Modeling Simultaneous Defaults: *A Top-Down Approach*

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Recent problems in America's subprime mortgage market have sparked a lively discussion on the pros and cons of credit derivatives. The main focus is on multi-name contracts such as collateralized debt obligations (CDOs) or CDO-squared contracts (a type of CDO where the underlying portfolio includes tranches of other CDOs). When evaluating these contracts, the crucial point is to capture the dependencies within the portfolio. However, this poses a big challenge to our existing pricing models.

Reduced-form models introduced by Lando [1998] and Duffie and Singleton [1999] have become popular tools for pricing credit derivatives. These models are adapted to some economic state variables so that they can relate correlation of credit spreads and ratings downgrades to the business cycle. However, conditioned on the state variables, defaults become independent events within standard doubly stochastic models. In particular, because of their inherent structure, these approaches are unable to explain clustering in defaults. As a consequence, they produce default correlations that are too small compared to empirical observations.[1] In fact, they would never predict events such as the subprime crisis in fixed income markets and therefore effectively underprice the risks in credit derivative contracts. To overcome this problem, Jarrow/Yu [2001] allow the intensity to depend on the state of other firms in the economy. Specifically, for some firm i they assume a piecewise constant intensity process with a jump at the default time of another firm j.[2] This approach captures default contagion but loses manageability if the number of firms increases. In particular, the valuation of defaultable claims within such a contagion model becomes quite complicated as the formulas do not only depend on the macroeconomic variables used but also on the current and possible future default status of the firms considered.

Instead of modelling causal relationships between firms directly, recent approaches put forward by Collin-Dufresne/Goldstein/Helwege [2003] and Schönbucher [2004] assume that markets dynamically learn from defaults. Building on unobserved frailty variables, new information is revealed through defaults and the market updates its prior distribution.

In this article we propose a new approach to modelling default dependencies in the context of reduced-form models. The main idea of our model is to incorporate contagion effects in addition to the dependence on common state variables in such a way that the model remains in a doubly stochastic setting. This allows us to derive valuation formulas for defaultable claims along the lines of Lando [1998] and therefore simplifies valuation in the presence of contagion effects considerably. Our approach differs by focusing on portfolios of firms instead of starting with a description of a default process for each firm individually.

By allowing two or more firms to default simultaneously the model captures default clusterings and more realistic levels of default dependencies compared to standard doubly stochastic approaches. To this end, we change the counting process to set-valued entities. The default risk is modeled through an economy-wide intensity process driven by macroeconomic factors. This leads to the default times in our economy but not to the identity of the defaulting entities. Therefore, we apply the concept of random-thinning to break down the economy-wide default risk to subportfolio or single-name level. By multiplying our economy-wide intensity process with thinning probabilities, i.e., a function that takes values in the interval [0,1], we obtain consistent default intensities for all entities. In order to reach an economic interpretation, we characterize the thinning probabilities explicitly using structural firm information. Consequently, a key feature is that both economy-wide default risk and correlations between firm's asset values induce default dependencies. This is consistent with the idea of markets learning from defaults as well as direct causal relationships between firms. As a result, we obtain a flexible tool for modelling default dependencies that combines the mathematical attractiveness of reduced-form models and the economic appeal of structural models.

Our starting point is an economy-wide doubly stochastic point process with the intensity for the default of at least one firm within the economy. Following a top-down approach, we next characterize the default intensity of sets and individual firms by applying random thinning. At this stage structural firm information and correlations between firms' asset values are incorporated. After having specified the default model we apply our framework to the valuation of defaultable claims. Then we explicitly derive valuation formulas for the spreads on a single-name credit default swap (CDS) and a first-to-default swap. The impact of a variation of the main dependency drivers on the selected dependence measure as well as on the fair spreads of credit derivatives are illustrated and compared to the selected benchmark models. We conclude with some final remarks and ideas for further research.

A DEFAULT MECHANISM COVERING DEFAULT DEPENDENCIES

Economic Framework

Consider an economy that consists of n firms, whose default status is represented by the vector-valued counting process $N = (N^{(1)}, \ldots, N^{(n)})$ with $N^{(i)}(t) = \Sigma_{l=1}^{\infty} \mathbf{1}_{\{\tau_l^{(i)} \leq t\}}$. The random variable $\tau_l^{(i)}$ indicates the l^{th} default time of company i. This implies that a firm can default more than once as it is the case after restructuring.[3] Alternatively one could think of replacing the defaulted firm with an otherwise identical non-defaulted firm. In any case, from an economic point of view, this ensures that the universe of firms is kept fixed in the economy instead of reducing the overall number of firms more and more after each default.[4]

Motivated by the work of Davis/Lo [2001] who model a direct default of firm j after the default of firm i with some probability q, we explicitly allow for common jumps in the counting process N. Since this is not in line with the assumption of a Poisson process for the individual events,[5] we focus on sets of firms instead of single companies. Defaults of firms that are directly caused by the default of another firm are interpreted as one common event. Therefore it is useful to transform the counting process into an equivalent one based on sets, which is denoted as $\tilde{N} = (\tilde{N}^{\{1\}}, \ldots, \tilde{N}^{\{1,\ldots,n\}})$. This process has now $2^n - 1$ components which represent possible default events. At default only one of these set-based counting processes jumps by one. In this sense these events are called independent.

The uncertainty in this economy is described by the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$, where Ω represents the set of all possible states, $\mathcal{F} = \mathcal{F}_T$ is the σ-algebra on Ω, and $\mathbb{P}$ is the actual probability measure on $\mathcal{F}$. Assuming arbitrage-free markets, an equivalent martingale measure $\mathbb{Q}$ exists.[6] Our valuation formulas for defaultable claims rely on this valuation measure.

The model is based on an economy-wide state vector X_t with macroeconomic and possibly firm-specific variables generating the information filtration $\mathcal{F}_t^X = \sigma(X_s : 0 \leq s \leq t)$. Default information concerning the n firms is captured by $\mathcal{H}_t = \vee_{i=1}^{n} \mathcal{H}_t^i$ with $\mathcal{H}_t^i = \sigma(N_s^{(i)} : 0 \leq s \leq t)$. The combined filtration $\mathcal{F}$ is the smallest sigma field that contains $\mathcal{F}^X$ and $\mathcal{H}$, i.e., $\mathcal{F}_t = \mathcal{F}_t^X \vee \mathcal{H}_t$.

Based on this economy, a general valuation framework can be developed in three steps. First, the economy-wide intensity has to be specified. In a second step, we have to split up this economy-wide default risk to the single entity or subportfolio level in order to get a representation for the default risk of a single firm. For this purpose we apply random thinning on the economy-wide intensity. In contrast to Giesecke/Goldberg [2007] and Errais/Giesecke/Goldberg [2007] we give an explicit construction for this thinning process based on one year default probabilities which we derive from a Merton [1974] framework. Finally, using the conditional independence of the event times of $\tilde{N}$ with respect to $\mathcal{F}_t^X$, valuation formulas along the lines of Lando [1998] can be derived.

Top-Down Approach

As a starting point of our top-down approach we model the sequence of economy-wide default times $(\tau_j)_{j\in\mathbb{N}}$ as jump times of a doubly stochastic Poisson process with intensity process $\bar{\lambda} = (\bar{\lambda}_t)_{0\le t\le T}$. The $\mathcal{F}^X$-adapted[7] process $\bar{\lambda} = (\bar{\lambda}_t)_{0\le t\le T}$ describing the economy-wide default intensity that at least one firm defaults corresponds to a jump in one component of $\tilde{N}$ or equivalently to a jump in at least one component of N.[8] Given a default time τ_j based on the economy-wide intensity process $\bar{\lambda}$, it is now necessary to identify the corresponding set of defaulted firms. In general, there exist $2^n - 1$ possible combinations of firms that could default. These sets form the "mark space" $\mathbb{X} = \mathcal{P}(\{1, \ldots, n\})\backslash ;$. At each default time τ_j we now draw one element $A_j \in \mathbb{X}$ from this mark space. Together with the default times, they form a marked Poisson process $\Phi = (\tau_j, A_j)_{j\in\mathbb{N}}$. Associated with each $A \in \mathbb{X}$ there is a counting process $\tilde{N}^A$ that is zero until the set A is drawn, then jumps to one and stays there until the next default of this set. By summing over all possible $\tilde{N}^A$ we get the number of default times in the economy. Therefore $\tilde{N}_t^\Sigma := \Sigma_{A\in\mathbb{X}}\tilde{N}_t^A$ is the counting process associated with default times. Consequently, this process obeys the same doubly stochastic Poisson process that governs the event times in the economy. Furthermore, it holds that $\tilde{N}_t^\Sigma - \int_0^t \bar{\lambda}_s ds$ is a $\mathcal{F}_t^X$-martingale. We can interpret this intensity $\bar{\lambda}$ as the systematic default arrival risk of a given economy. Based on this economy-wide process we next derive a relationship between the intensities of the components of $\tilde{N}$ and the economy-wide default intensity.

Suppose for all $A \in \mathbb{X}$ exists a $\mathcal{F}_t^X$-measurable intensity λ_t^A which is integrable, i.e., $\int_0^t \lambda_s^A ds < \infty$, $\mathbb{P}$ – *a.s.*, and for which $\tilde{N}^A - \int_0^t \lambda_s^A ds$ is a martingale. Then, the components $\tilde{N}^A$ of $\tilde{N}$ are independent $\mathcal{F}_t^X$-adapted doubly-stochastic Poisson processes[9] with intensities λ_t^A for all $A \in \mathbb{X}$.

In order to derive a relationship between the intensities λ^A and the economy wide intensity $\bar{\lambda}$, a theorem from Brémaud [1981] is helpful. Since every component $\tilde{N}^A$ has an intensity λ_t^A, then for all $A \in \mathbb{X}$ it holds that

$$\frac{\lambda_t^A}{\bar{\lambda}_t} = \mathbb{P}\left(A_j = A \middle| \mathcal{F}_{t-}^X, d\tilde{N}_t^\Sigma = 1\right) =: Z_t^A \tag{1}$$

As $\bar{\lambda}_t$ is the intensity of $\tilde{N}_t^\Sigma$, it also follows that $\bar{\lambda}_t = \Sigma_{A\in\mathbb{X}}\lambda_t^A$.

Roughly speaking, $\lambda_t^A/\bar{\lambda}_t$ in Equation (1) is the probability of having a jump in $\tilde{N}_t^A$ at time t given $\mathcal{F}_{t-}^X$ and given that $\tilde{N}_t^\Sigma$ has a jump of one[10] at time t.

This leads to a characterization of λ_t^A:

$$\lambda_t^A = Z_t^A \cdot \bar{\lambda}_t \tag{2}$$

The probability $\mathbb{P}(A_j = A \mid \mathcal{F}_{t-}^X, d\tilde{N}_t^\Sigma = 1) = Z_t^A$ defines the random thinning process for the component A. Thus, we can interpret Z_t^A as the conditional probability that the set of firms A defaults next given $\mathcal{F}_{t-}^X$ and there is a default in $\tilde{N}$ in the next time increment. Note that the $\mathcal{F}^X$-measurability of the economy-wide intensity is not sufficient to ensure the $\mathcal{F}^X$-measurability of λ^A. In addition, the thinning process must be $\mathcal{F}^X$-measurable. This issue is addressed in the next section where we focus on the question how to express the thinning probabilities Z_t^A using structural firm information.

SPECIFICATION OF THE THINNING PROCESS

The concept of random thinning is used to construct the intensity of a single default set A from the economy-wide default intensity. Formally, a thinning process for the economy-wide intensity $\bar{\lambda}$ with respect to a stopping default time τ_j is an adapted process $Z = (Z^{\{1\}}, \ldots, Z^{\{1,\ldots,n\}})$ for which the Stieltjes integral

$$\int_0^t Z_s^A \bar{\lambda}_s ds$$

is the cumulated intensity up to time t to the default time τ^A.[11] It can be easily shown that the thinning process Z^A satisfies the following properties almost surely:

a) $Z^A \in [0, 1] \quad (A \in \mathbb{X})$,
b) $\Sigma_{A\in\mathbb{X}} Z^A = 1$.

These properties can be interpreted in the following way. First, the values of the thinning process are bounded within the interval [0, 1]. This property ensures that the intensity λ^A remains positive and that no set carries more default risk than the economy as a whole.

Second, the entire economy-wide default risk has to be divided across the possible sets. As these properties agree with those of an ordinary probability measure, we refer to Z^A also as thinning probabilities. Usually,[12] a third property is to set the Z^A to zero where A defaults. This restriction is not necessary in our setup since multiple defaults are allowed. Technically, this ensures that the Z^A can be kept $\mathcal{F}^X$-adapted, and therefore do not depend on the default status of the n firms.

The central question is now how to specify this thinning process. As this process only has to admit to properties (a) and (b), there are several degrees of freedom remaining. In what follows we suggest connecting structural firm information, especially the dynamics of the firm's asset values, the structure of liabilities, and the asset correlations to the thinning probabilities. Using a simple structural model like the Merton [1974] model, we can calculate how likely a certain combination of firms ends up in a future state with debt values being higher than asset values (insolvency). This fits our idea of firms closer to insolvency being more likely to experience a sudden default event. With these thinning probabilities we can reasonably split up the economy-wide default intensity to the single entity level. Thus, the thinning process Z, which in general varies with time and state, is partitioning the economy-wide default intensity among all possible set-valued events with no default mass being lost.

Thinning Based on Structural Firm Information

To specify the thinning probabilities $Z^A(A \in \mathbb{X})$ based on structural firm information, we introduce the components of a structural model: Assume that the value of the total assets of each company follows an $\mathcal{F}^X$-adapted stochastic process $V^{(i)} = (V_t^{(i)})_{t \geq 0}(i = 1, \ldots, n)$ and that the asset value processes are correlated with each other. The matrix $\Sigma = (\varrho_{ij})_{i,j=1,\ldots,n}$ captures these asset correlations. As $L^{(i)}$ we denote the value of firm i's outstanding liabilities. Based on these specifications the actual likelihood for a set of firms to be insolvent within some given horizon T from now is

$$\hat{p}^A(t) = \mathbb{P}(V_{t+T}^{(i)} \leq L^{(i)}\ \forall i \in A,\ V_{t+T}^{(j)} > L^{(j)}\ \forall j \in A^c \mid \mathcal{F}_{t-}^X)$$

where $A^c = \{1, \ldots, n\} \backslash A$ is the complement of the default set A. To reach a partition, we standardize $\hat{p}^A(t)$ with $\sum_{A\in\mathbb{X}} \hat{p}^A(t)$, which yields to the desired thinning Z_t^A for set A. Obviously, $Z_t^A = \hat{p}^A(t)/\sum_{A\in\mathbb{X}} \hat{p}^A(t)$ fulfills properties (a) and (b) by construction. By applying Equation (2) we get the desired default intensity λ_t^A of set A. Note that this procedure holds for arbitrary dynamics of $V_t^{(i)}$ and $L^{(i)}$. In order to have a closed-form solution, adequate dynamics have to be specified.

Using a generalized Merton [1974] model, we are able to derive a closed-form solution. The dynamics of the asset value $V^{(i)}$ follow a geometric Brownian motion:

$$dV_t^{(i)} = \mu_i V_t^{(i)} dt + \sigma_i V_t^{(i)} dW_t^{(i)} \quad (i = 1, \ldots, n) \tag{3}$$

with constant drift and volatility parameters μ_i and σ_i. The liabilities are assumed to be constant. This leads to an extended version of the classical Merton framework that allows for correlation among the firm's asset value processes. In the next step, we compute the probability that the firms are insolvent at the fixed horizon T given the current value $V_t^{(i)}$. Because $V_{t+T}^{(i)}$ is log-normally distributed with

$$V_{t+T}^{(i)} = V_t^{(i)} \exp\left(\left(\mu_i - \frac{1}{2}\sigma_i^2\right)T + \sigma_i W_T^{(i)}\right) \tag{4}$$

it follows

$$\left\{V_{t+T}^{(i)} < L^{(i)}\right\} \Leftrightarrow \left\{U_t^{(i)} < -\underbrace{\frac{\ln(V_t^{(i)}/L^{(i)}) + (\mu_i - \frac{1}{2}\sigma_i^2)T}{\sigma_i\sqrt{T}}}_{d_i}\right\}$$

Note that the $U_t^{(i)}$ are standard normal random variables whose correlation is ρ_{ij}. Therefore, the probability $\hat{p}^A(t) = \Phi_n(-d_1, \ldots, -d_n; \Sigma)$ is the value of an n-dimensional normal distribution with correlation matrix Σ and integration limits $-d_i$. For the companies in the default set A we integrate from $-\infty$ to $-d_i$ and for companies that are not in this set A from d_i to ∞. Having calculated $\hat{p}^A(t)$, Z_t^A results as shown above. So, we obtain a thinning process that reflects how likely it is that the set A of firms defaults, given at least one sudden default event in the economy.

VALUATION FORMULA FOR DEFAULTABLE CLAIMS

Having specified our default mechanism, we next derive valuation formulas for single-name credit default swaps and first-to-default swaps within the proposed top-down approach. All the expectations are under the risk-neutral measure $\mathbb{Q}$. The change from the physical measure to the risk-neutral is analogous to that of Jarrow/Lando/Yu [2005]. Once the economy-wide default intensity $\bar{\lambda}$ is given under the measure $\mathbb{Q}$, we easily obtain the thinned process λ^A under $\mathbb{Q}$ using the same thinning process Z^A as before.[13]

Besides default risk a valuation model requires a model of the default-free term structure of interest rates. The term structure is defined by a spot-rate process $(r_t)_{0 \leq t \leq T}$ which is an $\mathcal{F}^X$-adapted stochastic process. The value of a default-free zero-coupon bond at time t with maturity T is:

$$b(r, t; T) = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T r(X_s, s)ds\right)\Big|\mathcal{F}_t^X\right]$$

For ease of notation, we simply write r_s instead of $r(X_s, s)$.

Our default model works on sets and not on the default status of the individual company. At first sight, this seems to be a problem because defaultable products in the market typically refer to a company and not to a set. Consider the event $\{\tau^{(i)} > T\}$, i.e., the outcome that company i does not default by T. This happens until T if and only if the sets that contain the company i have not suffered a default, i.e., $\tau^{(i)} = \Lambda_{A \in \mathbb{X}, i \in A} \tau^A$. Since the components of $\tilde{N}$ are conditionally independent, the intensity of default time of company i $\tau^{(i)}$ is then simply[14] $\Sigma_{A \in \mathbb{X}} \lambda_t^A \cdot \mathbf{1}_{\{i \in A\}}$. Note, this is an endogenous characterization within our framework of the so far unknown default intensity $\lambda_t^{(i)}$ which, besides the intensity that only company i defaults, consists of further summands representing the increase of the default risk due to possible contagion effects. Turning to pricing of defaultable claims, it is obvious that prices of the multi-name defaultable claims like n-th-to-default swaps depend on the underlying dependence structure. This is illustrated for a first-to-default swap. In what follows, even single-name CDS are affected by the dependence structure.

To derive expressions for single-name and multi-name credit default swaps, suppose that every firm in the economy has a zero-coupon bond with maturity T. A single-name CDS only protects against the default of firm i whereas a first-to-default swap refers to a basket containing all the outstanding zero-coupon bonds in the economy. In both cases the protection buyer pays a constant, continuous premium w to the protection seller up to maturity or default. The protection seller compensates the buyer in the case of default for his suffered losses. Here this payment is specified as $(1 - \delta)b(\cdot, T)$, where δ is the recovery rate and $b(\cdot, T)$ the value of an identical default-free zero-coupon bond at default. For the single name CDS the payments of the protection buyer have the value

$$\begin{aligned} \upsilon_B &= \mathbb{E}^{\mathbb{Q}}\left[\int_t^T \exp\left(-\int_t^s r_u du\right) w^{cds} \cdot \mathbf{1}_{\{\tau(i) > s\}} ds \Big| \mathcal{F}_t\right] \\ &= w^{cds}\mathbb{E}^{\mathbb{Q}}\left[\int_t^T \exp\left(-\int_t^s \left(r_u + \sum_{A \in \mathbb{X}, i \in A} \lambda_u^A\right) du\right) ds \Big| \mathcal{F}_t^X\right] \end{aligned}$$

and for the protection seller

$$\begin{aligned} \upsilon_s &= \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^{\tau^{(i)}} r_u du\right)(1-\delta) b(\tau^{(i)}, T) \cdot \mathbf{1}_{\{\tau^{(i)} \leq T\}} \Big| \mathcal{F}_t\right] \\ &= (1-\delta)\left(b(t,T) - \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T \left(r_u + \sum_{A \in \mathbb{X}, i \in A} \lambda_u^A\right) du\right) \Big| \mathcal{F}_t^X\right]\right) \end{aligned}$$

The detailed derivation can be found in Kunisch/Uhrig-Homburg [2007].

Since both cash-flow streams should have the same value for a fair CDS, we can solve for the fair CDS premium w^{cds}:

$$w^{cds} = \frac{(1-\delta)\left(b(t,T) - \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T \left(r_u + \sum_{A \in \mathbb{X}, i \in A} \lambda_u^A\right) du\right) \Big| \mathcal{F}_t^X\right]\right)}{\mathbb{E}^{\mathbb{Q}}\left[\int_t^T \exp\left(-\int_t^s \left(r_u + \sum_{A \in \mathbb{X}, i \in A} \lambda_u^A\right) du\right) ds \Big| \mathcal{F}_t^X\right]} \quad (5)$$

As we can see, the CDS premium for company i depends on the individual default intensity $\lambda^{(i)}$ and therefore on the thinning process. Consequently, one change in a single component of the thinning process Z will lead to a change in the premium of all outstanding CDS contracts due to the necessary standardization.

In the first-to-default contract the specification of the default event changes. Now the first default that ever occurs in the economy triggers the payment of the contract. This default time, simply denoted with τ, has the intensity $\bar{\lambda}$ since it is the minimum of all jump times of $\tilde{N}$.

Analogous to the single-name CDS we get an expression for the values of the cash-flow stream to the protection buyer[15]

$$\upsilon_B = \mathbb{E}^{\mathbb{Q}}\left[\int_t^T \exp\left(-\int_t^s r_u du\right) w^{ftd} \cdot \mathbf{1}_{\{\tau>s\}} ds \middle| \mathcal{F}_t\right]$$
$$= w^{ftd} \cdot \mathbb{E}^{\mathbb{Q}}\left[\int_t^T \exp\left(-\int_t^s \left(r_u + \bar{\lambda}_u\right) du\right) ds \middle| \mathcal{F}_t^X\right]$$

and to the protection seller

$$\upsilon_S = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^\tau r_u du\right)(1-\delta) b(\tau,T) \cdot \mathbf{1}_{\{\tau \le T\}} \middle| \mathcal{F}_t\right]$$
$$= (1-\delta)\left(b(t,T) - \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T \left(r_u + \bar{\lambda}_u\right) du\right) \middle| \mathcal{F}_t^X\right]\right)$$

Again the identity for the values of both streams leads to the fair premium:

$$w^{ftd} = \frac{(1-\delta)\left(b(t,T) - \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T \left(r_u + \bar{\lambda}_u\right) du\right) \middle| \mathcal{F}_t^X\right]\right)}{\mathbb{E}^{\mathbb{Q}}\left[\int_t^T \exp\left(-\int_t^s \left(r_u + \bar{\lambda}_u\right) du\right) ds \middle| \mathcal{F}_t^X\right]} \tag{6}$$

Comparing Equation (6) with Equation (5) we see that Equation (6) does not need the thinning process. This fact could be useful in an empirical estimation of $\bar{\lambda}$ as it only depends in this framework on the short rate process r and $\bar{\lambda}$. That is, $\bar{\lambda}$ can be estimated from quotations of first-to-default contracts on an index such as the iTraxx.

COMPARISON OF DIFFERENT APPROACHES: A SIMULATION STUDY

In order to provide insights into the dependence structure induced by the proposed top-down approach, we compare the results from the top-down approach to several benchmark models. We compare an extended Merton [1974] model with correlated asset value processes and an extension of the Lando [1998] model with correlated default intensities. Note that our top-down model combines elements of both benchmark models in one approach: the intensity process as economy-wide default risk and the extended Merton model for the thinning values. In this sense our comparison contrasts the combined top-down approach with its individual components.

While this comparison illustrates how the combination of these components performs in contrast to the individual ones, it should not be understood as a comparison to the most sophisticated alternatives, such as models with jumps (e.g., Zhou [2001]).

We compare the top-down approach via simulation to these two benchmarks in two different ways: first, we analyze the differences between the models with respect to the default correlation and the prices of credit derivatives such as first-to-default swaps and single-name CDS. Also, we investigate what drives the dependence structure in the models. To this end, we perform a comparative static analysis focusing on the asset correlation and the intensity level as the most important parameters of the thinning and the economy-wide default process.

Before presenting the simulation study and the comparison in more detail, the next section briefly describes the two benchmark models chosen and also specifies the processes used within the top-down approach. To simplify matters, we only consider a setting with two firms.

Description of the Benchmark Models and Specification of Our Top-Down Approach

In the following we will specify the dynamics of the processes in our approach and will give a brief description of the benchmark models. In order to use the models for valuation of defaultable claims, we specify them under the risk neutral measure $\mathbb{Q}$. Consequently, we also compute the default correlation under this measure.

Generalized Merton model. Consider two firms with asset values $V^{(i)} (i = 1, 2)$ following a geometric Brownian motion each under the risk-neutral measure:

$$dV_t^{(i)} = rV_t^{(i)} dt + \sigma_i V_t^{(i)} dW^{(i)} \quad (i = 1, 2) \tag{7}$$

$W^{(i)}$ is a standard Brownian motion. r is the instantaneous short-term interest rate, which is assumed to be constant, and σ_i is the constant proportional volatility of the return on firm i's asset value. Moreover, the Brownian motions are correlated with ϱ_{12}^W. Default occurs when the asset value process is below the respective values of liabilities L_i $(i = 1, 2)$ at the end of the horizon T. Given this default definition and the dynamics of the asset values in Equation (7), the individual default probabilities $p^{(i)}$

($i = 1, 2$) as well as the joint probability $p^{(1,2)}$ that both companies default at time T are easily obtained:

$$p^{(i)} = \Phi\left(-\frac{\ln(V_0^{(i)} / L^{(i)}) + (r - \frac{1}{2}\sigma_i^2)\cdot T}{\sigma_i\sqrt{T}}\right)$$
$$= \Phi(-d_i) \quad (i = 1, 2) \tag{8}$$

$$p^{(1,2)} = \Phi_2\left(-d_1, -d_2;\, \varrho_{12}^W\right) \tag{9}$$

where $\Phi_2(-d_1, -d_2; \varrho_{12}^W)$ denotes the distribution function of the bivariate standard normal distribution with correlation ϱ_{12}^W at $(-d_1, -d_2)$. Then we get as default correlation

$$\rho_{12} = \frac{\text{Cov}(\mathbf{1}_{\{\tau^{(1)}\le T\}}, \mathbf{1}_{\{\tau^{(2)}\le T\}})}{\sqrt{\text{Var}(\mathbf{1}_{\{\tau^{(1)}\le T\}})}\cdot\sqrt{\text{Var}(\mathbf{1}_{\{\tau^{(2)}\le T\}})}}$$
$$= \frac{p^{(1,2)} - p^{(1)}\cdot p^{(2)}}{\sqrt{p^{(1)}(1-p^{(1)})}\sqrt{p^{(2)}(1-p^{(2)})}} \tag{10}$$

In order to compare the approaches on a pricing level, we also need pricing formulas similar to those for the benchmark models. In the Merton model the interest rate is constant and therefore interest rate risk and default risk are independent. Moreover, default can only occur at T. So, the values of the payment stream of protection buyer v_B and protection seller v_S result in:

$$v_B^{(i)} = w_i^{cds}\cdot\frac{1}{r}(e^{-rt} - e^{-rT})$$
$$v_S^{(i)} = (1-\delta)e^{-r(T-t)}\left(1 - \mathbb{Q}\left(V_T^{(i)} \ge L^{(i)}\middle|\mathcal{F}_t^X\right)\right)$$

The identity of both streams leads to fair premium of a CDS in the Merton case:

$$w_i^{cds} = \frac{(1-\delta)e^{-r(T-t)}\left(1 - \mathbb{Q}\left(V_T^{(i)} \ge L^{(i)}\middle|\mathcal{F}_t^X\right)\right)}{\frac{1}{r}(e^{-rt} - e^{-rT})} \tag{11}$$

Analogously to this, we get the following formula for the first-to-default swap:

$$w_{Merton}^{ftd} = \frac{(1-\delta)e^{-r(T-t)}\left(\begin{array}{l}\sum_{i=1}^{2}\mathbb{Q}\left(V_T^{(i)} < L^{(i)}\middle|\mathcal{F}_t^X\right)\\ \quad - \mathbb{Q}\left(V_T^{(i)} < L^{(i)}(i=1,2)\middle|\mathcal{F}_t^X\right)\end{array}\right)}{\frac{1}{r}(e^{-rt} - e^{-rT})} \tag{12}$$

Generalized Lando model. As a second benchmark, we consider a simple reduced-form model with correlated intensities. This can be considered as an extension of the classical Lando [1998] model. We assume that each individual intensity $\lambda^{(i)}$, $i = 1, 2$ follows a mean-reverting square-root process

$$d\lambda_t^{(i)} = \kappa_i(\theta_i - \lambda_t^{(i)})dt + \eta_i\sqrt{\lambda_t^{(i)}}dB_t^{(i)}$$

under the risk-neutral probability measure $\mathbb{Q}$. κ_i, θ_i, and η_i, $i = 1, 2$ are constant parameters and $B^{(i)}$, $i = 1, 2$ are two correlated standard Brownian motions with correlation $\varrho_{12}^B \ge 0$. This ensures positive intensity values. Default then occurs at $\tau^{(i)} = \inf\{t \ge 0 : \int_0^t \lambda_s^{(i)}ds \ge E_i\}$, where $E_i \sim \exp(1)$ iid. Within this setting, individual default probabilities $p^{(i)}$, $i = 1, 2$ and the joint probability $p^{(1,2)}$ result as

$$p^{(i)} = 1 - \mathbb{E}\left[\exp\left(-\int_0^T \lambda_s^{(i)}ds\right)\right] \tag{13}$$

$$p^{(1,2)} = \mathbb{E}\left[\mathbf{1}_{\{\tau^{(1)}\le T\}}\cdot\mathbf{1}_{\{\tau^{(2)}\le T\}}\right]$$
$$= p^{(1)} + p^{(2)} + \mathbb{E}\left[\exp\left(-\int_0^T\left(\lambda_s^{(1)} + \lambda_s^{(2)}\right)ds\right)\right] - 1 \tag{14}$$

Using Equation (10) the required default correlations are obtained. The necessary valuation formulas can be derived using the original three building blocks from Lando [1998]. The formulas will be the same as our Equations (5) and (6) if one replaces the intensities through $\lambda^{(i)}$ and $\Sigma_{i=1}^2\lambda^{(i)}$ in these two expressions.

Specification of the top-down approach. In order to fix a specific top-down model, we have to specify the economy-wide default intensity $\bar{\lambda}$. For simplicity, we again assume that $\bar{\lambda}$ follows a one-factor mean-reverting square-root process:

$$d\bar{\lambda}_t = \kappa\left(\theta - \bar{\lambda}_t\right)dt + \eta\sqrt{\bar{\lambda}_t}\,dB_t$$

with constant parameters κ, θ, and η and a standard Brownian motion B. Because this process is $\mathcal{F}_t^X$-adapted, we can consider the correlation ϱ_{i3} of the Brownian Motion B to the $W^{(i)}$'s of Equations (3) that drive the thinning process. The thinning is specified as previously discussed. Given these specifications, we are able to compute variances and covariances of the default times between each of the firms. Using

$$\rho_{12} = \frac{\mathrm{Cov}(N^{(1)}, N^{(2)})}{\sqrt{\mathrm{Var}(N^{(1)})} \cdot \sqrt{\mathrm{Var}(N^{(1)})}} \tag{15}$$

gives us the desired default correlation. The detailed derivation can be found in Kunisch/Uhrig-Homburg [2007].

Simulation and Comparison Setting

To evaluate these expressions for the correlations and the prices of credit derivatives we perform a Monte-Carlo simulation. For this purpose we have to discretize the continuous-time stochastic processes by applying an Euler approximation on the economy-wide intensity process, the individual intensities, and the asset values. Then, we simulate the values of these processes up to the horizon T. We choose equidistant time steps with 50 time steps per year. At each time step, we compute the values for the thinning process and calculate the integrated intensity processes. Based on 10,000 simulation runs for a given set of parameters, we compute the default probabilities, the default correlation and the fair spreads for the credit derivatives.

In order to ensure consistency between the economy-wide intensity $\bar{\lambda}$ and the individual intensities of the standard doubly-stochastic model, starting values are chosen such that an aggregation of the individual values at time $t = 0$ reproduces the initial value of $\bar{\lambda}$. A similar assumption is made with respect to the long-term means, $\theta = \theta_1 + \theta_2$, while we assume that volatilities and the mean-reversion parameter are identical for all three processes, i.e., $\eta = \eta_1 = \eta_2$ and $\kappa = \kappa_1 = \kappa_2$. Furthermore, we select the values for the different processes such that the individual default probability of the companies in the extended Merton and Lando models are equal and the probability of at least one default is the same in all three approaches. In the base case, we select the individual default probability to be 1.23%. The default probability of 1.23% typically[16] corresponds to a lower investment-grade rating such as A or Baa.

As the probability of at least one default is the sum of the individual default probabilities, we get a probability of 2.46% for at least one default. These assumptions lead to the following parameters for the different processes as shown in Exhibit 1.

The parameter values for the mean-reverting square-root process $\bar{\lambda}$ are taken from Bühler/Trapp [2006]. With these parameter values[17] we get the desired individual default probability of 1.23%. Furthermore, we allow correlation of $\bar{\lambda}$ with the asset values $V^{(i)}$. We set these parameter $\varrho_{i,3}$ to 0.2. To specify the asset-value processes needed for the extended Merton model, we orientate our parameter selection on Eom et al. [2004]. For their sample they estimate[18] a mean asset volatility of 0.236 and a range for the leverage ratios from 0.086 to 0.864. Here we select 0.2 as asset volatilities. Our chosen default probability of 1.23 is consistent with a leverage ratio $L^{(i)}/V_0^{(i)}$ of 0.727, which is in the range of Eom et al. [2004]. For the asset correlation ϱ_{12}^W we choose 0.2, which is in the interval of [0.12, 0.24] given by the Basel Committee on Banking Supervision. This parameter will be varied in the next

EXHIBIT 1

Selected Parameter Values for the Simulation Study

Benchmark models

Parameter values for the extended Merton model:

$V_0^{(1)}$	$V_0^{(2)}$	r	σ_1	σ_2	$L^{(1)}$	$L^{(2)}$	ϱ_{12}^W
1.375	1.375	0.03	0.2	0.2	1	1	0.2

Parameter values for the extended intensity model:

$\lambda_0^{(1)}$	$\lambda_0^{(2)}$	κ_1	κ_2	θ_1	θ_2	η_1	η_2	$\varrho_{1,2}^B$
0.01	0.01	0.005	0.005	1	1	0.01	0.01	0.2

Top-down model

Parameter values for economy-wide default intensity:

$\bar{\lambda}_0$	κ	θ	η	$\varrho_{1,3}$	$\varrho_{2,3}$
0.02	0.005	2	0.01	0.2	0.2

Parameter value for the thinning process:

$V_0^{(1)}$	$V_0^{(2)}$	μ_1	μ_2	σ_1	σ_2	$L^{(1)}$	$L^{(2)}$	ϱ_{12}	T
1.375	1.375	0.15	0.15	0.2	0.2	1	1	0.3	1

section to illustrate changes in the default correlation and its impact on premiums for credit derivatives.

Results and Interpretation

Exhibit 2 illustrates the impact on default correlation for all three models over varying time horizons.

As we can see, the extension of the reduced-form model is completely unable to produce significantly positive default correlations. Its default correlation is on the order[19] of 10^{-5}. For short maturities, both the structural and the reduced-form models produce a negligible correlation. However, with increasing time horizon the default correlation in the structural model increases. This arises from an increasing volatility in the distribution of the asset-value process, i.e., it is more likely that the asset-value process will be below the default barrier. Such a behavior is consistent with the empirical results reported by Lucas [1995]. The top-down approach produces these effects and in addition significantly positive default correlations even for short horizons. So, our parametrization shows that the top-down approach dominates the basic model extensions.

A main driver of the dependence between the firms is the correlation between their asset value processes (asset correlation). In the extended Merton setting the impact of the asset correlation on the default correlation is well understood. In our top-down approach the asset correlation impacts the default correlation indirectly through the thinning process. To obtain a similar impact in the extended intensity models the only way is through the correlation of the individual intensity processes $\lambda^{(i)}$. Exhibit 3 illustrates the impact of the microstructure correlation on the default correlation.

Not surprisingly, the increase of the correlation between the two intensities has no significant effect on the default correlation. The opposite is true when turning to the extended Merton model and our top-down approach. As we can see, these models behave similarly when this value increases. The strong increase in the Merton model is especially due to the fact that we have two symmetric firms and that therefore an increase in the asset correlation has the strongest effects on the event that both will default within the horizon.

In the next step, we vary the starting values of the intensities $\bar{\lambda}_0$ and $\lambda_0^{(i)}$. To make the extended Merton model comparable to these changes in the intensity, we vary the level of the asset value-debt ratio so that the individual default probabilities will be equal. The change of the intensity level for the doubly-stochastic model has no impact on the default correlation, as shown in Exhibit 4. This is related to the fact that although the probability of default increases as expected covariance and variance increase by the same magnitude. In our top-down model we can observe an increase in the default correlation whereas the increase in the extended Merton model is more pronounced. This is due to the fact that default events become more likely, and related to this also the intensity for joint events increases, which results in a higher default correlation. With the increase of the asset value-

EXHIBIT 2

Default Correlation in the Three Approaches with Different Time Horizons

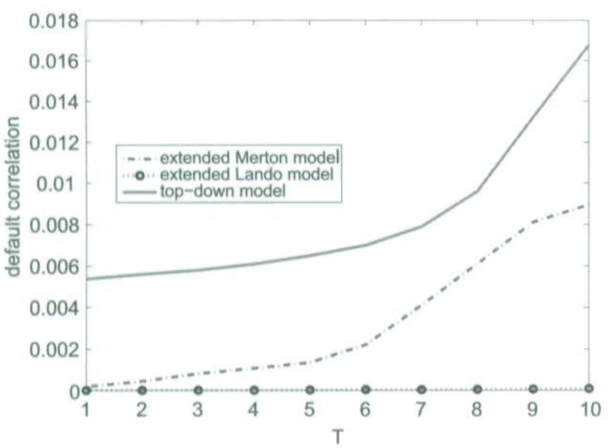


EXHIBIT 3

Default Correlation in the Three Approaches with Different Asset Correlations

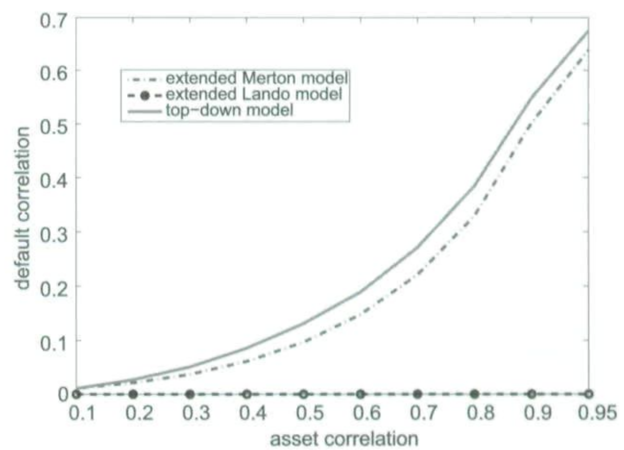
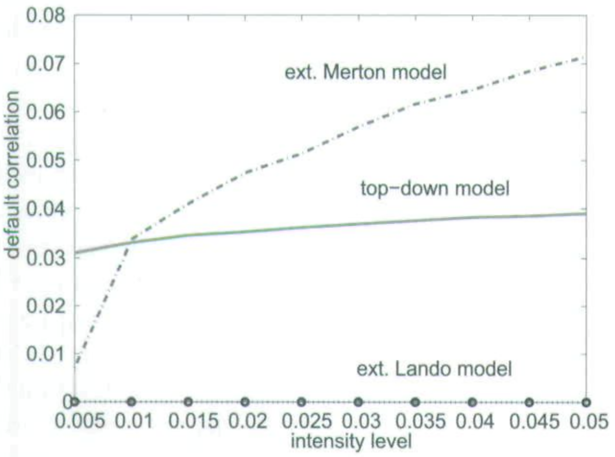


EXHIBIT 4

Default Correlation with Different Intensity Levels for All Three Models

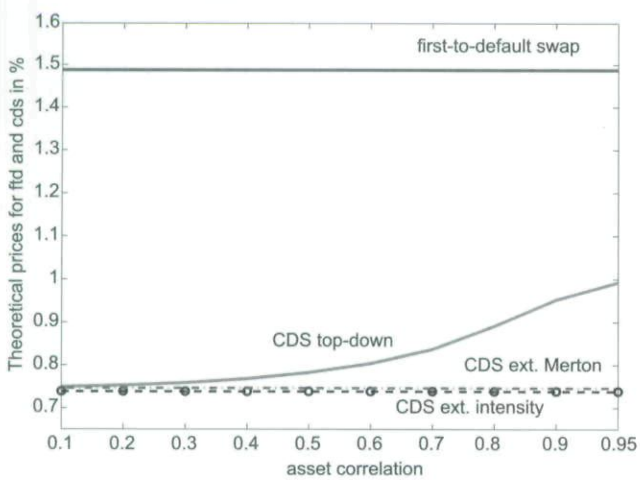


debt ratio in the extended Merton model, it is more likely that both companies could be in default at maturity. This results in an increase of the joint probability and leads directly to an increasing default correlation.

Besides the effect on the dependence measure, a more interesting fact from a practical point of view is how such changes in the default dependencies are affecting fair values of credit derivatives. Here, we use the formulas for CDS and first-to-default swaps derived previously. We assume the risk-free rate is constant at 3% in order to isolate effects of changes in model parameters. Exhibit 5 shows the theoretical prices for these two credit derivatives in all three models as we vary the microstructure correlation.

EXHIBIT 5

Impact of the Asset Correlation on Theoretical Prices of CDS and First-to-Default Swap



By construction the first-to-default swap has the same value for all three approaches because all models are parameterized in such a way that they have the same probability of at least one default. This probability does not depend on the asset correlation. For low values of the asset correlation we only observe a slight difference between the models for the value of a single-name CDS. This owes to the fact that all three models have nearly the same individual default probability.

When asset correlation increases, the value for a CDS in the extended Merton model and the extended Lando model does not vary with the microstructure correlation because the individual default probabilities are not affected by this correlation. However, in our top-down approach the probability that a simultaneous default could happen increases, which is equivalent to an increase in the default intensity. This increasing default intensity directly leads to increasing spreads for the contract.[20] We can conclude that only in our model is the CDS spread sensitive to the changes in the asset correlation. The asset correlation in our model controls for the likelihood that the firm defaults as a result of a contagion effect, while the benchmarks neglect this contagion risk.

We also check the influence of variation in the intensity levels as we have in Exhibit 4 but now evaluate the impact on the theoretical prices of CDS and first-to-default swaps. We observe increases in the spreads for all contracts in all models simply because the likelihood of a default increases with higher intensity or higher asset value-debt ratios. With this increase in our model the intensities for a default only by company 1 and the joint default of both companies increase as well. Therefore this leads to a slightly stronger increase of the prices in our model than in the other two benchmarks.

Finally, we also want to comment on the most popular model in the market, the Gaussian copula model. As all copula models this model is driven by a fixed set of parameters whose values have to be selected at the beginning. In the Gaussian copula model this parameter is the default correlation matrix. By fixing this parameter set, the whole dependence structure is fixed together with the reaction in case of a default at a future point in time. Therefore, increasing default correlations as observed in the subprime crisis are not captured by this framework. In fact, in contrast to the proposed top-down approach the copula models provide no link to economic variables that could explain a dynamic dependence structure.

CONCLUSION AND OUTLOOK

In this article we have introduced a new approach towards modelling default dependencies that is based on a top-down construction. The central element of this construction is the economy-wide default intensity $\bar{\lambda}$. By using random thinning on this process we have been able to derive the intensities for the basic entities. In this case, these entities are not single companies, but sets of companies that can default. This new entity allows for simultaneous defaults of firms and opens a new way in modelling dependencies. To deduce the intensities for the independent default sets we derive the components in the thinning process by using structural elements in a setting similar to Merton [1974]. This procedure incorporates structural information in the setting of a reduced-form model while maintaining the advantages of a reduced-form model. Furthermore, we can capture two main facts for default dependencies with this top-down construction: common risk factors and contagion effects. The common risk factors drive the stochastic intensity and the asset correlation between the asset values included in the thinning process. The contagion effects are modelled through the default sets. All sets with more than one firm represent events in which contagion causes multiple defaults. The tendency to simultaneous defaults is mainly driven through the thinning process. Here, the asset correlation is the main driver of this component as illustrated in our comparison. With increasing correlation between companies this combination becomes more likely to default because of the increased value of the associated thinning component. In contrast to existing contagion models this top-down approach can handle those effects more easily as we remain in a doubly stochastic framework. Therefore we have no looping effects and are able to value under the filtration of the macroeconomic state variables given that our events are conditionally independent.

Another major advantage of our framework is that we can easily derive formulas for defaultable claims and evaluate them. We explicitly derive formulas for single-name CDS and first-to-default swaps and investigate their behavior if parameters change. In fact, the model of Lando [1998] is a special case of our approach. When thinning factors for default sets with two or more elements are set to zero, the contagion factors in the representation of the intensity of company i vanishes and default correlation in that case is only driven by the state variables. Therefore, our approach can be understood as an extension of the standard doubly stochastic default models but the approach allows for more flexibility in specifying the dependence between firms and therefore produces dynamic default correlations.

We can also apply our framework to synthetic tranches of credit derivatives indices, such as the Dow Jones CDX North American Investment Grade Index or the Dow Jones iTraxx Europe. For this purpose we would have to build a model for the losses that arise in case of a default. This could be easily done similar to Longstaff/Rajan [2008] and may be a promising area of future research on the correlation skew of CDO tranches.

We have not yet discussed which combinations of companies should be grouped together to form a common default event. The crux is how to sensibly reduce the number of combinations to a manageable number. A simple approach could be to set $\hat{p}^A(t)$ to zero for sets that have more than two elements. A more economic approach is to allow common events only for companies that have a direct economic relationship because they belong to the same industrial sector or are linked in their supply chains.

Furthermore, it remains an open question as to whether the Merton-like approach is a promising specification for computing the desired probabilities of joint defaults in the thinning process. An alternative approach, which we leave for future research, is to use scores based on structural firm information such as leverage ratios or the distance-to-default. Finally, the economy-wide intensities could be easily extended to multiple factors. In principle, jumps could also be included, however it is an empirical question whether this additional complexity is warranted.

ENDNOTES

[1]See Hull and White [2001] and Das, Duffie, Kapadia, and Saita [2007].

[2]In Schönbucher/Schubert [2001] jumps at default times are captured through a pre-specified copula.

[3]Credit events by the ISDA are bankruptcy, obligation acceleration, obligation default, failure to pay, repudiation, and restructuring.

[4]This is similar to the case of the iTraxx portfolio in which defaulted CDS will be replaced by non-defaulted ones each six month.

[5]At this point neither the type of process driving N nor a dependence structure among the intensities $\lambda^{(i)}$ belonging to $N^{(i)}$ is assumed.

[6]Under $\mathbb{Q}$ discounted asset prices are martingales with respect to the filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$. See Harrison/Pliska [1981] for details.

[7]Instead of assuming an intensity adapted to the underlying state vector only, we could also allow an $\mathcal{F}_t$-adaption. This would mean

that the value of the economy-wide intensity can also be affected by individual default events.

[8]Therefore the jumps in $\tilde{N}$ follow a Poisson process as only one event can happen at default. Note this does not imply an assumption about the underlying point process of N because multiple simultaneous jumps can occur in N.

[9]See Brémaud [1981], Theorem 6, p. 26.

[10]The jump-size of one is no restriction because default times τ_j are identified with a unique set. Therefore only the default of one set $A \in \mathbb{X}$ can occur. A simultaneous default of an additional set B in the same short period of time is not possible. The event that both sets default at the same time is already governed by the set $A \cup B = C \in \mathbb{X}$.

[11]See Dellacherie/Meyer [1982].

[12]See Giesecke/Goldberg [2007], p. 10.

[13]Of course the alternative way to change the measure of λ^A directly can be done as shown in Jarrow/Lando/Yu [2005]. The results stay the same, due to the multiplicative structure of thinned intensities $Z^A \cdot \bar{\lambda}$.

[14]See Duffie [1998], p. 3.

[15]Again refer to Kunisch/Uhrig-Homburg [2007] for a detailed derivation.

[16]See Moody's Investor Report [2008], Exhibit 11, p. 11, and Exhibit 18, p. 16.

[17]See Bühler/Trapp]2006], p. 11 and p. 21.

[18]Compare Eom et al. [2004], p. 505.

[19]The value does not change significantly when using other parameters. In particular, an increase in the correlation between the intensities as high as 1 scarcely changes the basic result.

[20]Compare with Equation (5).

REFERENCES

Brémaud, P. *Point Processes and Queues: Martingale Dynamics.* Berlin, Heidelberg, New York: Springer, 1981.

Bühler, W., and M. Trapp. "Credit and Liquidity Risk in Bond and CDS Markets." Working paper, University of Mannheim, 2006.

Collin-Dufresne, P., R. Goldstein, and J. Helwege. "Is Credit Event Risk Priced? Modeling Contagion via the Updating of Beliefs." Working paper, Carnegie Mellon University, 2003.

Das, S., D. Duffie, N. Kapadia, and L. Saita. "Common Failings: How Corporate Defaults are Correlated." *Journal of Finance*, 62 (2007), pp. 93–117.

Davis, M., and V. Lo. "Infectious Defaults." *Quantitative Finance*, 1 (2001), pp. 382–386.

Dellacherie, C., and P.A. Meyer. *Probabilities and Potential: Theory of Martingales.* Amsterdam: North-Holland, 1982.

Duffie, D. "First-to-Default Valuation." Working paper, Graduate School of Business Stanford University and Université de Paris, Dauphine, 1998.

Duffie, D., and K. Singleton. "Modeling Term Structures of Defaultable Bonds." *Review of Financial Studies*, 12 (1999), pp. 687–720.

Eom, Y.H., J. Helwege, and J. Huang. "Structural Models of Corporate Bond Pricing: An Empirical Analysis." *Review of Financial Studies*, 17 (2004), pp. 499–544.

Errais, E., K. Giesecke, and L. Goldberg. "Pricing Credit from the Top Down with Affine Point Processes." Working paper, Graduate School of Business Stanford University, 2007.

Giesecke, K., and L. Goldberg. "A Top-Down Approach to Multi-Name Credit." Working paper, Cornell University, 2007.

Harrison, J., and S. Pliska. "Martingales and Stochastic Integrals in the Theory of Continious Trading." *Journal of Economic Theory*, 11 (1981), pp. 215–260.

Hull, J., and A. White. "Valuing Credit Default Swaps II: Modeling Default Correlations." *Journal of Derivatives*, 8 (2001), pp. 12–22.

Jarrow, R., D. Lando, and F. Yu. "Default Risk and Diversification: Theory and Empirical Implications." *Mathematical Finance*, 15 (2005), pp. 1–26.

Jarrow, R., and F. Yu. "Counterparty Risk and the Pricing of Defaultable Securities." *Journal of Finance*, 56 (2001), pp. 1765–1799.

Kunisch, M., and M. Uhrig-Homburg. "A Top-Down Framework for Modelling Default Dependencies." Working paper, Universität Karlsruhe (TH), 2007.

Lando, D. "On Cox. Processes and Credit Risky Securities." *Review of Derivatives Research*, 2 (1998), pp. 99–120.

Longstaff, F., and A. Rajan. "An Empirical Analysis of the Pricing of Collateralized Debt Obligations." *Journal of Finance*, 63 (2008), pp. 529–564.

Lucas, D. "Default Correlation and Credit Analysis." *Journal of Fixed Income*, 4 (1995), pp. 76–87.

Merton, R.C. "On the Pricing of Corporate Debt: The Risk Structure of Interest Rates." *Journal of Finance*, 29 (1974), pp. 449–470.

Moody's Global Corporate Finance. "Corporate Default and Recovery Rates, 1920–2007." Special Comment Moody's Investor Service, 2008.

Schönbucher, P. J. "Frailty Models, Contagion, and Information Effects." Working paper, ETH Zürich, 2004.

Schönbucher, P. J., and D. Schubert. "Copula-Dependent Default Risk in Intensity Models." Working paper, University of Bonn, 2002.

Zhou, C. "An Analysis of Default Correlation and Multiple Defaults." *Review of Financial Studies*, 14 (2001), pp. 555–576.