.

[9]See Appendix C for the derivations of both duration measures.

[10]Garman [1985] shows that an option's elasticity is a measure of its interest-rate sensitivity. Similar to our paper, Chance [1990] uses this approach to model the duration of the limited-liability option imbedded in a corporate bond.

[11]We do not use the spline methodology or the Nelson and Siegel [1987] procedure to estimate the term structures for nominal and TIPS bonds because of the low number of data points for individual TIPS.

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